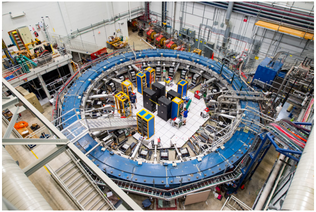
Short-distance constraints on the hadronic light-by-light

Nils Hermansson-Truedsson

In collaboration with J. Bijnens (Lund) and A. Rodríguez–Sánchez (Valencia)

September 2, 2024

THE UNIVERSITY *of* EDINBURGH



Experiment Figure from Fermilab

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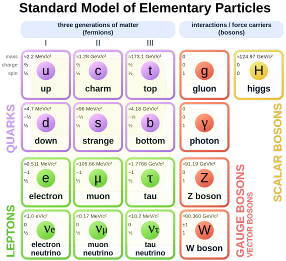


Theory prediction

+ ?

Unknown

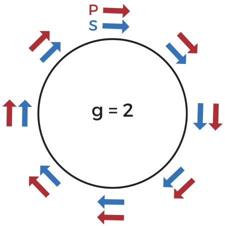
Search for new particles with low-energy precision calculations

- If real tension found, search for new particles in direct detection
- Need good control of theoretical/experimental uncertainties
- Flavour physics $|V_{us}|$ and $|V_{ud}|$ [RM123/S 2019; RBC/UKQCD 2023; Talk by MDC]
- Today: Muon anomalous magnetic moment

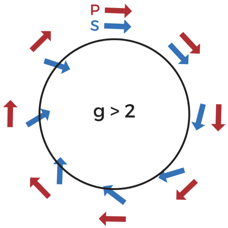
- Muon has spin **S** and magnetic moment

$$\vec{\mu} = g_\mu \frac{Q_\mu e}{2m_\mu} \vec{S}$$

- $g_\mu = 2$ is the Dirac value for gyromagnetic ratio
- Spin interacts with external magnetic field
- Let muon move circularly in homogeneous magnetic field



Not real life



Anomalous [Schwinger 1948]

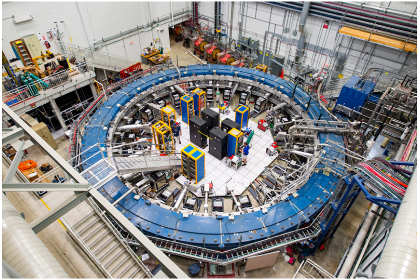


Figure from Fermilab

I. Muons injected into ring

II. Angular frequency of motion

$$\omega_c = \frac{e\,B}{m_\ell \gamma}$$

III. Spin Larmor precession frequency

$$\omega_s = \omega_c + \underbrace{\frac{(g_\ell - 2)}{2}}_{a_\mu} \frac{e\,B}{m_\ell}$$

IV. Detect electrons from muon decay

Fermilab experiment is done! Final data analysis now (early 2025)

- JPARC experiment will make independent valuable measurement

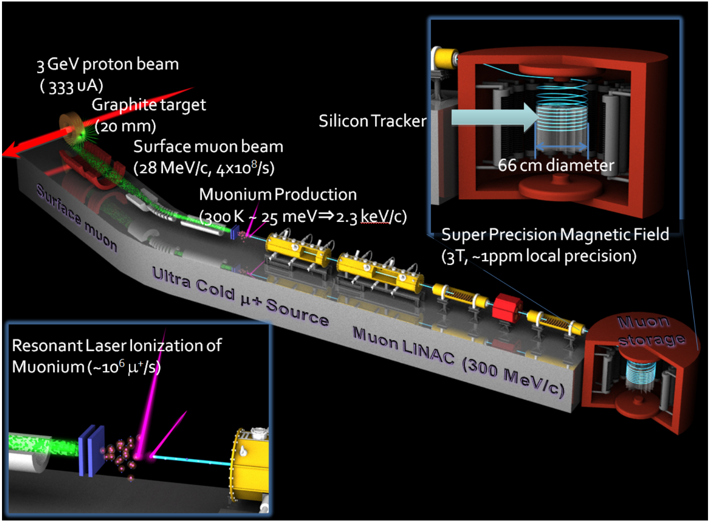
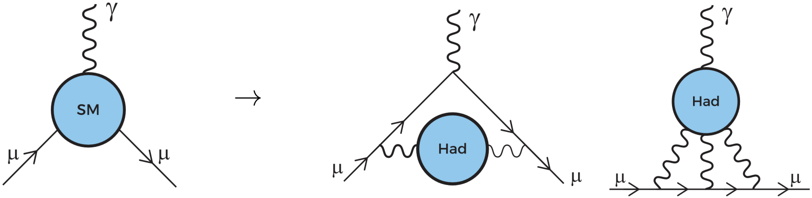


Figure from JPARC

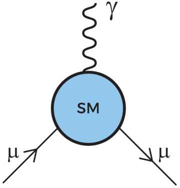
Why anomalous? Virtual corrections



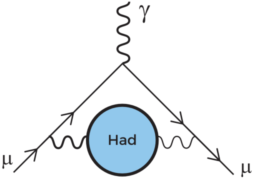
$\Longrightarrow$ We can use the Standard Model for this

If tension: Sensitive to "light" new physics

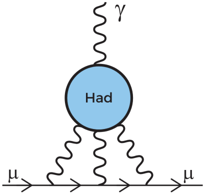
$$a_\mu = \frac{(g-2)_\mu}{2} \sim \frac{m_\mu^2}{M_{NP}^2}$$



Standard Model



Hadronic vacuum polarisation



Hadronic light-by-light

$$a_\mu^{\mathrm{SM},\,2020} = 116\,591\,810(43) \times 10^{-11}$$ [White Paper 20]

$$a_\mu^{\mathrm{QED}} = \underbrace{116\,584\,718.931(104) \times 10^{-11}}_{\text{5 QED loops!}}$$ [Aoyama, Hayakawa, Kinoshita, Nio 12/19]

- Compare: $a_\mu^{\mathrm{QED}} = \alpha/(2\pi) \approx 116\,2\ldots \times 10^{-11}$ [Schwinger 48]
- Hadronic corrections biggest uncertainty
- Analytical methods, dispersion theory, Lattice QCD

$$a_\mu^{\mathrm{HVP}} = 6\,931(40) \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL}} = 92(19) \times 10^{-11}$$

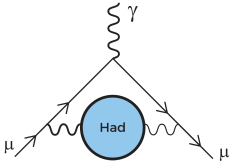
- Standard Model (SM) [White Paper, 2020] vs. Experimental value Brookhaven/Fermilab

$$a_\mu^{\mathrm{SM,\,2020}} = 116\,591\,810(43) \times 10^{-11}\,,$$
$$a_\mu^{\mathrm{exp}} = 116\,592\,059(22) \times 10^{-11}\,,$$
$$\Delta a_\mu = 249(48) \times 10^{-11}$$

- 5.2σ deviation between experiment and theory [White Paper 2020; Fermilab 2023]

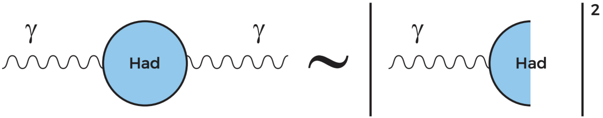
$\Delta a_\mu > 5\sigma$: Did we discover new physics!?

- Not really... Lattice QCD does not agree with dispersion theory [BMW 20/24]

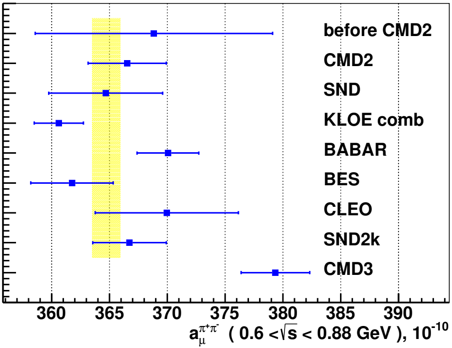


Hadronic vacuum polarisation

- Dispersion theory: HVP from experimental data:
  $e^+e^- \to \gamma^* \to \text{hadrons}$



- Inconsistent data for $e^+e^- \to \gamma^* \to \pi\pi$



$\longrightarrow$ Current status

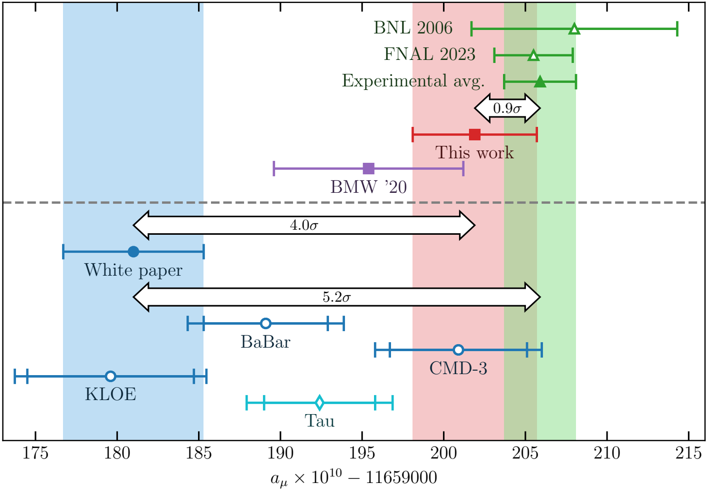
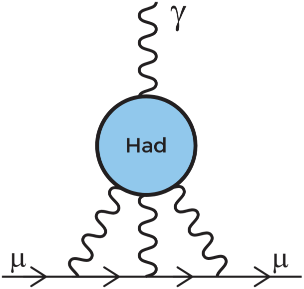


Figure from [BMW 24]

- Has to be better understood
- Lattice QCD: Parts of BMW calculation cross-checked
- Dispersion theory + experiments under scrutiny
- Should be settled in coming few years
- Still need to improve precision on HLbL: Today



- Hadronic light-by-light
- White paper value:

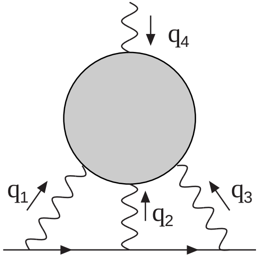
$$a_\mu^{\mathrm{HLbL}} = 92(19) \times 10^{-11}$$

- $\approx 20\%$ relative uncertainty
- Precision goal is 10%

- Weighted average of dispersion theory and lattice QCD
- New lattice results:

$$a_\mu^{\mathrm{HLbL,\ RBC/UKQCD\ 23}} = 124.7(14.9) \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL,\ Mainz\ 21/22}} = 109.6(15.9) \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL,\ BMW\ prel.}} = 126.8(13.0) \times 10^{-11}$$

- Today: Short-distance constraints from QCD on HLbL tensor
- Relevant for dispersive approach



- Problematic since several momentum scales involved in the diagram
- Four momenta q_1, q_2, q_3 and q_4
- $q_4 \to 0$ (static limit) and $q_{1,2,3}$ integrated over
- Loop integral has different regions ($Q_i^2 = -q_i^2$):
    - $Q_i^2 \gg \Lambda_{\mathrm{QCD}}^2$ all large: short-distance region
    - $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda_{\mathrm{QCD}}^2$: Melnikov-Vainshtein limit
    - $Q_i^2 \ll \Lambda_{\mathrm{QCD}}^2$: low-energy limit

- Dispersive approach: sum over intermediate states [Colangelo et al 15/17]
- States: π, K, η, η', $\phi(1020)$, $h_1(1170)$, $b_1(1235)$, ...

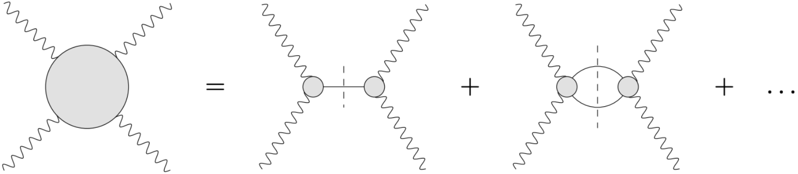
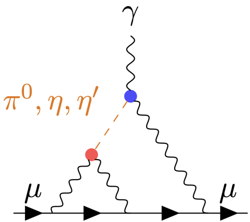


Figure from [Colangelo et al. 17]

- This is an infinite tower of states, ordered by masses
- Let us look how this enters the HLbL diagram

- Leading contribution is the π, η, η' pole:
  $a^{\text{HLbL}}_{\mu,\pi,\eta,\eta'} = 93.8(4.0) \times 10^{-11}$
- Compare to $a^{\text{HLbL}}_{\mu} = 92(19) \times 10^{-11}$



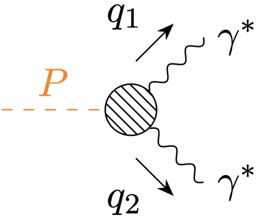


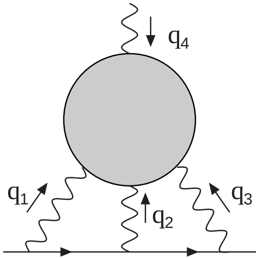
Figure from S. Holz @ Chiral Dynamics 2024

- Not Feynman diagrams, the meson is on-shell
- Transition form factors $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ and $F_{P\gamma^*\gamma}(Q_3^2, 0)$
- Need data to know these for all states in dispersive method

- Experiments can measure some form factors (e.g. BESIII)
- Lattice QCD can be used to determine some: ETMC, Mainz, BMW
- Dispersive method requires control of remaining contributions
- Model calculations: axial vectors, tensors, scalars

  [Bern group; Rebhan group; Cappiello group]
- Models have to match QCD: Short-distance constraints (SDCs)
- Goal: Combine dispersive + models + SDCs
- Biggest part of uncertainty from short-distance:

$$a_\mu^{\text{HLbL, SDC}} = 15(10) \times 10^{-11} \qquad \text{vs} \qquad a_\mu^{\text{HLbL}} = 92(19) \times 10^{-11}$$



- HLbL rigorously constrained by QCD: SDCs
- Operator product expansion (OPE) techniques in kinematical limits
- $Q_i^2 \gg \Lambda^2_{\mathrm{QCD}}$ all large: short-distance region 1 soft photon (SDC1)
- $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda^2_{\mathrm{QCD}}$: Melnikov-Vainshtein 2 soft photons (SDC2)
- Separately: SDCs on form factors [Brodsky-Lepage-Radyushkin; Light cone sum rules; ...]

Motivation:

- Want to make a systematic OPE to derive SDCs
- Can reduce systematic uncertainties in the HLbL
- One soft photon: Always *assumed* that pQCD quark loop is first term in an OPE and sufficiently good
  - → Which OPE is it?
  - → What about the higher order terms in the OPE?
  - → Paper I: [Bijnens, NHT, Rodríguez–Sánchez 19]
    Paper II: [Bijnens, NHT, Laub, Rodríguez–Sánchez 20]
    Paper III: [Bijnens, NHT, Laub, Rodríguez–Sánchez 21]
- Two soft photons: Leading-order OPE known [Melnikov, Vainshtein 04]
  - → What about higher-order corrections?
  - → Paper 4: [Bijnens, NHT, Rodríguez–Sánchez 23]
  - → In preparation: [Bijnens, NHT, Rodríguez–Sánchez]
- Future: Where should we start relying on SDCs?
- Future: How to combine everything?

- Fundamental object: HLbL tensor [Bardeen, Tung 71; Tarrach 75; Colangelo et al. 15/17]

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3) = -i \int \frac{d^4 q_4}{(2\pi)^4} \left(\prod_{i=1}^{4} \int d^4 x_i \, e^{-iq_i x_i} \right) \times \langle 0 | T \left\{ J^{\mu_1}(x_1) J^{\mu_2}(x_2) J^{\mu_3}(x_3) J^{\mu_4}(x_4) \right\} | 0 \rangle \overset{\mathrm{WI}}{=} q_{4\,\nu_4} \frac{\partial \, \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}}$$

- Need to obtain 54 scalar functions $\hat{\Pi}_i$

$$\lim_{q_4 \to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4\nu_4}} = \lim_{q_4 \to 0} \sum_{i=1}^{54} \frac{\partial \hat{T}_i^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4\nu_4}} \hat{\Pi}_i \quad \leftarrow \text{Projection}$$

- $g-2$: Only 6 independent $\hat{\Pi}_i$ contribute: $i = 1, 4, 7, 17, 39, 54$
- Once you know $\hat{\Pi}_i$ you can get the HLbL $\leftarrow$ Projection

- We can get a_μ^{HLbL} in the SD regime by putting restrictions in the integration consistent with $\hat{\Pi}_i$ from an OPE

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau\, \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} f_i\left(\{\hat{\Pi}_j\}\right)$$

$$\longrightarrow \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau\, \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \underbrace{\mathcal{R}(Q_1, Q_2, \tau))}_{\text{kin. reg.}} \sum_{i=1}^{12} f_i\left(\{\hat{\Pi}_j\}\right)$$

- How can we get at the $\hat{\Pi}_i$ in SDC kinematics?

- Start with SDC1, then SDC2

- SDC1: $Q_1^2, Q_2^2, Q_3^2 \gg \Lambda_{\mathrm{QCD}}^2$
- Euclidean space: Short-distance $\leftrightarrow$ large Q_i^2
- Usual OPE: Expand for large Q_i^2
- Problem: $Q_4^2 \to 0 \implies$ problems in OPE of $\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3)$

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3) = -i \int \frac{d^4 q_4}{(2\pi)^4} \left(\prod_{i=1}^{4} \int d^4 x_i \, e^{-iq_i x_i} \right) \times \langle 0 | T \left\{ J^{\mu_1}(x_1) J^{\mu_2}(x_2) J^{\mu_3}(x_3) J^{\mu_4}(x_4) \right\} | 0 \rangle$$

- How to see this? Only leading perturbative term is finite
- Need to rethink the problem for a convenient approach Paper I (2019)

- Problem because of soft photon for $(g-2)_\mu$:
  Do an OPE in an external EM field

$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2) = -\frac{1}{e}\int \frac{d^4q_3}{(2\pi)^4}\left(\prod_{i=1}^{3}\int d^4x_i\, e^{-iq_ix_i}\right) \times \langle 0|\, T\left(J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3)\right)|\gamma(q_4)\rangle$$

$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2) = i\epsilon_{\nu_4}(q_4)q_{4,\,\mu_4}\lim_{q_4\to 0}\frac{\partial\,\Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_4^{\nu_4}}\,.$$

- We can thus obtain a_μ^{HLbL} from $\Pi^{\mu_1\mu_2\mu_3}$
- Such an OPE introduced for baryon magnetic moment sum rules [Balitsky,Yung, 83], [Ioffe,Smilga, 84], and later for the EW $g-2$ contributions as well [Czarnecki,Marciano,Vainshtein, 03]
- Different from vacuum OPE [Shifman, Vainshtein, Zakharov, 79]

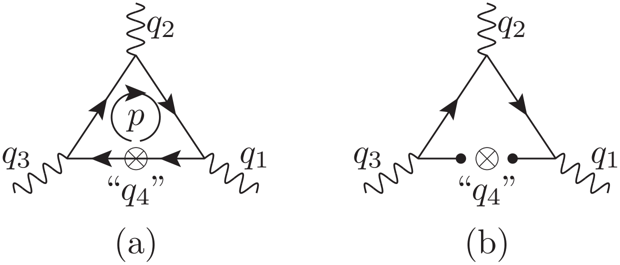
- Let us look at how to do it

$$J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3) = \prod_{i=1}^{3} \overline{q}(x_i)\gamma^{\mu_i}q(x_i)$$

- Expand $q(x_i) = q(0) + x_i^{\mu}\,\partial_\mu q(0) + \ldots$
- Radial gauge: $A_\mu(z) = \frac{1}{2}z^\nu F_{\nu\mu}(0)$
- Promote $\partial_\mu \leftrightarrow D_\mu$ [Pascual, Tarrach 84]

$$J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3) = \prod_{i=1}^{3} \left[\overline{q}(0) + \overline{q}(0)\,\overleftarrow{D}_\mu\, x_i^\mu\right]\gamma^{\mu_i}\left[q(0) + x_i^\mu\, D_\mu q(0)\right]$$

- Expand Dyson series in $\langle 0| \, T(J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3)) \, |\gamma(q_4)\rangle$
- Contract and do not contract
- Leads to a bunch of OPE terms. Tedious in position space
- Can do it in momentum space with Feynman diagrams instead



- (a) Confirms old perturbative quark loop expectation for LO OPE
- (b) Leading non-perturbative term: $\langle \bar{q}\sigma_{\mu\nu}q\rangle$ Magnetic susceptibility

- The OPE is an expansion in $1/Q_i$ and α_s
- Through a given order, we can thus extract

$$\hat{\Pi}_i = P^i_{\mu_1\mu_2\mu_3\mu_4\nu_4} \lim_{q_4\to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4\nu_4}}$$

- Quark loop: $1/Q_i^2$ scaling
- With $\langle \bar{q}\sigma_{\mu\nu}q\rangle = e_q\, X_q\, F_{\mu\nu}$ the non-perturbative correction is

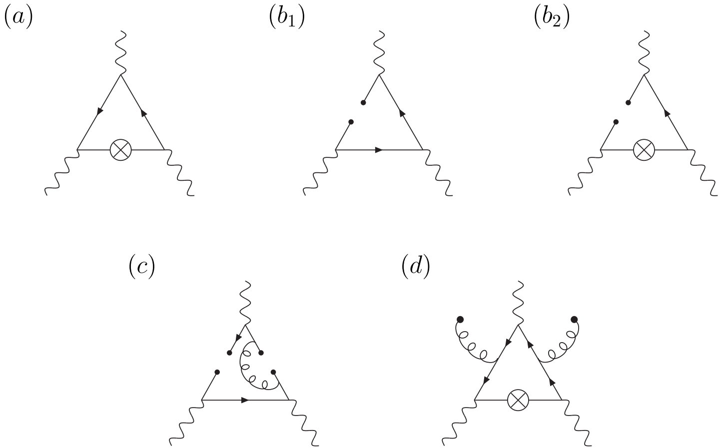
$$\hat{\Pi}_1 = m_q X_q e_q^4 \frac{-4\left(Q_1^2+Q_2^2-Q_3^2\right)}{Q_1^2 Q_2^2 Q_3^4}\,, \qquad \hat{\Pi}_7 = 0\,,$$

$$\hat{\Pi}_4 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2}\,, \qquad \hat{\Pi}_{39} = 0\,,$$

$$\hat{\Pi}_{17} = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^4}\,,$$

$$\hat{\Pi}_{54} = m_q X_q e_q^4 \frac{-4\left(Q_1^2-Q_2^2\right)}{Q_1^4 Q_2^4 Q_3^2}\,.$$

- Papers I and II: Perturbative: (a) quark loop $1/Q^2$
- Non-perturbative: $1/Q^4$, $1/Q^6$:
  (b$_1$) $\langle \bar{q}\sigma_{\mu\nu}q\rangle$, (b$_2$) $\langle \bar{q}q\rangle$, (c) $\langle \bar{q}\,\Gamma_1 q\,\bar{q}\,\Gamma_2 q\rangle$, (d) $\langle \alpha_s GG\rangle$



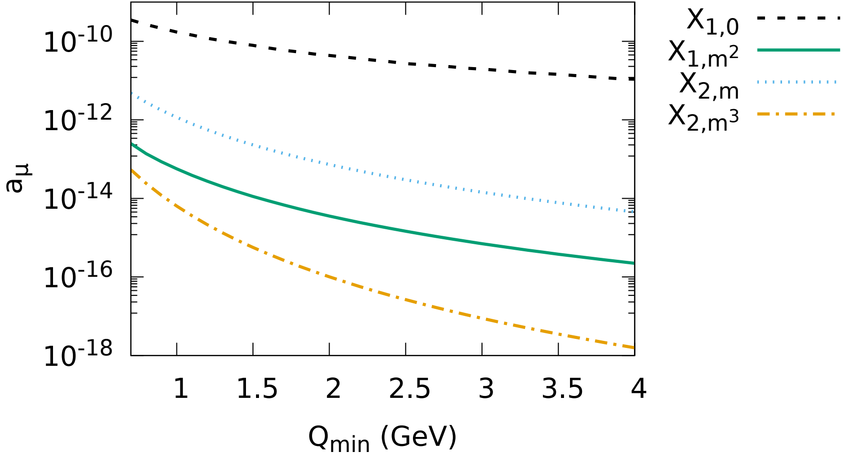
- In general can then write the OPE as

$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = \underbrace{\vec{C}^{\mu_1\mu_2\mu_3\mu_4\nu_4}_{\overline{\mathrm{MS}}}(q_1,q_2,\mu)}_{\text{pert. Wilson}} \cdot \underbrace{\vec{X}(\mu)\ \langle F_{\mu_4\nu_4}\rangle}_{\text{non-pert. ME}}$$

- Need to define matrix elements (ME) at renormalisation scale μ
- Can be calculated in lattice QCD, VMD, ...
- $\langle \overline{q}q \rangle$ [FLAG (JLQCD, ...)]; $\langle GG \rangle$ [Shifman, Vainshtein, Zakharov 78]; $\langle \overline{q}\sigma_{\mu\nu}q \rangle$ [Bali et al. 20]; ...
- How good is the quark loop?

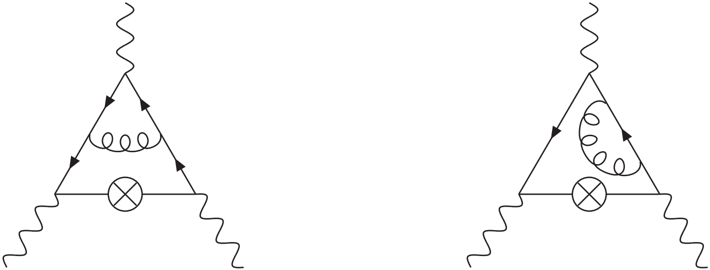
$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_{Q_{\min}}^{\infty} dQ_1 \int_{Q_{\min}}^{\infty} dQ_2 \int_{-\tau(Q_{\min})}^{\tau(Q_{\min})} d\tau\, \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} f_i\left(\{\hat{\Pi}_j\}\right)$$

- Numerically studied these as well, for $Q_i^2 > Q_{\min}^2$



- Non-perturbative condensates suppressed by two orders of magnitude in general in SDC1

- We thus see that non-perturbative corrections to the massless quark loop are very small
- Paper III: What about the massless perturbative $\mathcal{O}(\alpha_s)$ correction to the quark loop? 2 loops



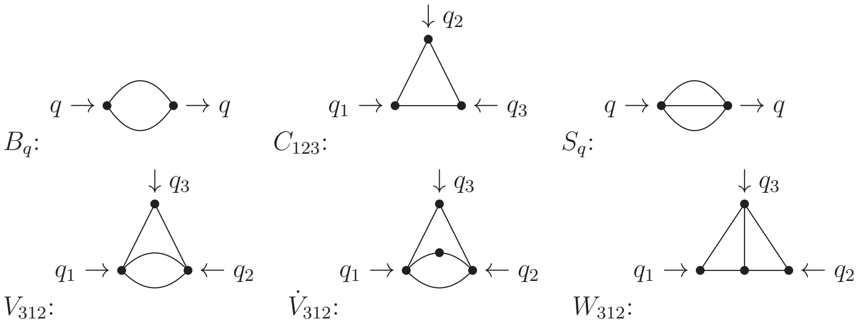
- Generate $\partial_{q_4}^{\nu_4} \Pi^{\mu_1\mu_2\mu_3\mu_4}$ including two QCD vertices from the Dyson series

  $\longrightarrow$ Project onto $\hat{\Pi}_i$

- Result: $\sim$ 6000 scalar 2-loop integrals on the form ($d = 4 - 2\varepsilon$)

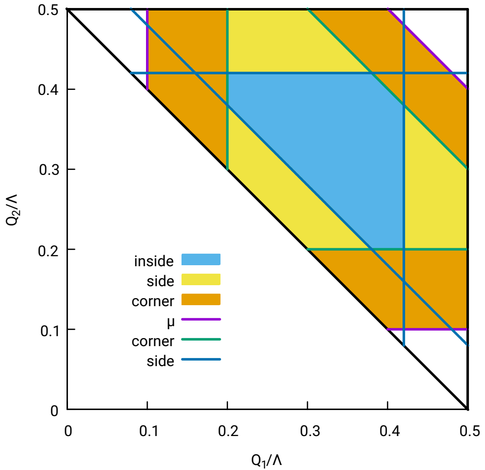
$$M(i_1, ..., i_7) = \frac{1}{i^2} \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \frac{1}{p_1^{2i_1}(p_1 - q_1)^{2i_2}(p_1 + q_2)^{2i_3} p_2^{2i_4}(p_2 - q_1)^{2i_5}(p_2 + q_2)^{2i_6}(p_1 - p_2)^{2i_7}}$$

- Use Kira to reduce these to a minimal set of 21 master integrals

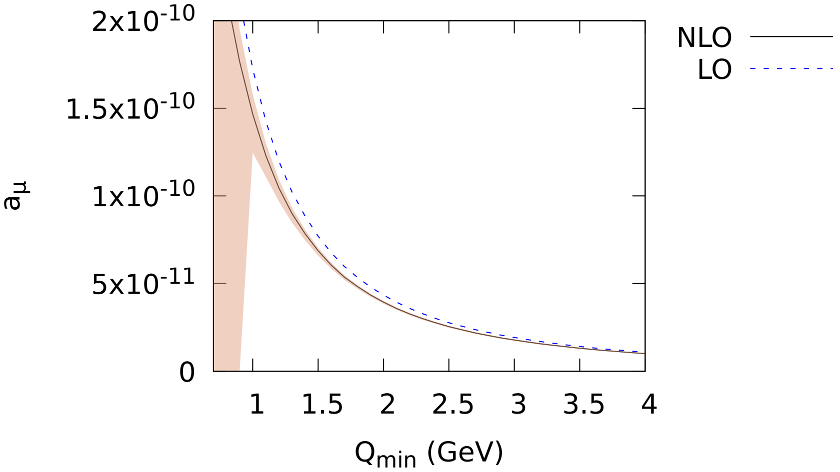


- Renormalisation not needed (first order in α, α_s, massless)
- Individual integrals divergent up to $1/\varepsilon^3$ (expansions known [Birthwright et al., 2004; Chavez et al., 2012])
- The finiteness of $\hat{\Pi}_i$ is a strong check on our calculation

- Numerically difficult around e.g. $\lambda = (Q_1^2 + Q_2^2 - Q_3^2)^2 - 4Q_1^2Q_2^2 \approx 0$
- We had to expand the analytical results in all different regions



- $\Lambda = Q_1 + Q_2 + Q_3$, and we have up to $1/\lambda^4$ that cancel

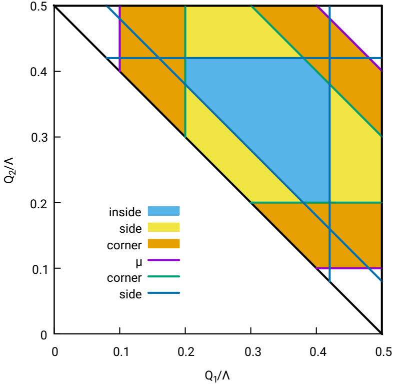


- Main uncertainty: $\alpha_s(\mu)$ where $\mu \in (Q_{\min}/\sqrt{2}, \sqrt{2}Q_{\min})$, run $\alpha_s(m_Z)$ with 5-loops to m_τ and μ
- In general see about -10% of the quark loop (LO)

Conclusions so far

- The massless quark-loop is the leading term in an OPE and a decent representation of the short-distance behaviour with $Q_1^2 \sim Q_2^2 \sim Q_3^2 \gg \Lambda_{\mathrm{QCD}}^2$
- We have shown that higher-order terms in the OPE are numerically small (suppressed by m_q and small condensates)
- 2-loop correction is about -10% of the quark-loop
- This can now be used by the dispersive/model studies [Colangelo et al., 2020]
- Currently: SDC2 $Q_1^2 \sim Q_2^2 \gg Q_3^2, \Lambda_{\mathrm{QCD}}^2$ very important to constrain models. Only leading order used in White Paper
- Higher order terms and α_s needed

- Caveat: Overlap regions
- $Q_i^2 \gg \Lambda_{\mathrm{QCD}}^2$ SDC1
- $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda_{\mathrm{QCD}}^2$ SDC2
- In SDC1 we have the case $Q_i^2 \sim Q_j^2 \gg Q_k^2 \gg \Lambda_{\mathrm{QCD}}^2 \in \mathrm{SDC2}$
- We must find the same result in the corner limits (perturbative)



OPE overview

- Recall for one soft photon we study

$$\Pi^{\mu_1\mu_2\mu_3}(q_1, q_2) \sim \int \frac{d^4 q_3}{(2\pi)^4} \left(\prod_{i=1}^{3} \int d^4 x_i \, e^{-iq_i x_i} \right) \times \langle 0 | T \left(J^{\mu_1}(x_1) J^{\mu_2}(x_2) J^{\mu_3}(x_3) \right) | \gamma(q_4) \rangle$$

$Q_{1,2,3} \gg \Lambda_{\mathrm{QCD}}$: Keep $F_{\mu\nu}$–like operators ($\bar{q}\sigma_{\mu\nu} q \sim F_{\mu\nu}$)

- Melnikov-Vainshtein limit: For two soft photons we instead consider

$$\Pi^{\mu_1\mu_2}(q_1) \sim \int \frac{d^4 q_3}{(2\pi)^4} \left(\prod_{i=1}^{2} \int d^4 x_i \, e^{-iq_i x_i} \right) \times \langle 0 | T \left(J^{\mu_1}(x_1) J^{\mu_2}(x_2) \right) | \gamma(q_3) \gamma(q_4) \rangle$$

$Q_{1,2} \gg Q_3, \Lambda_{\mathrm{QCD}}$: Keep operators with the right quantum numbers
Can relate it to the HLbL as well

- Define new variables

$$\hat{q} = \frac{1}{2}(q_1 - q_2)\,, \qquad q_{1,2} = \pm\hat{q} - \frac{1}{2}(q_3 + q_4)$$

$$\underbrace{\overline{Q}_3}_{\text{large}} = Q_1 + Q_2\,, \qquad \underbrace{\delta_{12}}_{\text{small}} = Q_1 - Q_2$$

- Related through

$$\underbrace{\hat{Q}^2}_{\text{large}} = \frac{1}{4}\left(\overline{Q}_3^2 + \delta_{12}^2 - Q_3^2\right)$$

- Rewrite $\hat{Q}^2$ in terms of δ_{12}^2, $\overline{Q}_3^2$ and Q_3^2 for $\hat{\Pi}_i$
- Needed to do expansions in corners of SDC 1

- Our object is now

$$\Pi^{\mu_1\mu_2\mu_3\mu_4} = \sum_j \frac{ie_{q_j}^2}{e^2} \int \frac{d^4 q_4}{(2\pi)^4} \int d^4 x_1 \int d^4 x_2 \, e^{-i(q_1 x_1 + q_2 x_2)} \times \langle 0 | T(J_j^{\mu_1}(x_1) J_j^{\mu_2}(x_2)) | \gamma^{\mu_3}(q_3) \gamma^{\mu_4}(q_4) \rangle$$

$$\equiv \sum_j \frac{ie_{q_j}^2}{e^2} \left\langle e^{-i(q_1 x_1 + q_2 x_2)} \, \bar{q}(x_1) \gamma^{\mu_1} q(x_1) \; \bar{q}(x_2) \gamma^{\mu_2} q(x_2) \right\rangle_{q_4,\, x_1,\, x_2}^{j, \mu_3,\, \mu_4}$$

- Leading order must be proportional to axial current $J_A^\mu = \overline{q}\gamma_5\gamma^\mu q$
[Melnikov, Vainshtein 03]
- Leading behaviour: $1/\hat{Q}^2$
- Want to derive perturbative and non-perturbative corrections to this

- A bunch of operators: $\lim_{q_4\to 0} \partial_{q_4}^{\mu_4} \langle 0|\mathcal{O}_{i,\,D}^{\alpha\beta}|\gamma(3)\gamma(4)\rangle$

$$
\begin{aligned}
D = 3: \quad & \mathcal{O}_{1,\,D=3}^{\alpha\beta\rho} = \overline{q}\left[\gamma^\alpha\gamma^\rho\gamma^\beta - \gamma^\beta\gamma^\rho\gamma^\alpha\right]q \\
D = 4: \quad & \mathcal{O}_{1,\,D=4}^{\alpha\beta} = \overline{q}\gamma^\beta\left[\vec{D}^\alpha - \overleftarrow{D}^\alpha\right]q \\
& \mathcal{O}_{2,\,D=4}^{\alpha\beta} = F^{\alpha\gamma}F_\gamma^\beta \\
& \mathcal{O}_{3,\,D=4}^{\alpha\beta} = F^{\gamma\delta}F_{\gamma\delta}\,g^{\alpha\beta} \\
& \mathcal{O}_{4,\,D=4}^{\alpha\beta} = G^{\alpha\gamma}G_\gamma^\beta \\
& \mathcal{O}_{5,\,D=4}^{\alpha\beta} = G^{\gamma\delta}G_{\gamma\delta}\,g^{\alpha\beta} \\
& \mathcal{O}_{6,\,D=4}^{\alpha\beta} = \overline{q}\left[\gamma^\alpha\gamma^\gamma\gamma^\beta + \gamma^\beta\gamma^\gamma\gamma^\alpha\right]\left[\vec{D}^\gamma + \overleftarrow{D}^\gamma\right]q
\end{aligned}
$$

- Can do all of the OPE terms here:

$$
\lim_{q_4\to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}} = \sum \mathrm{Wilson}(\hat{Q},\mu) \times \mathrm{ME}(Q_3,\mu)
$$

$$\lim_{q_4\to 0}\frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}} = \sum_j \frac{e_{q,j}^2}{e^2}\lim_{q_4\to 0}\partial_{q_4}^{\nu_4}\left\langle \bar{q}\left[\Gamma^{\mu_1\mu_2}(-\hat{q}) - \Gamma^{\mu_2\mu_1}(-\hat{q})\right] q\right\rangle^{j,\mu_3,\mu_4}$$

$$+\sum_j \frac{ie_{q_j}^2}{e^2\hat{q}^2}\left(g^{\mu_1\delta}g^{\mu_2}_{\beta} + g^{\mu_2\delta}g^{\mu_1}_{\beta} - g^{\mu_1\mu_2}g^{\delta}_{\beta}\right)\left(g^{\delta}_{\alpha} - 2\,\frac{\hat{q}^{\delta}\hat{q}_{\alpha}}{\hat{q}^2}\right)$$

$$\times \lim_{q_4\to 0}\partial_{\nu_4}^{q_4}\left\langle \bar{q}(\overrightarrow{D}^{\alpha} - \overleftarrow{D}^{\alpha})\gamma^{\beta}q\right\rangle^{j,\mu_3,\,\mu_4}_{\overline{\mathrm{MS}}(\mu)}$$

$$+\sum_j \frac{ie_{q_j}^2}{e^2\hat{q}^2}\left(g^{\mu_1\delta}g^{\mu_2}_{\beta} + g^{\mu_2\delta}g^{\mu_1}_{\beta} - g^{\mu_1\mu_2}g^{\delta}_{\beta}\right)\left(g^{\delta}_{\alpha} - 2\,\frac{\hat{q}^{\delta}\hat{q}_{\alpha}}{\hat{q}^2}\right)$$

$$\times \lim_{q_4\to 0}\partial_{q_4}^{\nu_4}\left\langle Z^j_{DF}(\mu)\,\frac{\alpha}{4\pi}\left(F^{\mu\nu}F_{\mu\nu}g^{\alpha\beta} + d\,F^{\alpha\gamma}F^{\beta}_{\gamma}\right)\right.$$

$$\left. + Z^j_{DG}(\mu)\,\frac{\alpha_s}{4\pi}\left(G^{\mu\nu}_a G^a_{\mu\nu}\,g^{\alpha\beta} + d\,G^{\alpha\gamma}_a G^{a,\,\beta}_{\gamma}\right)\right\rangle^{j,\mu_3\mu_4}$$

$$+\sum_j \frac{e_{q_j}^2}{8e^2}\lim_{q_4\to 0}\left[\partial_{q_4}^{\nu_4}\left\langle e^2 e_{q_j}^2 F_{\nu_3'\mu_3'}F_{\nu_4'\mu_4'} + \frac{1}{2N_c}\,g_s^2\,G^a_{\nu_3'\mu_3'}G^a_{\nu_4'\mu_4'}\right\rangle^{j,\mu_3\mu_4}\right]$$

$$\times \lim_{q_3,q_4\to 0}\partial_{q_3}^{\nu_3'}\,\partial_{q_4}^{\nu_4'}\,\Pi^{\mu_1\mu_2\mu_3'\mu_4'}_{\mathrm{ql},j}$$

- Here we need to calculate $\hat{\Pi}_i$ from the OPE
- Question is what the matrix elements are
- Case 1: $Q_1^2 \sim Q_2^2 \gg Q_3^2 \gg \Lambda^2_{\mathrm{QCD}} \in \mathrm{SDC1}$ Perturbative regime
- Case 2: $Q_1^2 \sim Q_2^2 \gg Q_3^2 \sim \Lambda^2_{\mathrm{QCD}}$ Non-perturbative regime

- Perturbative dimension $D = 3$:

$$\hat{\Pi}_1 = -\frac{4N_c \sum_j e_{q,j}^4}{\pi^2 Q_3^2 \overline{Q}_3^2}$$

- Non-perturbative dimension $D = 3$:

$$\lim_{q_4 \to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_{4,\,\mu_4}} = \frac{1}{2\pi^2} \frac{q_3^2}{\hat{q}^2} \epsilon^{\mu_1\mu_2\hat{q}\delta} \Big(\epsilon_{\mu_3\mu_4\nu_4\delta}\, \omega_T(q_3^2) - \frac{1}{q_3^2} \epsilon_{q_3\mu_4\nu_4\delta}\, q_{3\mu_3}\, \omega_T(q_3^2) + \frac{1}{q_3^2} \epsilon_{\mu_3\mu_4\nu_4 q_3}\, q_{3\delta} \left[\omega_L(q_3^2) - \omega_T(q_3^2) \right] \Big)$$

$$\hat{\Pi}_1 = \frac{2}{\pi^2 \overline{Q}_3^2}\, \omega_L(q_3^2)$$

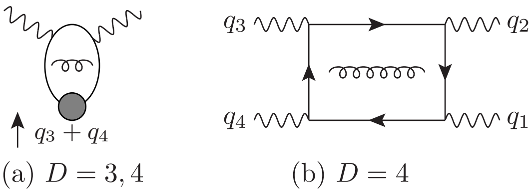
- Perturbative case for $D = 4$

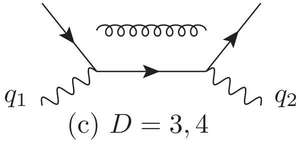
$$\hat{\Pi}_1 = -\frac{4}{\pi^2 \, Q_3^2 \, \overline{Q}_3^2} \quad \hat{\Pi}_4 = -\frac{16}{3\pi^2 \overline{Q}_3^4}, \quad \hat{\Pi}_7 = \mathcal{O}\left(\frac{1}{\overline{Q}_3^6}\right),$$

$$\hat{\Pi}_{17} = \frac{16}{3\pi^2 Q_3^2 \overline{Q}_3^4}, \quad \hat{\Pi}_{39} = \frac{16}{3\pi^2 Q_3^2 \overline{Q}_3^4}, \quad \hat{\Pi}_{54} = \mathcal{O}\left(\frac{1}{\overline{Q}_3^5}\right).$$

- Agree with corner expansion of SDC1
- Non-perturbative dimension $D = 4$ depends on new form factors such as $\omega_{(8)}^{D,1}$, $\omega_{(8)}^{D,5}$ and $\omega_{(8)}^{D,6}$, but also $\omega_{T,L}(q_3^2)$
- Preliminary estimates: Chiral model + resonance saturation
- Not big corrections but work in progress

- We still have to consider gluonic





- $D = 3$: Reproduce known result [Lüdtke, Procura 20]

$$\Pi^{\mu_1\mu_2}_{D=3,\,\mathrm{NLO}} \approx -\frac{e_q^2}{e^2}\left(1-\frac{\alpha_s}{\pi}\right)\left\langle \bar{q}[\Gamma^{\mu_1\mu_2}(-\hat{q})-\Gamma^{\mu_2\mu_1}(-\hat{q})]q \right\rangle^{3,4}$$

- $D = 4$: Reproduce our corner expansion in SDC1 in Paper III

Conclusions and outlook

- OPEs to derive short-distance constraints for the HLbL

  One soft photon limit

  Two soft photons limit

- For $Q_1, Q_2, Q_3 \gg \Lambda_{\rm QCD}$:

  Quark loop is the leading term

  Non-perturbative corrections small

  Gluon corrections: -10% on the quark loop

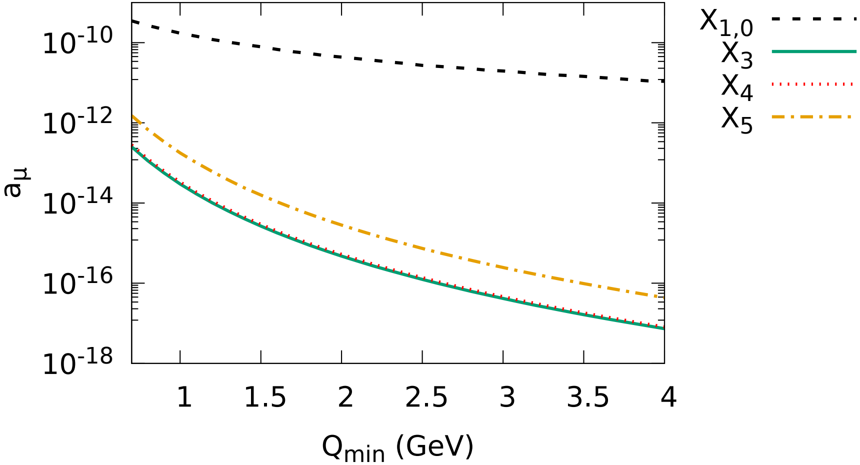
- For $Q_1, Q_2 \gg Q_3, \Lambda_{\rm QCD}$:

  Limit $Q_3 \gg \Lambda_{\rm QCD}$: Agreement with the quark loop $\tilde{\Pi}_i$ through $D = 4$

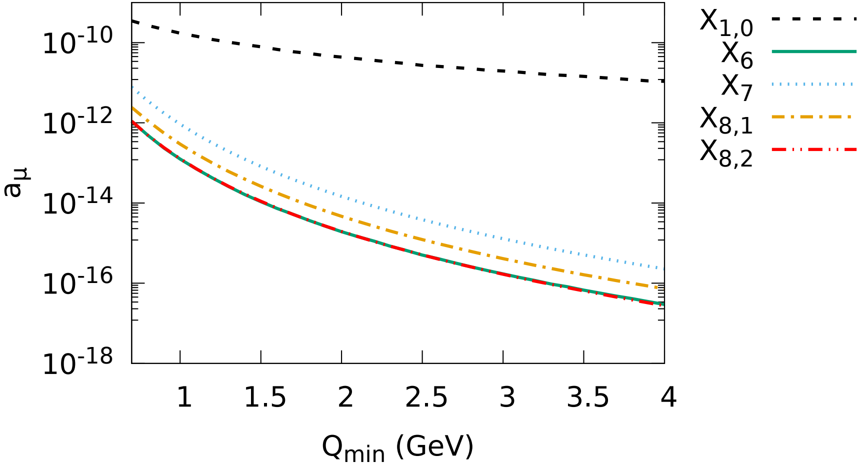
  Need also $Q_3 \sim \Lambda_{\rm QCD}$: Non-perturbative extrapolations

  Impact of the short-distance constraints?

Backup slides



10^-10
10^-12
10^-14
10^-16
10^-18
a_μ
1
1.5
2
2.5
3
3.5
4
Q_min (GeV)
X_1,0
X_3
X_4
X_5



10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
a_μ
1
1.5
2
2.5
3
3.5
4
Q_{min} (GeV)
$X_{1,0}$
X_6
X_7
$X_{8,1}$
$X_{8,2}$

Values of the condensates

- $X_{5,6,7}$ related to the quark and gluon condensates: Known in literature
- X_2 is the magnetic susceptibility $\langle \bar{q}\sigma_{\mu\nu}q\rangle$: Known from the lattice
- X_8 are four-quark condensates: We do leading order in large N_c

$$\overline{X}_{8,1}^{N_c\to\infty} = \overline{X}_{8,2}^{N_c\to\infty} = -2\frac{\pi\alpha_s}{9}X_2\langle\bar{q}q\rangle$$

- $X_{2,3,4}$: Do vacuum OPE of 2-point functions and match to large N_c form

$$X_2 = \frac{2}{M_\rho^2}\langle\bar{q}q\rangle \longleftarrow \text{Agrees well with LQCD}$$

$$X_3 = -\frac{m_0^2}{6M_\rho^2}\langle\bar{q}q\rangle \longleftarrow \text{New}$$

$$X_4 = -\frac{m_0^2}{6M_\rho^2}\langle\bar{q}q\rangle \longleftarrow \text{New}$$

$$\frac{1}{2}(\bar{q}_i\lambda_1 q_j\bar{q}_k\lambda_2 q_l - \bar{q}_i\lambda_2 q_j\bar{q}_k\lambda_1 q_l) = \frac{1}{64}\left(\gamma^\mu_{ji}\gamma^\nu_{lk} - \gamma^\nu_{ji}\gamma^\mu_{lk}\right)\left[\bar{q}\lambda_1\gamma_\mu q\bar{q}\lambda_2\gamma_\nu q - \bar{q}\lambda_2\gamma_\mu q\bar{q}\lambda_1\gamma_\nu q\right]$$
$$+ \frac{1}{64}\left((\gamma^\mu\gamma_5)_{ji}(\gamma^\nu\gamma_5)_{lk} - (\gamma^\nu\gamma_5)_{ji}(\gamma^\mu\gamma_5)_{lk}\right)\left[\bar{q}\lambda_1\gamma_\mu\gamma_5 q\bar{q}\lambda_2\gamma_\nu\gamma_5 q - \bar{q}\lambda_2\gamma_\mu\gamma_5 q\bar{q}\lambda_1\gamma_\nu\gamma_5 q\right]$$
$$+ \frac{1}{64}g_{\lambda\alpha}\left(\sigma^{\mu\lambda}_{ji}\sigma^{\alpha\nu}_{lk} - \sigma^{\nu\lambda}_{ji}\sigma^{\alpha\mu}_{lk}\right) \times \frac{1}{2}g^{\rho\beta}\left[\bar{q}\sigma_{\mu\rho}\lambda_1 q\,\bar{q}\sigma_{\beta\nu}\lambda_2 q - \bar{q}\sigma_{\mu\rho}\lambda_2 q\,\bar{q}\sigma_{\beta\nu}\lambda_1 q\right]$$

$$\bar{q}_i\lambda_1 q_j\bar{q}_k\lambda_1 q_l = \frac{1}{32}\left(\sigma^{\mu\nu}_{ji}\delta_{lk} + \delta_{ji}\sigma^{\mu\nu}_{lk}\right)\left[\bar{q}\lambda_1 q\bar{q}\lambda_1\sigma_{\mu\nu} q\right]$$
$$+ \frac{1}{64}\epsilon^{\mu\nu\mu_1\nu_1}\epsilon_{\mu_1\nu_1\mu_2\nu_2}\left((\gamma_\mu\gamma_5)_{ji}(\gamma_\nu)_{lk} - (\gamma_\mu)_{ji}(\gamma_\nu\gamma_5)_{lk}\right)\left[\bar{q}\lambda_1\gamma^{\mu_2}\gamma_5 q\bar{q}\lambda_1\gamma^{\nu_2} q\right]$$
$$+ \frac{1}{32}\left(\sigma^{\mu\nu}_{ji}\gamma_{5\,lk} + \gamma_{5\,ji}\sigma^{\mu\nu}_{lk}\right)\left[\bar{q}\lambda_1\sigma_{\mu\nu} q\,\bar{q}\lambda_1\gamma_5 q\right],$$

$$\bar{q}_i\lambda_8 q_j\bar{q}_k q_l \pm \bar{q}_i q_j\bar{q}_k\lambda_8 q_l = \frac{1}{32}\left(\sigma^{\mu\nu}_{ji}\delta_{lk} \pm \delta_{ji}\sigma^{\mu\nu}_{lk}\right)\left[\bar{q}\sigma_{\mu\nu}\lambda_8 q\,\bar{q}q \pm \bar{q}\sigma_{\mu\nu} q\,\bar{q}\lambda_8 q\right]$$
$$+ \frac{1}{64}\epsilon^{\mu\nu\mu_1\nu_1}\epsilon_{\mu_1\nu_1\mu_2\nu_2}\left((\gamma_\mu\gamma_5)_{ji}(\gamma_\nu)_{lk} \mp (\gamma_\mu)_{ji}(\gamma_\nu\gamma_5)_{lk}\right)$$
$$\times \left[\bar{q}\lambda_8\gamma^{\mu_2}\gamma_5 q\,\bar{q}\gamma^{\nu_2} q \pm \bar{q}\gamma^{\mu_2}\gamma_5 q\,\bar{q}\lambda_8\gamma^{\nu_2} q\right]$$
$$+ \frac{1}{32}\left(\sigma^{\mu\nu}_{ji}\gamma_{5\,lk} \pm \gamma_{5\,ji}\sigma^{\mu\nu}_{lk}\right)\left[\bar{q}\sigma^{\mu\nu}\lambda_8 q\,\bar{q}\gamma_5 q \pm \bar{q}\sigma^{\mu\nu} q\,\bar{q}\lambda_8\gamma_5 q\right].$$

This reduces the original set of 1679616 matrix elements to a basis of 12 non-zero ones.

Analytical form of 2-loop result

- Analytical form (finite master integral coefficients and logarithms)

$$\tilde{\Pi}_m = f^{pqr}_{m,ijk} F_{ijk}(2) Q_1^{2p} Q_2^{2q} Q_3^{2r} + w^{pqr}_{m,ijk} W_{ijk}(0) Q_1^{2p} Q_2^{2q} Q_3^{2r}$$
$$+ c^{pqr}_{m,ijk} C_{ijk}(0) Q_1^{2p} Q_2^{2q} Q_3^{2r}$$
$$+ n^{pqr}_{m,1} Q_1^{2p} Q_2^{2q} Q_3^{2r} \log\frac{Q_1^2}{Q_3^2} + n^{pqr}_{m,2} Q_1^{2p} Q_2^{2q} Q_3^{2r} C_{ijk}(0) \log\frac{Q_2^2}{Q_3^2}$$
$$+ l^{pqr}_{m,ijk1} Q_1^{2p} Q_2^{2q} Q_3^{2r} C_{ijk}(0) \log\frac{Q_1^2}{Q_3^2} + l_{m,ijk2} Q_1^{2p} Q_2^{2q} Q_3^{2r} C_{ijk}(0) \log\frac{Q_2^2}{Q_3^2}$$

- $\tilde{\Pi}_m$ related to our $\hat{\Pi}_i$
- Again we can evaluate a_μ^{HLbL} in SDC1 kinematics

Examples of technical details

- Example: Renormalisation of $\mathcal{O}^{\alpha\beta}_{1,\,D=4,\,j} = \bar{q}\left[\overrightarrow{D}^{\alpha} - \overleftarrow{D}^{\alpha}\right]\gamma^{\beta} q$

$$\mathcal{O}^{\alpha\beta}_{1,\,D=4\,j} = \mathcal{O}^{\alpha\beta}_{j}(\mu) + Z^{j}_{DF}(\mu)\,\frac{\alpha}{4\pi}\,\left(F^{\mu\nu}F_{\mu\nu}g^{\alpha\beta} + d\,F^{\alpha\gamma}F^{\beta}_{\gamma}\right) + \ldots$$

$$Z^{j}_{DF}(\mu) = -i\,\frac{2}{3}\,N_c\,e^{2}_{q,j}\,\frac{\mu^{-2\epsilon}}{\hat{\epsilon}}$$

- Example: Form-factor decomposition

$$\frac{i\sum_j\left(e^{2}_{q_j} - \sum_k \frac{e^{2}_{q,k}}{3}\right)}{e^{2}\hat{q}^{2}} \lim_{q_4\to 0}\,\partial^{\nu_4}_{q_4}\left\langle \bar{q}\left[\overrightarrow{D}^{\alpha} - \overleftarrow{D}^{\alpha}\right]\gamma^{\beta} q\right\rangle^{j,\mu_3,\,\mu_4}_{\overline{\mathrm{MS}}(\mu)}$$

$$= \sum_{i=1}^{6} \omega^{D,i}_{(8)}\,L^{\alpha\beta\mu_3\mu_4\nu_4}_{i}$$