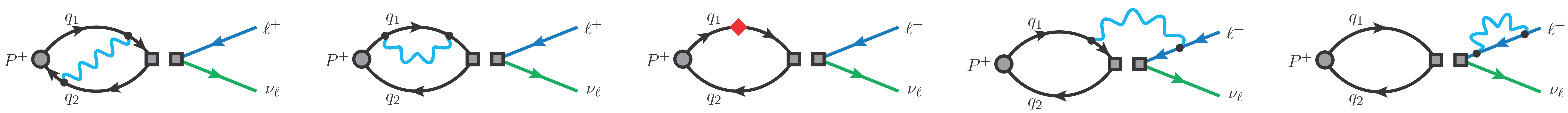


Leptonic decays of pseudoscalar mesons from lattice QCD+QED calculations

Matteo Di Carlo

~~31st Aug 2024~~ › › 28th Aug 2024

Funded by the European Union

RIKEN Center for Computational Science

Testing the Standard Model with flavour physics

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

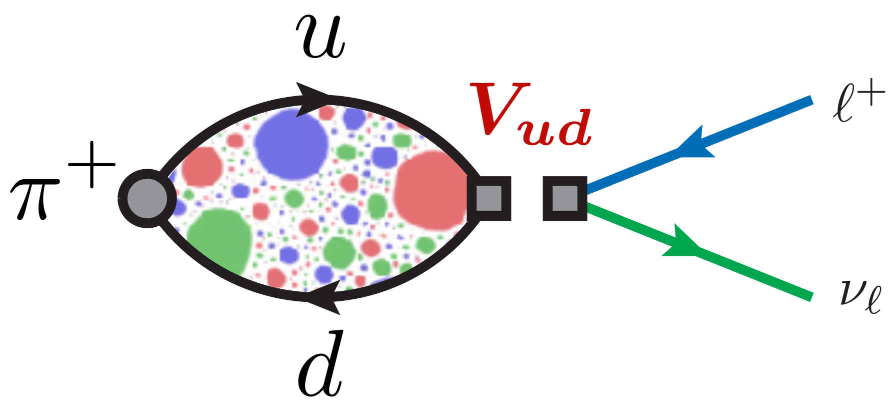
in the Standard Model:

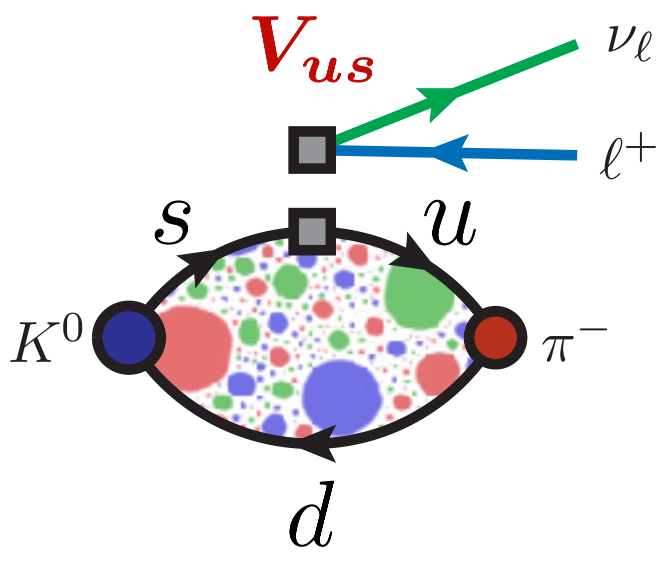
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of mesons

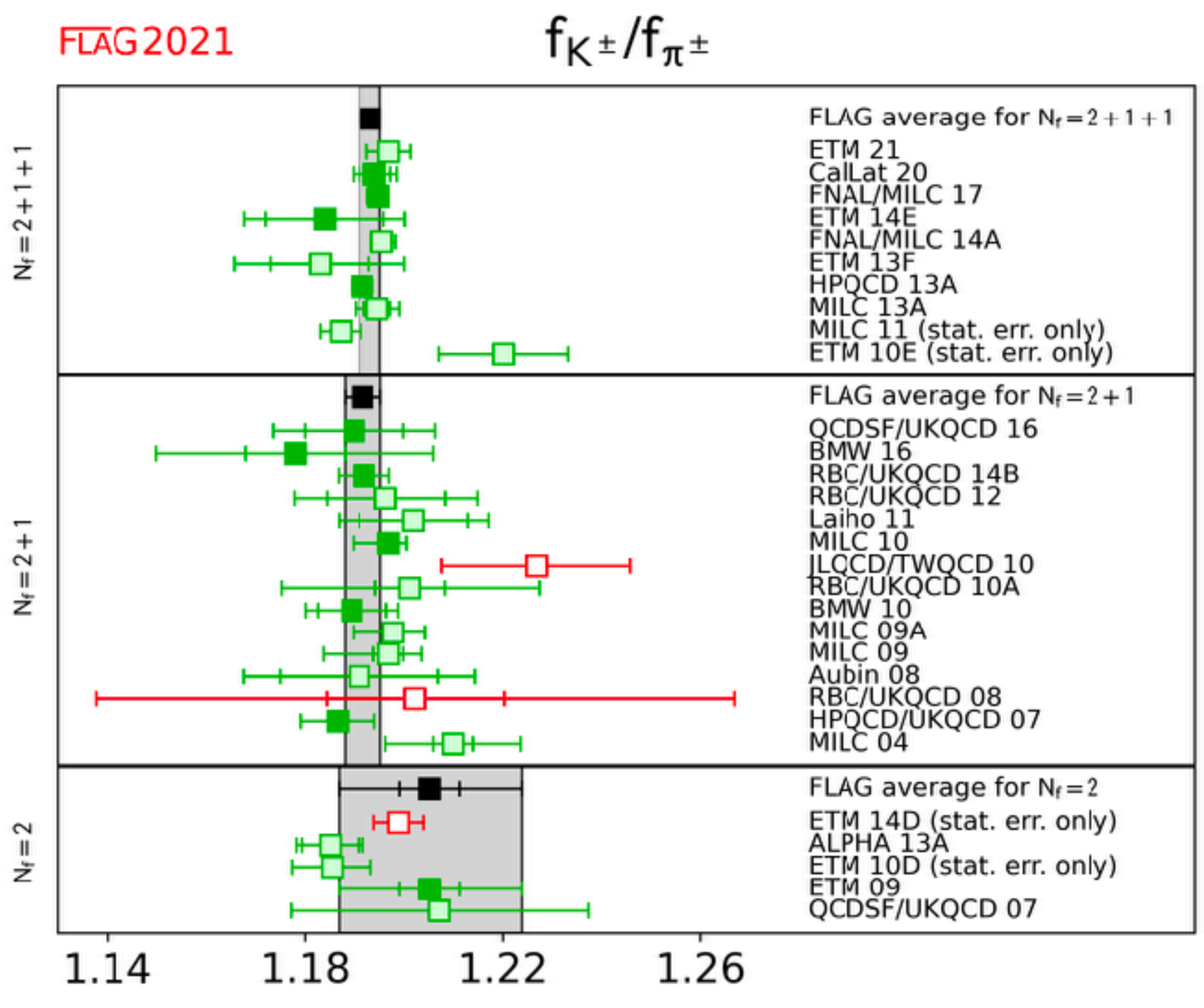
$$\underbrace{\frac{\Gamma[K \to \ell\nu_\ell(\gamma)]}{\Gamma[\pi \to \ell\nu_\ell(\gamma)]}}_{\text{experiments}} \propto \boxed{\left|\frac{V_{us}}{V_{ud}}\right|^2} \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

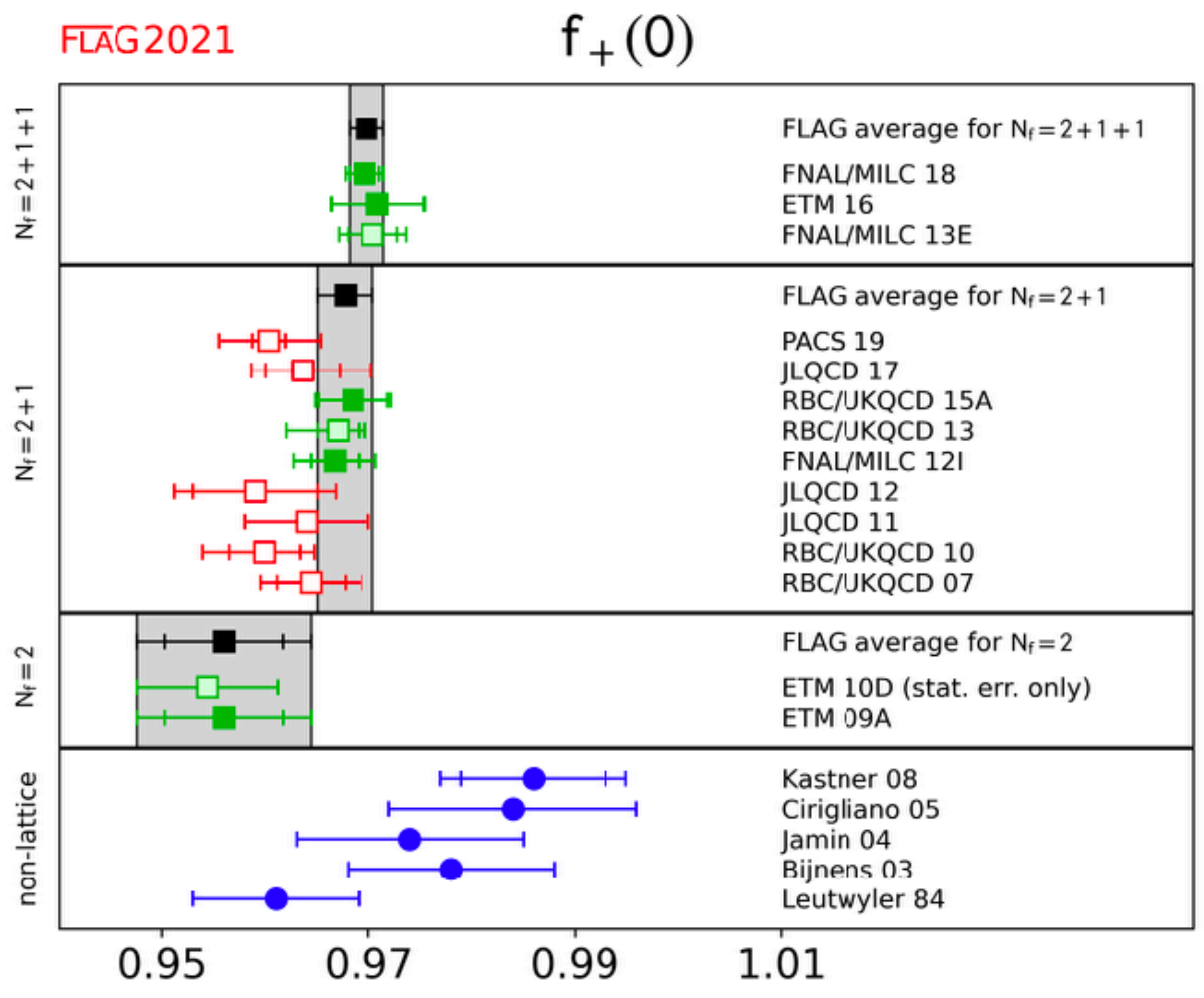
$$\underbrace{\Gamma[K \to \pi\ell\nu_\ell(\gamma)]}_{\text{experiments}} \propto \boxed{|V_{us}|^2} \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





Leptonic and semi-leptonic decays from lattice QCD





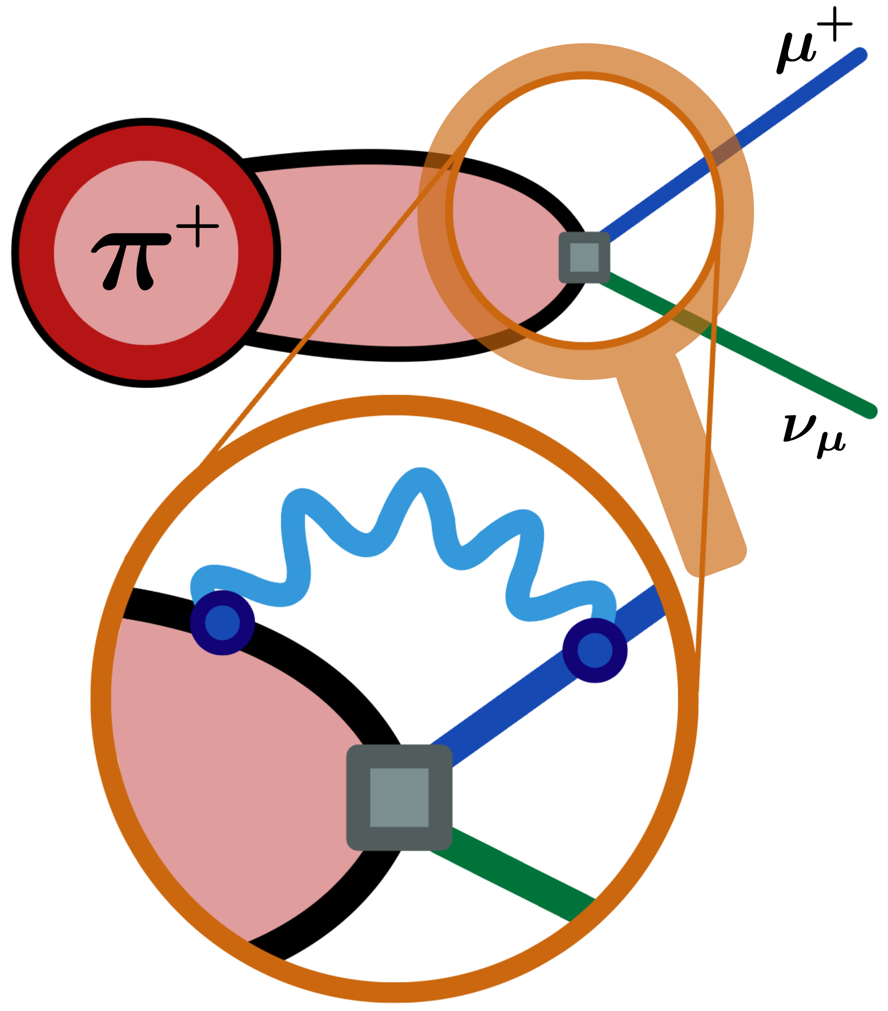
$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19) \qquad f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2021 — Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision!**

FLAG Review 2021. EPJC **82**, 869 (2022)

QED and isospin-breaking effects



Current level of precision requires the inclusion of isospin-breaking corrections due to

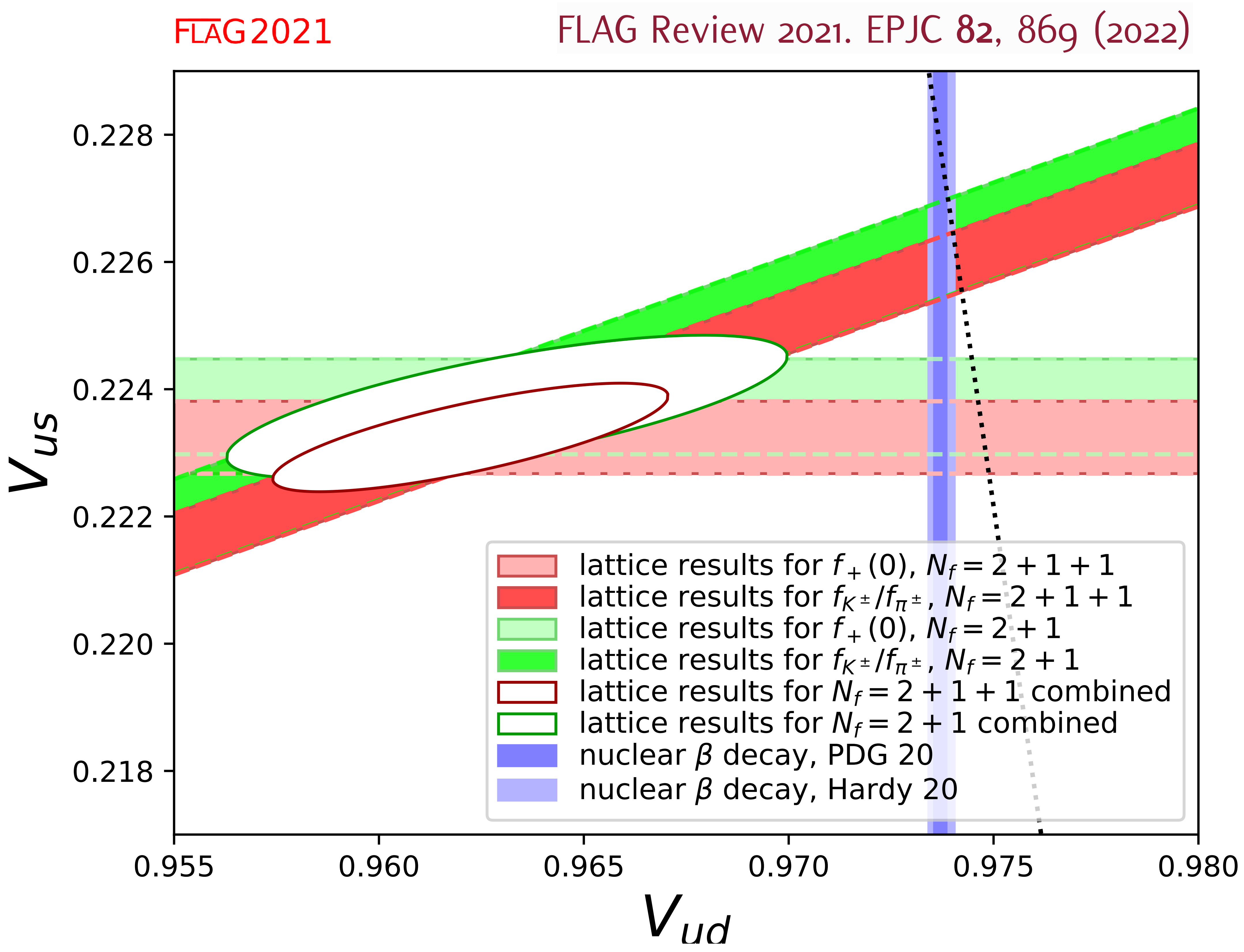
- strong effects $[m_u - m_d]_{\text{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

$\sim \mathcal{O}(1\%)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R^\ell_{K\pi})$$

- results currently quoted in the PDG come from χPT — V.Cirigliano & H.Neufeld, PLB **700** (2011)
- these are **non-perturbative** (i.e. structure dependent) quantities
- can be obtained through first-principle **lattice calculations!**

First-row CKM unitarity tests



Different **tensions** in the V_{us}-V_{ud} plane:

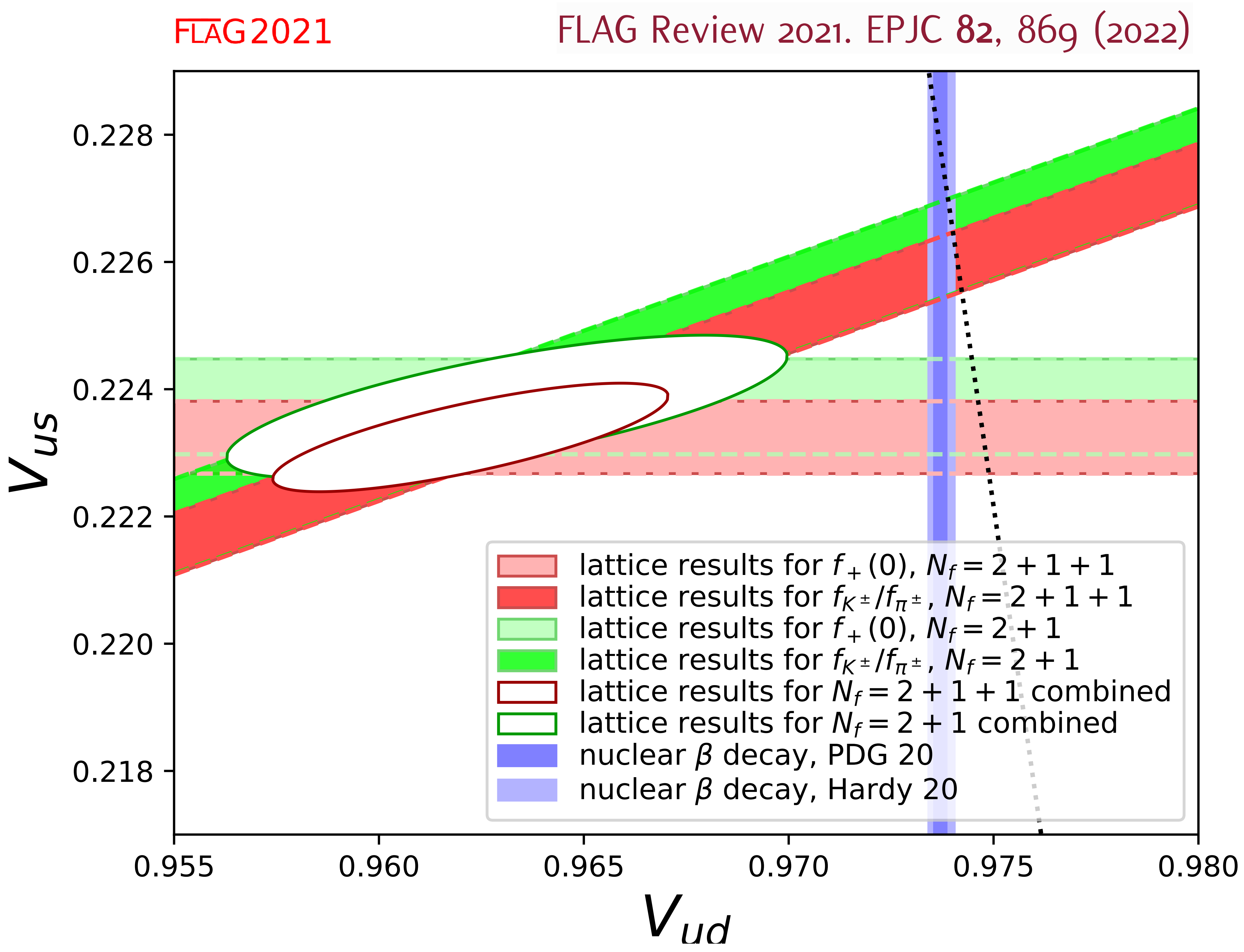
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

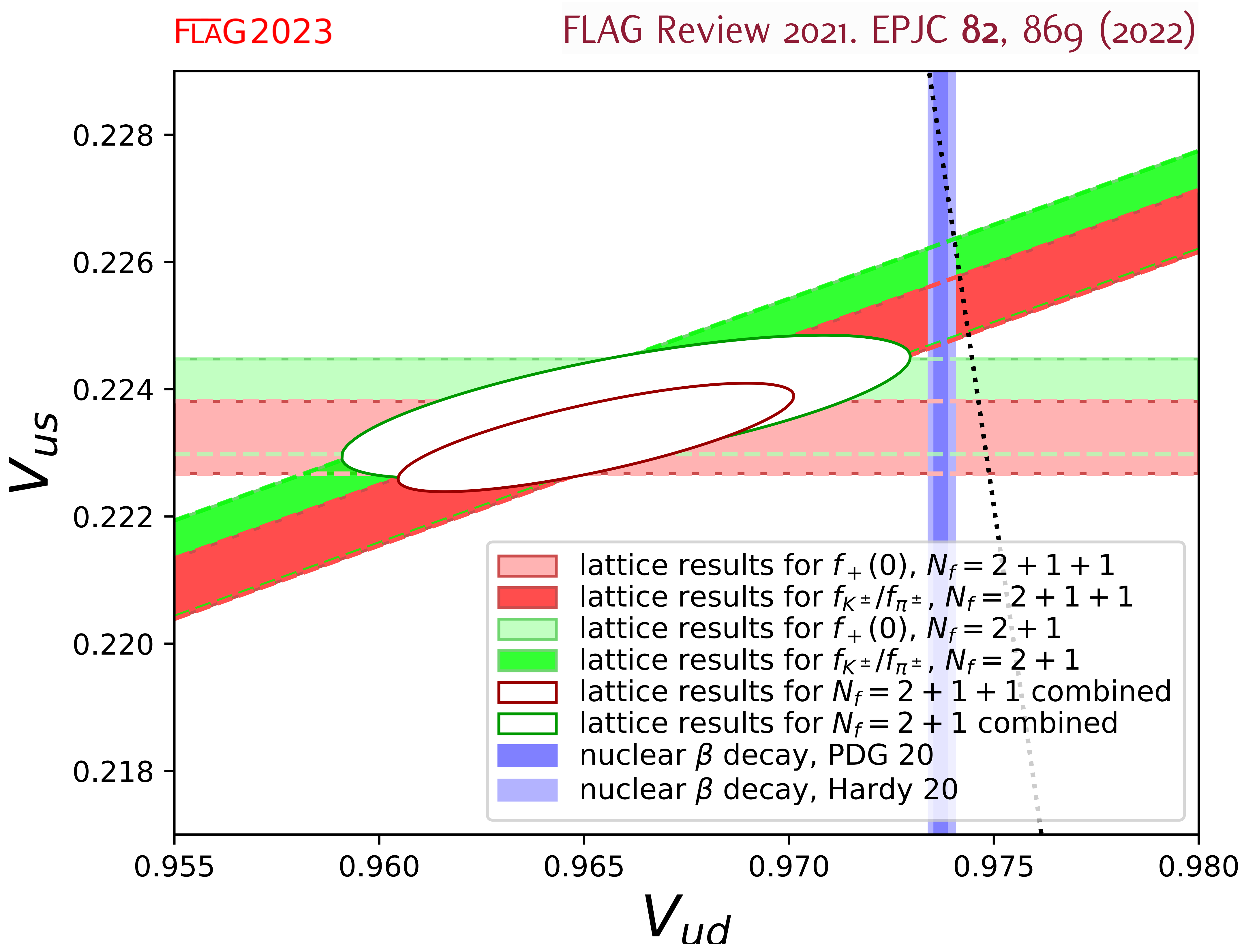
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

First-row CKM unitarity tests



with QED corrections
from **lattice calculation**
(RM123S, 2019)



with QED corrections
from **chiral perturbation theory**

Charged states in a finite box

Computing **QED corrections on a finite-sized lattice** is challenging:

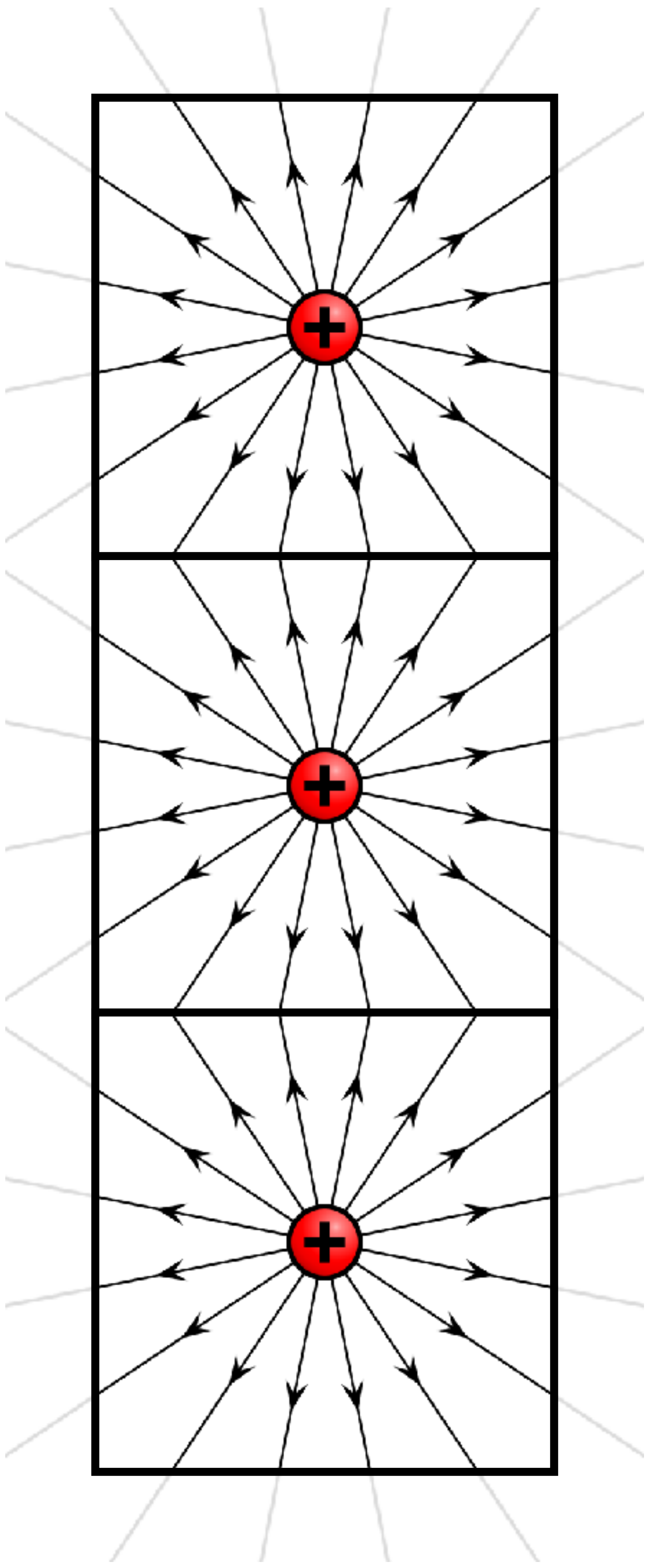
- **long-range interactions** don't like **finite volumes** with **periodic boundary conditions**
- **finite-volume effects** can be sizeable and power-like

  M.Hayakawa & S.Uno, PTP **120** (2008) / Z.Davoudi & M.Savage, PRD **90** (2014) / S.Borsanyi et al., Science **347** (2015)
- logarithmic **infrared divergences** arise in virtual/real decay rates

  V.Lubicz et al., PRD **95** (2017)

There are also recent proposals to compute radiative corrections as **convolutions** of hadronic correlators with **infinite-volume** QED kernels

N.Asmussen et al., [1609.08454] / T.Blum et al., PRD **96** (2017) / X.Feng & L.Jin, PRD **100** (2019) / N.Christ et al., [2304.08026]



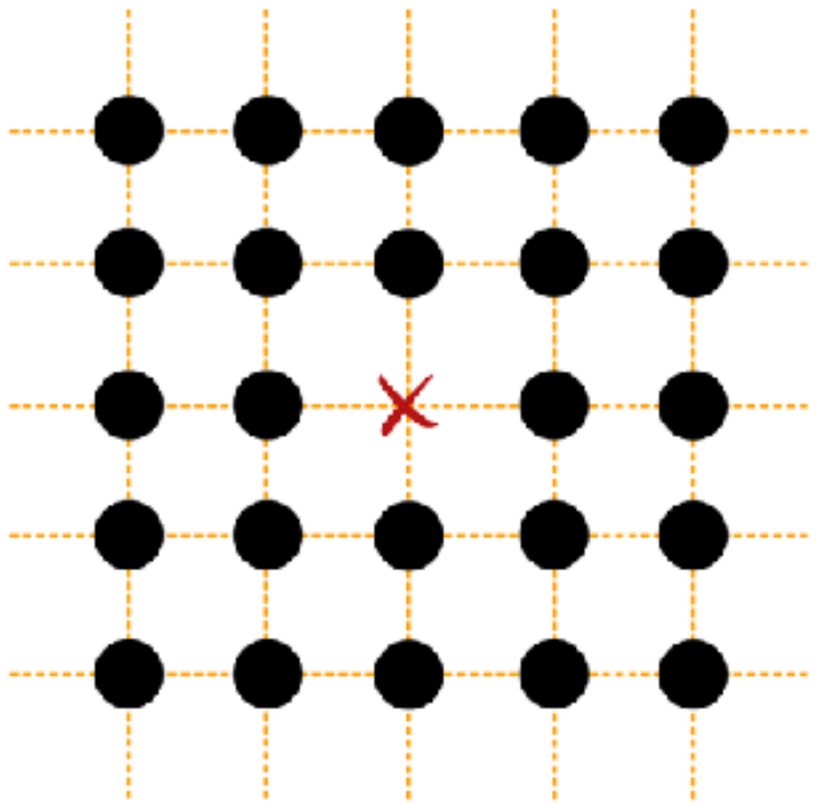
Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions:

$\mathrm{QED_L}$

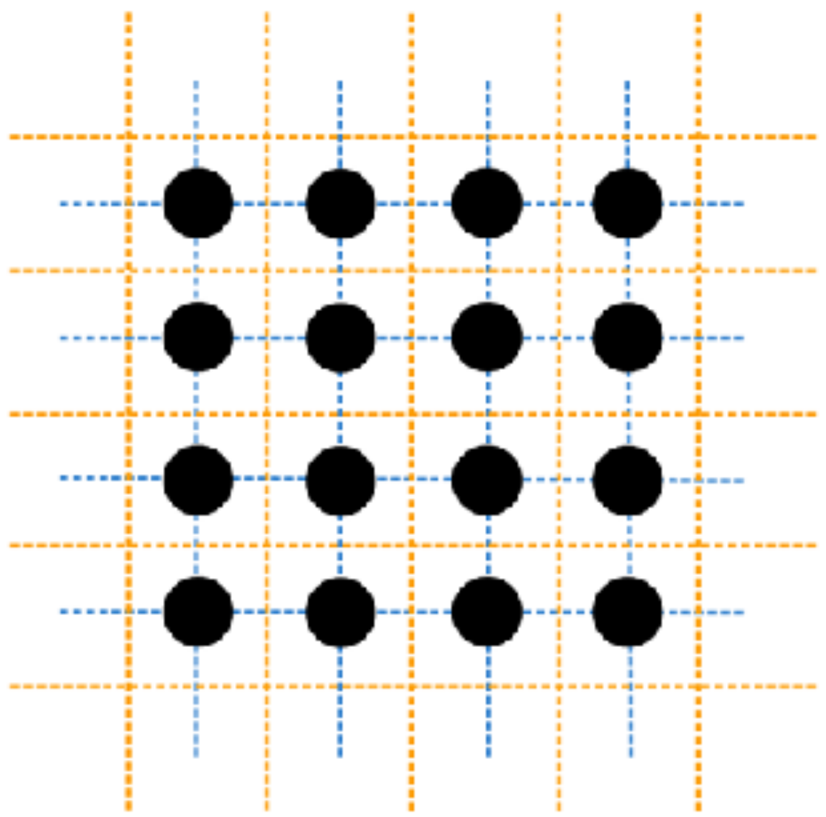


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

$\mathrm{QED_{C^*}}$

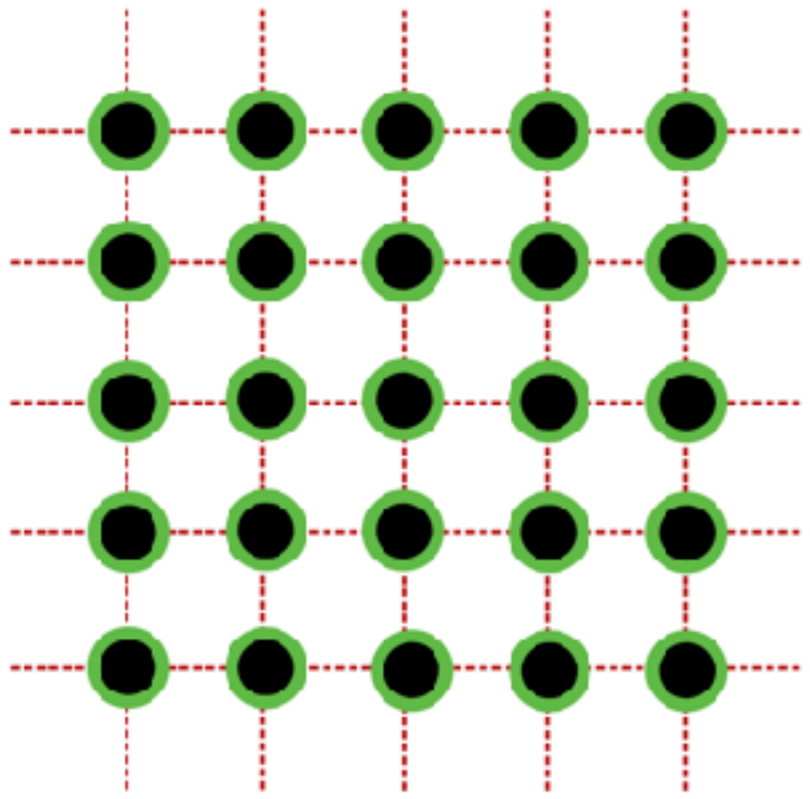


$\Omega_3 = 2\pi\mathbb{Z}^3/L \quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)
B.Lucini et al., JHEP **02** (2016)

$\mathrm{QED_m}$

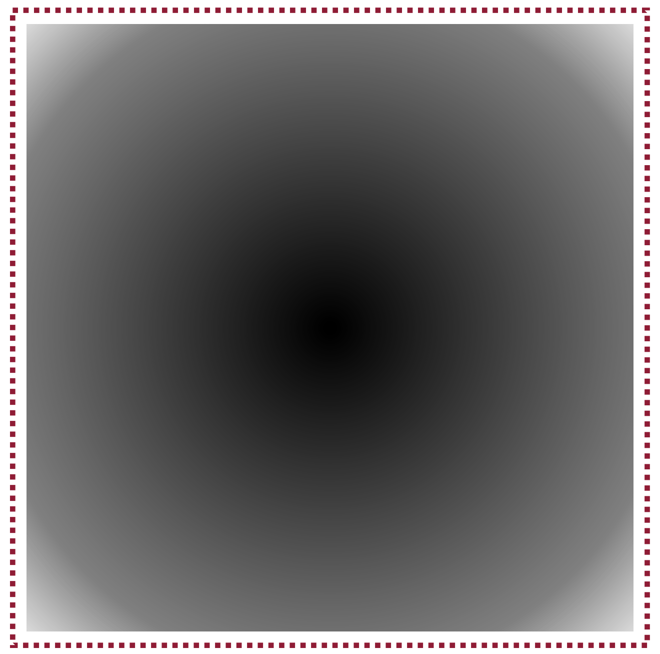


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_\infty}$



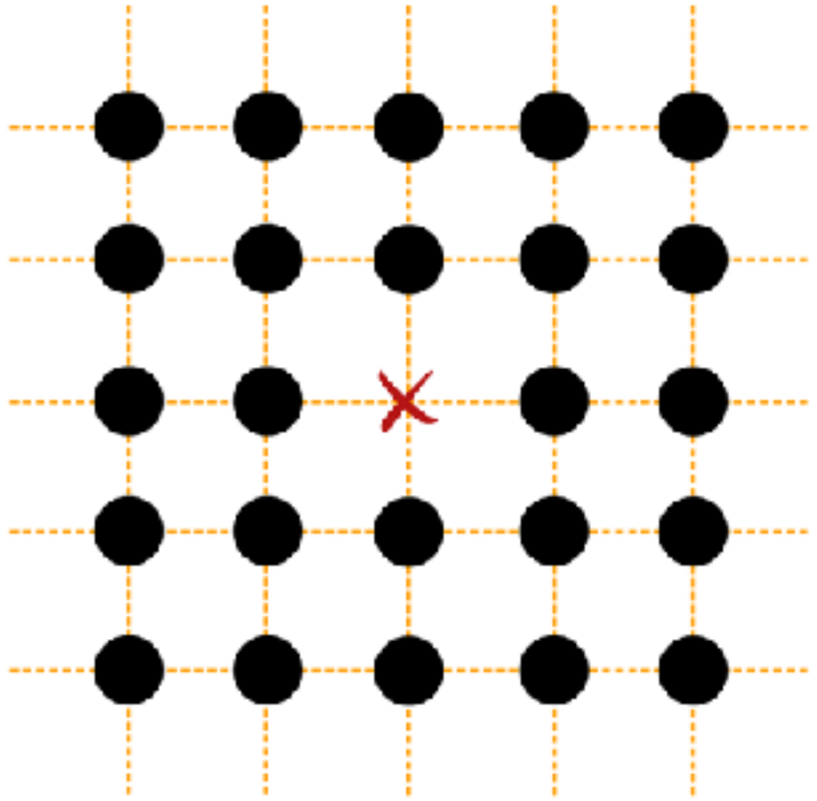
$\Omega_4 = \mathbb{R}^4$

infinite-volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

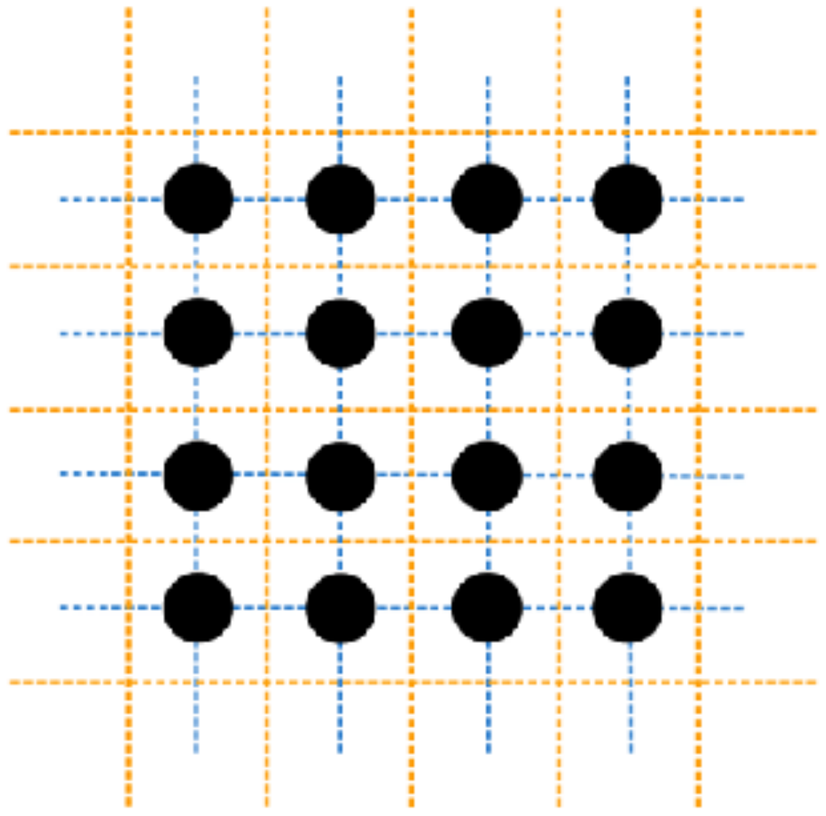
Charged states in a finite box

QED$_L$



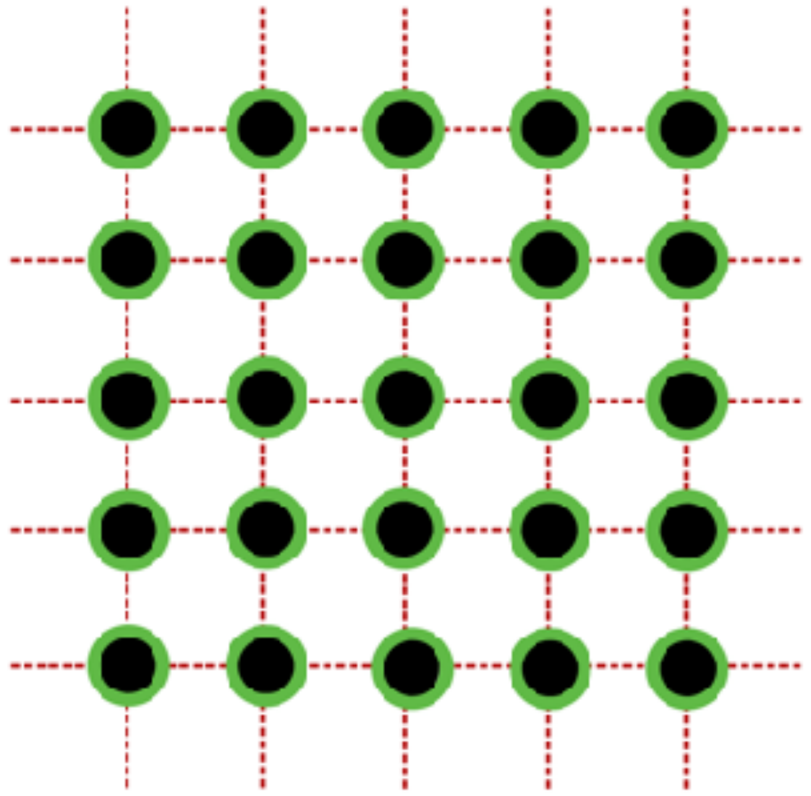
$\Omega_3 = 2\pi\mathbb{Z}^3/L$

QED$_{C^*}$



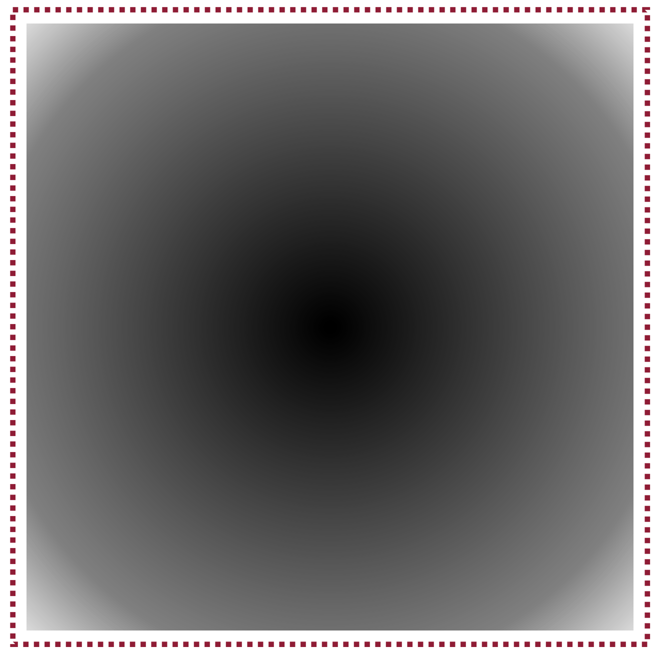
$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

QED$_m$



$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

QED$_\infty$



$\Omega_4 = \mathbb{R}^4$

finite-volume photon

non-local | local

∞-volume photon

power-like finite-volume effects | exponential finite-volume effects

UV / IR mixing | dedicated ensembles | two IR regulators | observable-dependent

Implementing QCD+QED on the lattice

- RM123 perturbative approach

G.M.de Divitiis et al. (RM123), PRD 87 (2013)

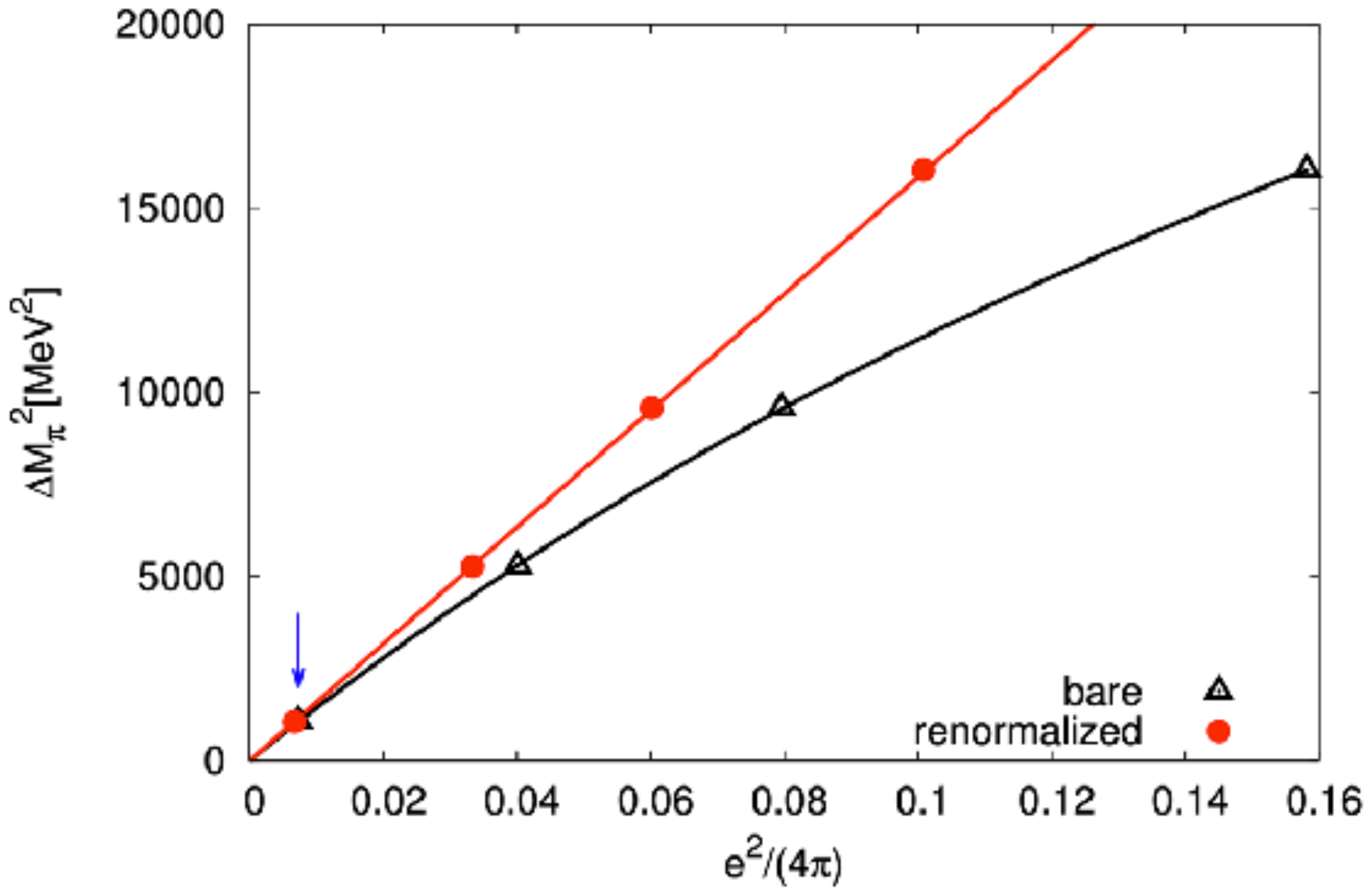
$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S_{\rm iso} - \Delta S} = \langle \mathcal{O} \rangle_{\rm iso} + \langle \Delta S \, \mathcal{O} \rangle_{\rm iso} + \ldots$$

Pros: only evaluate QCD observables

Cons: need to compute many diagrams, also disconnected: 

- Full QCD+QED lattice simulations

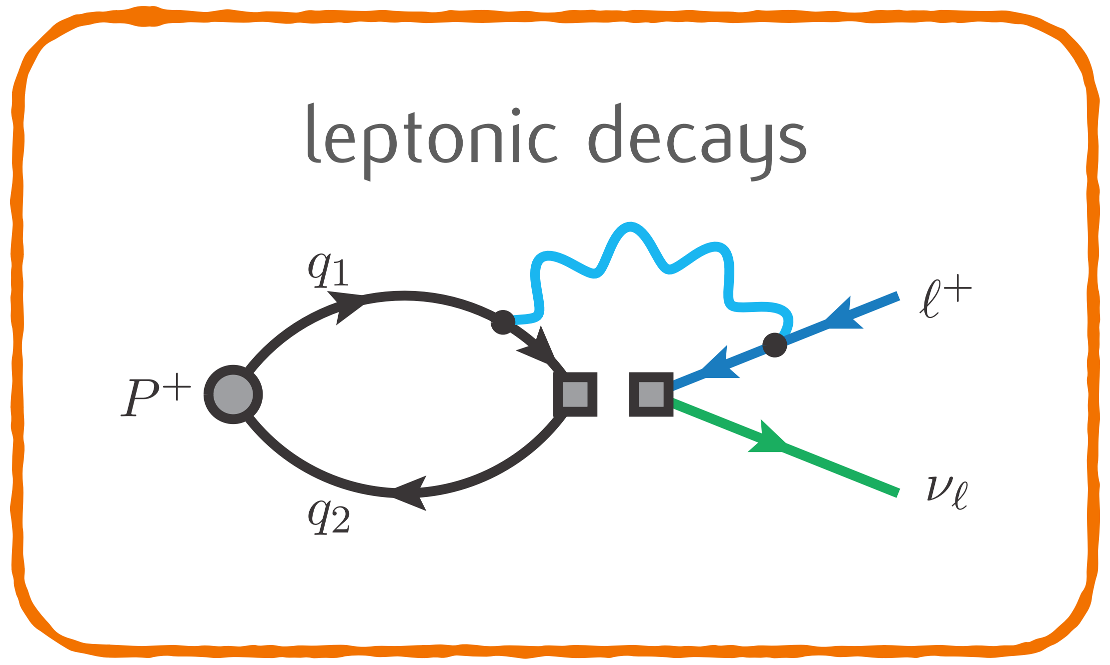
S.Borsanyi et al., Science 347 (2015)



Pros: simpler observables: 

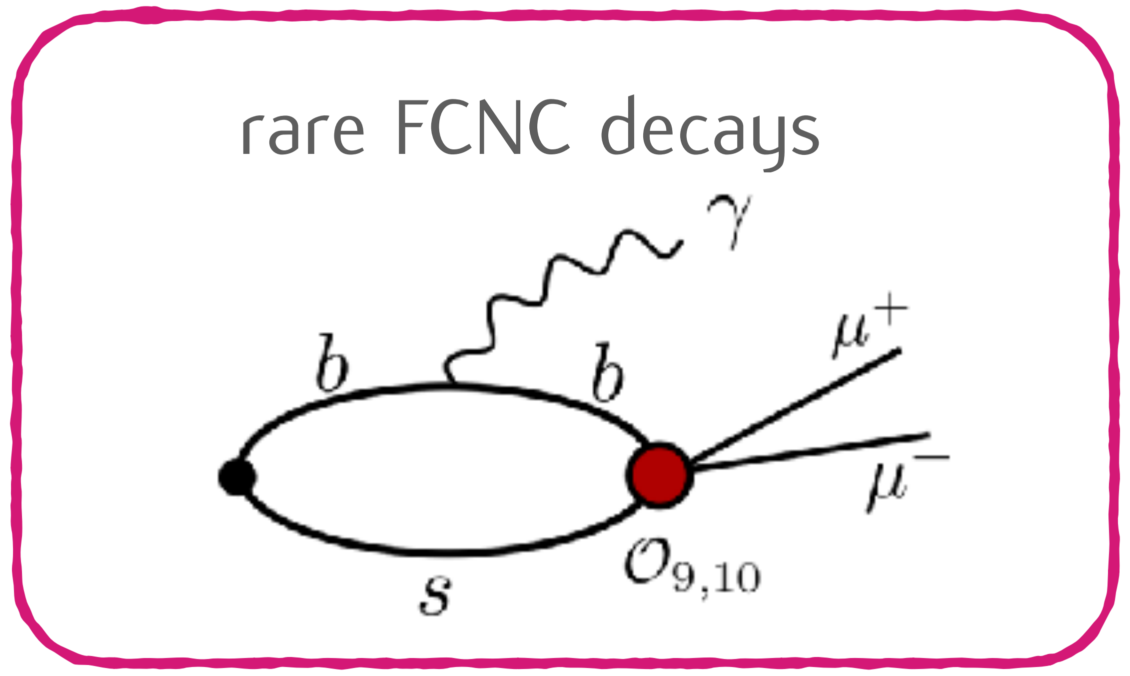
Cons: need of dedicated gauge configurations

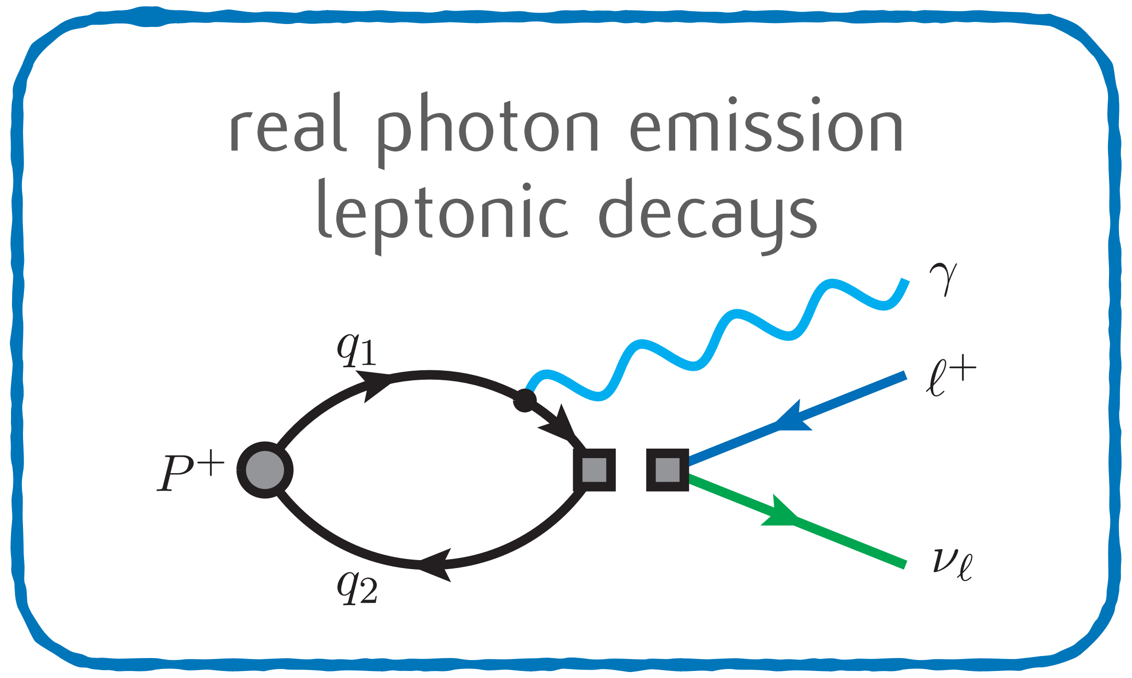
Weak decays — some recent works



N. Carrasco et al., PRD 91 (2015)
V. Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D. Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]

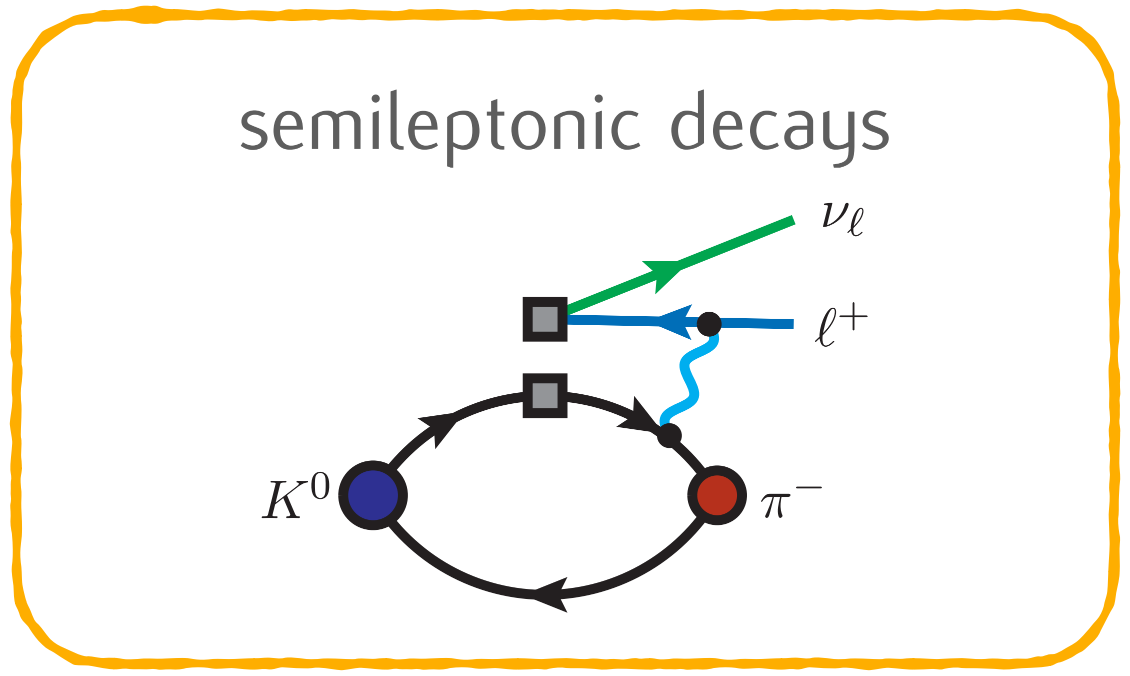
R.Frezzotti et al., [2402.03262]

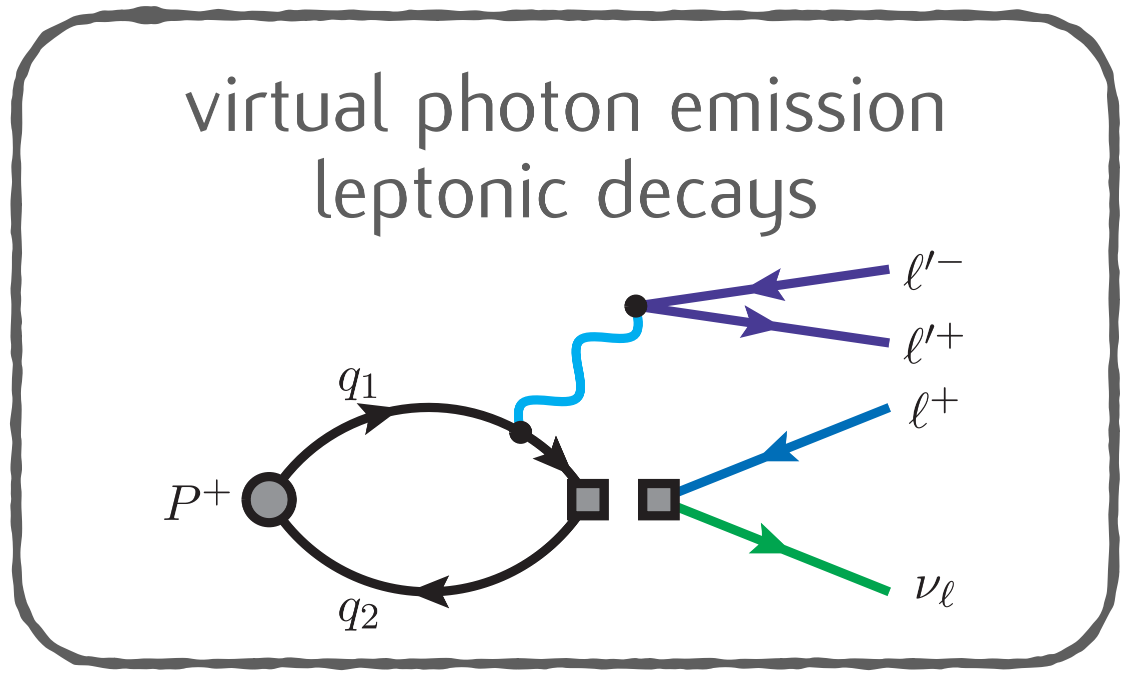




G.M. de Divitiis et al., [1908.10160]
C. Kane et al., [1907.00279 & 2110.13196]
R. Frezzotti et al., PRD 103 (2021)
A.Desiderio et al., PRD 102 (2021)
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

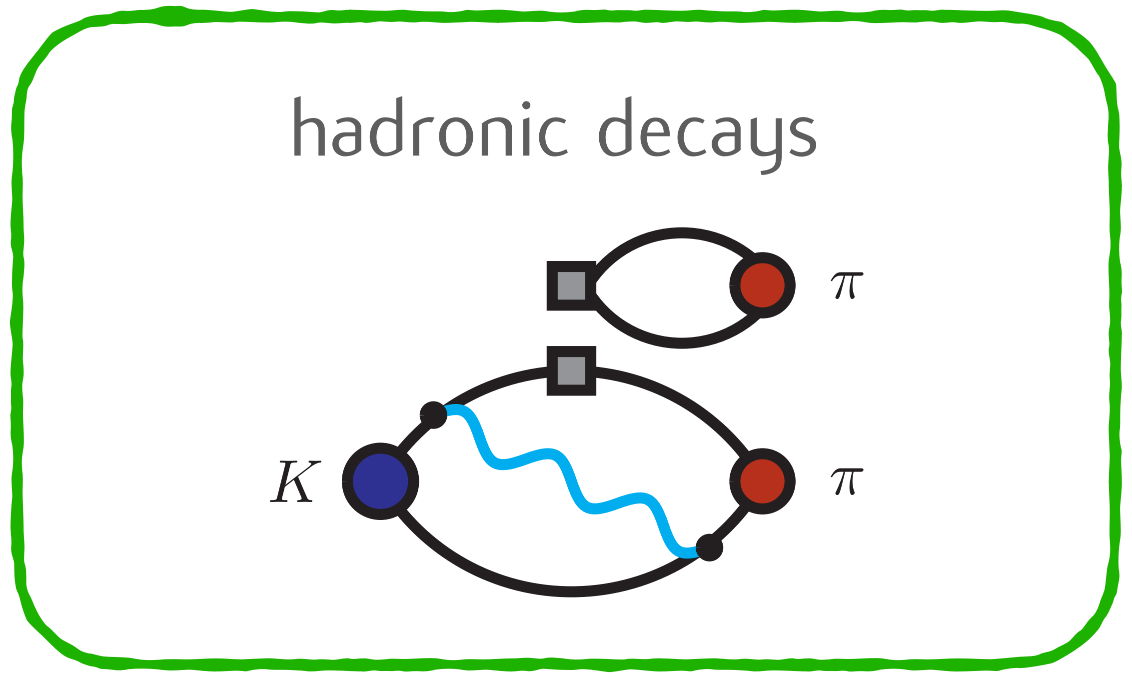
C.Sachrajda et al., [1910.07342]
N.Christ et al., [2304.08026]



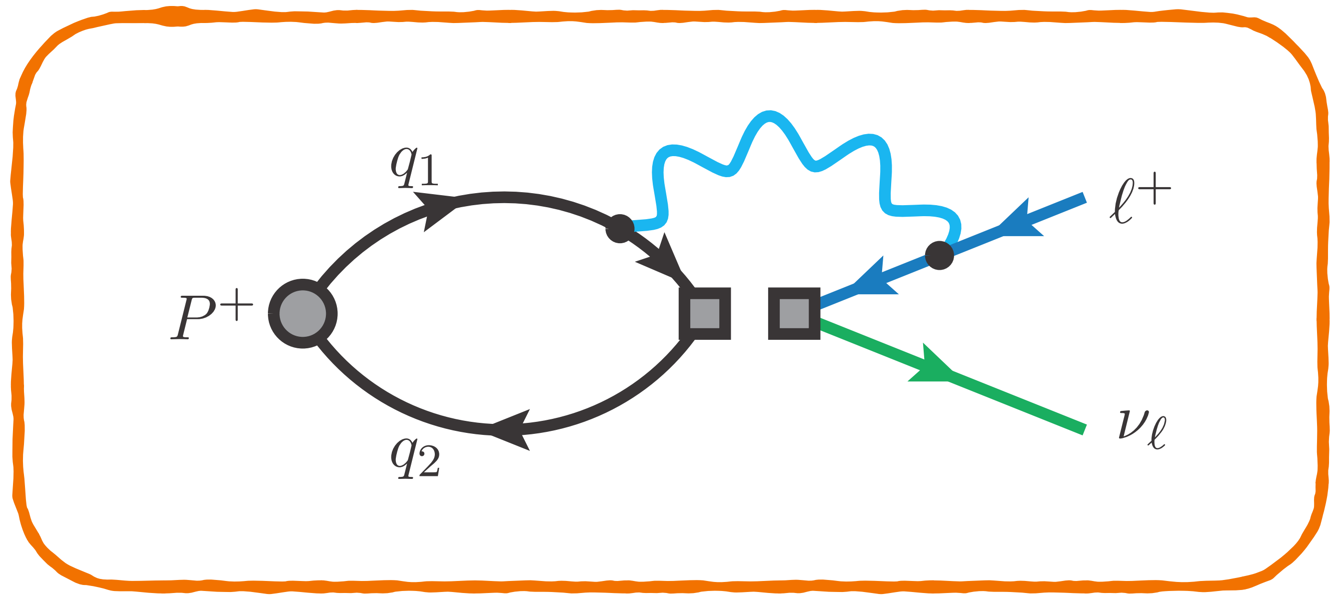


G.Gagliardi et al., Phys. Rev. D 105 (2022)
R.Frezzotti et al., [2306.07228]

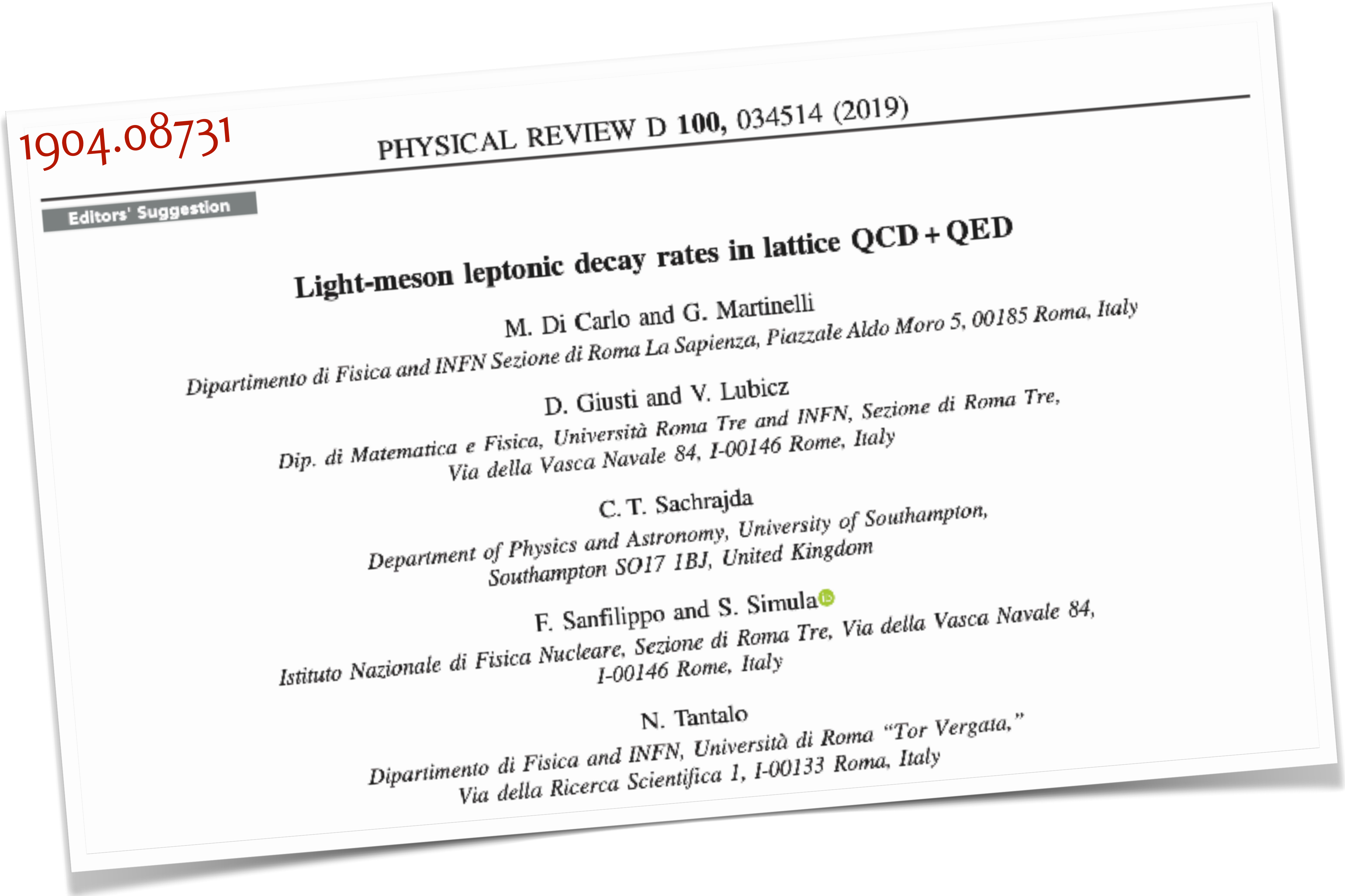
R.Abbott et al., PRD 102 (2020)
Z.Bai et al., PRL 115 (2015)
N.Christ et al., PRD 106 (2022)
N.Christ & X.Feng, EPJ Web Conf. 175 (2018)
Y.Cai & Z.Davoudi, [1812.11015]



leptonic decays of light pseudoscalar mesons



1904.08731


PHYSICAL REVIEW D **100**, 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD + QED

M. Di Carlo and G. Martinelli
Dipartimento di Fisica and INFN Sezione di Roma La Sapienza, Piazzale Aldo Moro 5, 00185 Roma, Italy

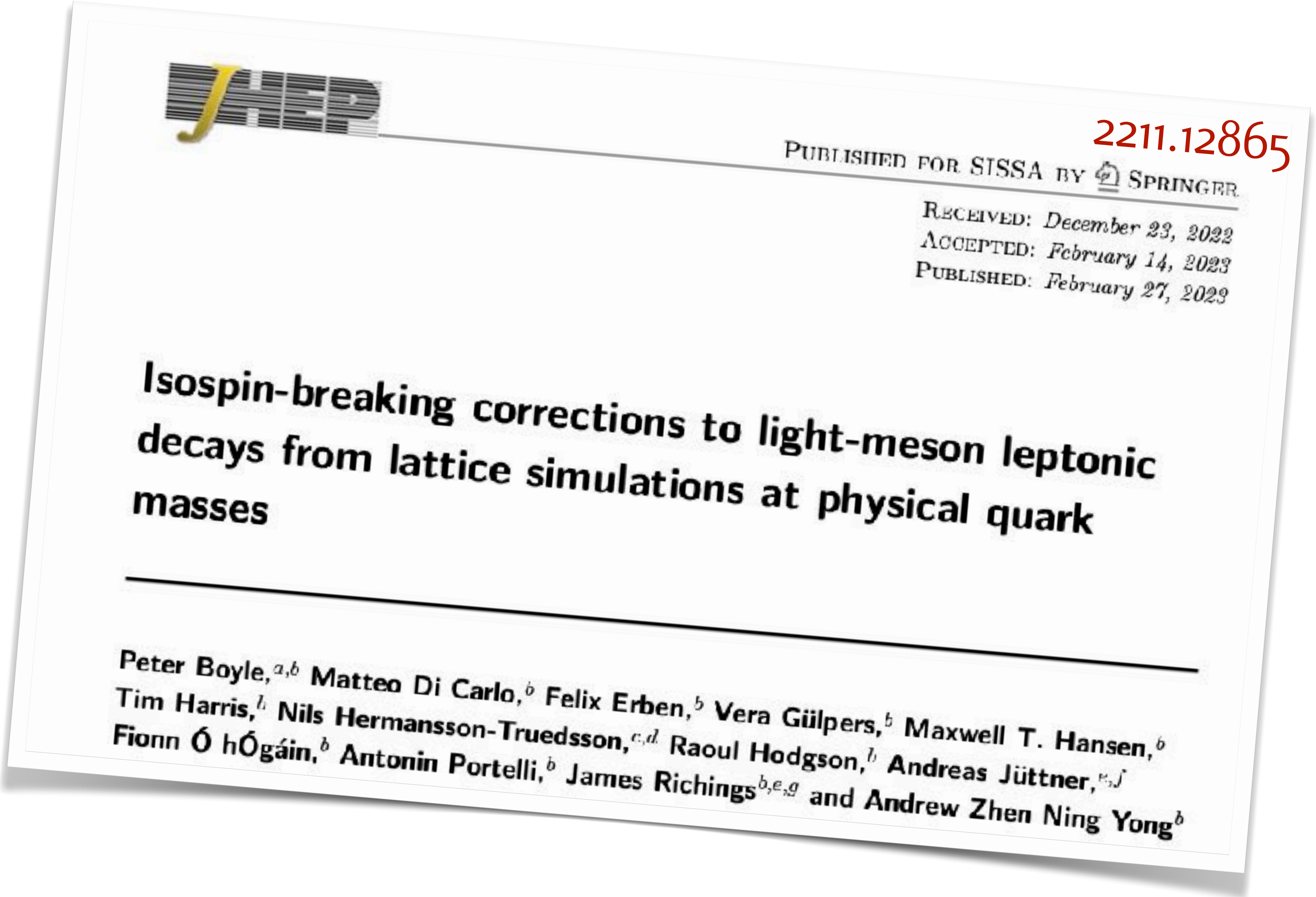
D. Giusti and V. Lubicz
Dip. di Matematica e Fisica, Università Roma Tre and INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy

C. T. Sachrajda
Department of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom

F. Sanfilippo and S. Simula
Istituto Nazionale di Fisica Nucleare, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy

N. Tantalo
Dipartimento di Fisica and INFN, Università di Roma "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Roma, Italy

2211.12865


JHEP

PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *December 23, 2022*
ACCEPTED: *February 14, 2023*
PUBLISHED: *February 27, 2023*

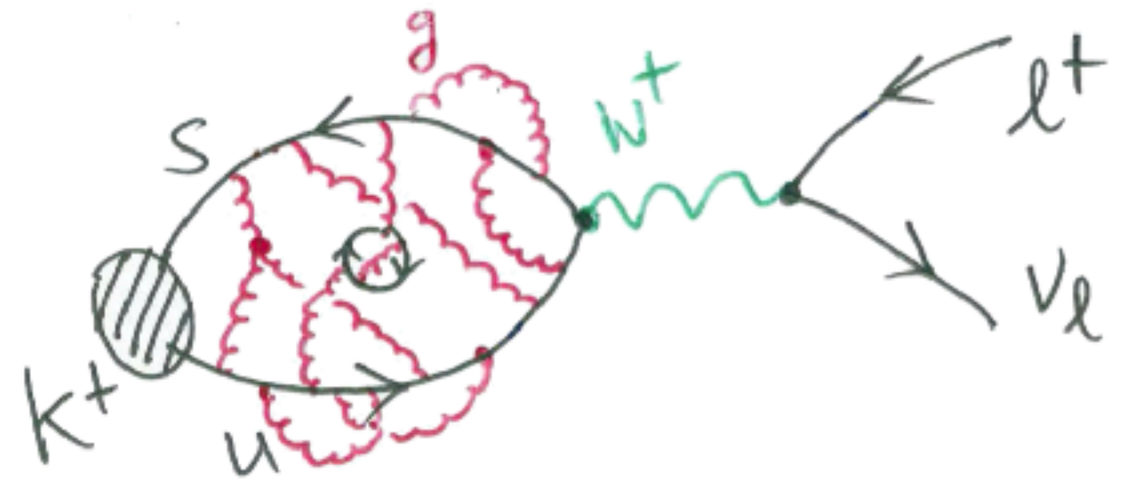
Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

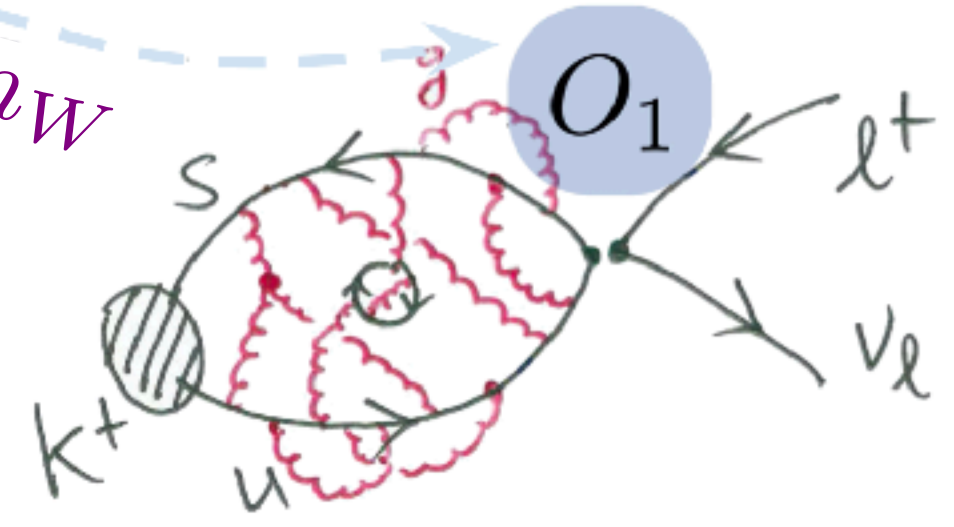
Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$

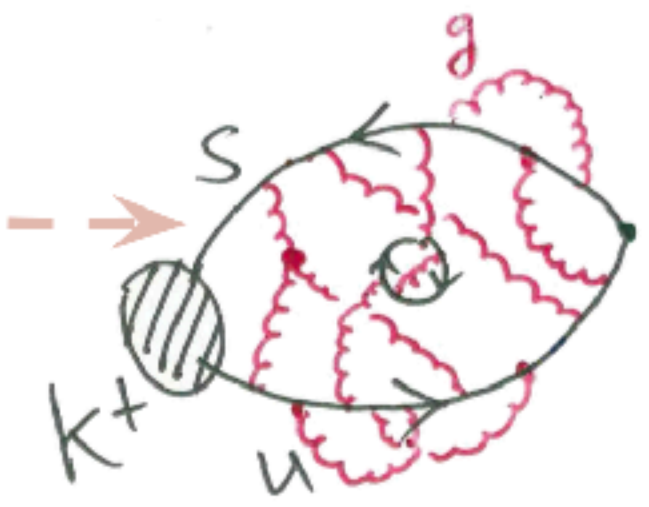


$1/a \ll m_W$



In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$



with the non-perturbative dynamic encoded in the **decay constant**

$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

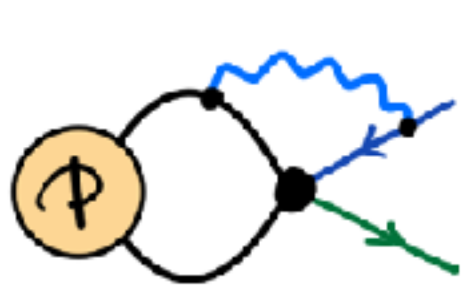
Leptonic decay rate at $\mathcal{O}(\alpha)$

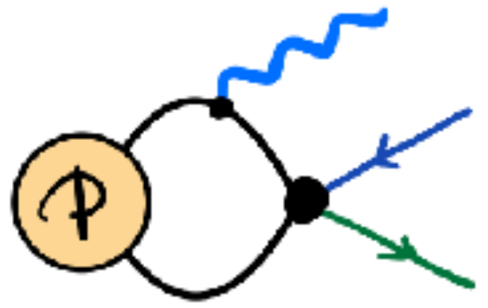
- The **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity

- **IR divergences** appear in intermediate steps of the calculation

F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{ \quad + \quad \Big\}$$





IR finite — IR divergent — IR divergent

- **UV divergences**: need to include QED corrections to the renormalization of the weak Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha_{\rm em}}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underbrace{\left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]}_{O_1^{\rm W\text{-}reg}(M_W)}$$

A.Sirlin, NPB **196** (1982)

E.Braaten & C.S.Li, PRD **42** (1990)

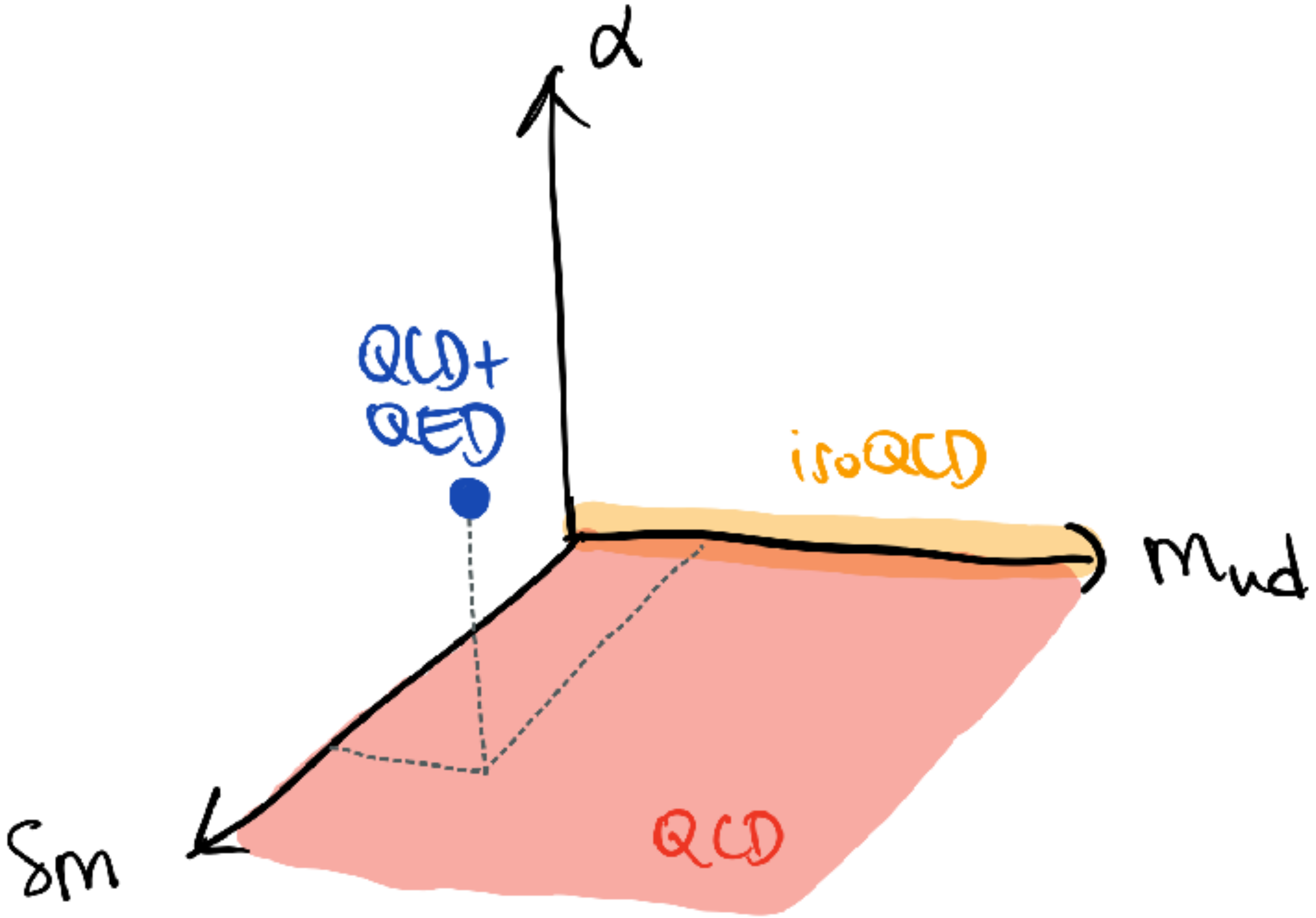
$$O_1^{\rm W\text{-}reg}(M_W) = Z^{\rm W\text{-}S}\left(\frac{M_W}{\mu}, \alpha_s(\mu), \alpha_{\rm em}\right) O_1^{\rm S}(\mu)$$

- perturbative @ 2 loops in QCD+QED
- non-perturbative in lattice QCD+QED

MDC et al., PRD **100** (2019)

Leptonic decay rate at $\mathcal{O}(\alpha)$

Defining the isospin symmetric world



- The full QCD+QED theory is unambiguously defined after **matching** a set of observables to the real world

$$\left[\frac{\hat{\mathrm{M}}_j}{\hat{\Lambda}}\right]^2 (g, e^\phi, \hat{\mathbf{m}}^\phi) = \left(\frac{\mathrm{M}_j^\phi}{\Lambda^\phi}\right)^2 \longrightarrow \hat{\mathbf{m}}^\phi(g)$$

$$j = 1, \ldots, N_f$$

- The definition of QCD or isoQCD requires a prescription, i.e. some renormalization conditions to **fix the bare parameters** of the action

$$\boldsymbol{\sigma}^{\mathrm{QCD}} = (g^{\mathrm{QCD}}, 0, \hat{\mathbf{m}}^{\mathrm{QCD}}) \qquad \hat{\mathbf{m}}^{\mathrm{QCD}} = (\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}, \ldots)$$

$$\boldsymbol{\sigma}^{(0)} = (g^{(0)}, 0, \hat{\mathbf{m}}^{(0)}) \qquad \hat{\mathbf{m}}^{(0)} = (\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}, \ldots)$$

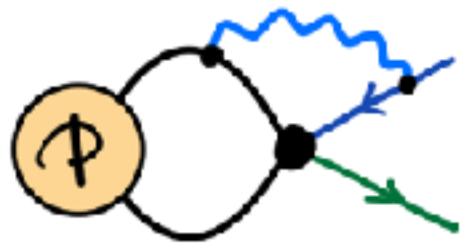
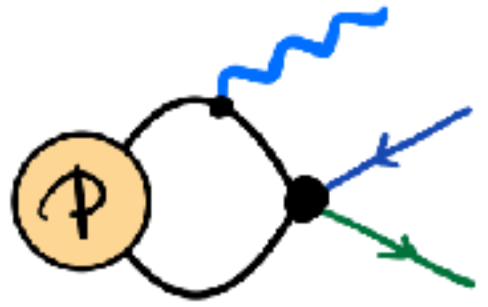
BMW hadronic scheme in **RBC-UKQCD (2022)** compatible with **GRS quark mass** scheme in **RM123S (2019)**

BMW, PRL 111 (2013), PRL 117 (2016)　　Gasser, Rusetsky & Scimemi, EPJC 32 (2003)

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

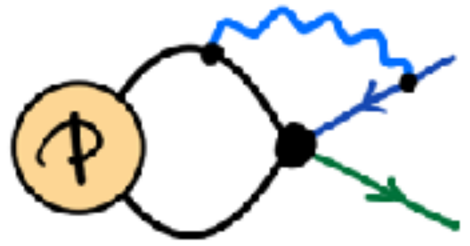
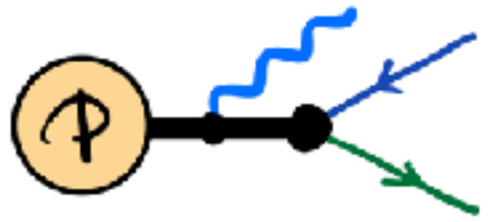
$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Bigg\{$$  $$+$$  $$\Bigg\}$$

IR finite — IR divergent — IR divergent

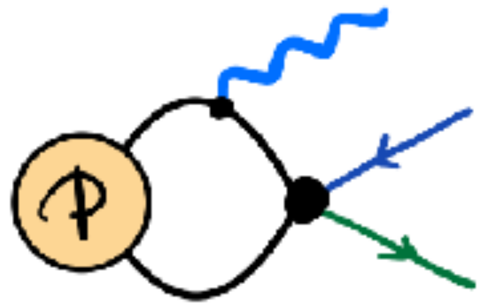
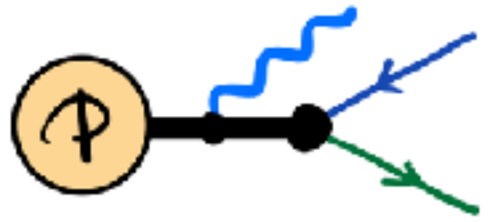
Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $-$  $\Big\} + \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $+$  $\Big\}$

IR finite — IR finite — IR finite

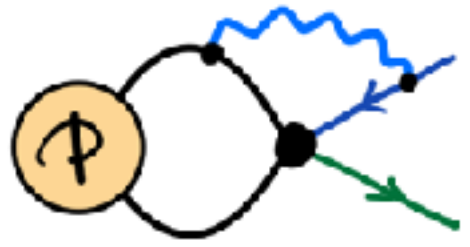
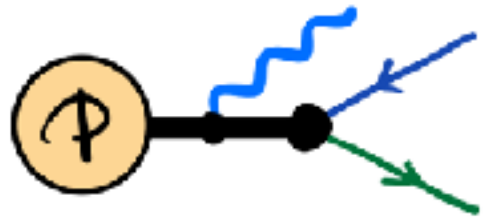
$+ \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $-$  $\Big\}$

IR finite

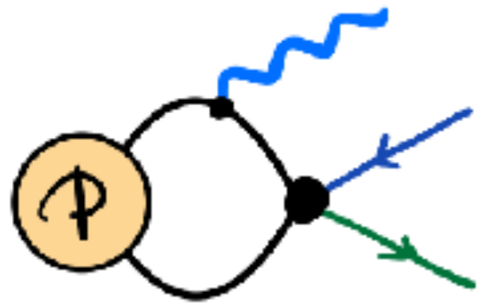
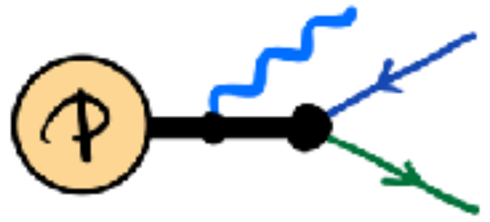
Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) = \lim_{L\to\infty} \Big\{$  $-$  $\Big\} + \lim_{m_\gamma \to 0} \Big\{$  $+$  $\Big\}$

on the lattice — in perturbation theory

$+ \lim_{L\to\infty} \Big\{$  $-$  $\Big\}$

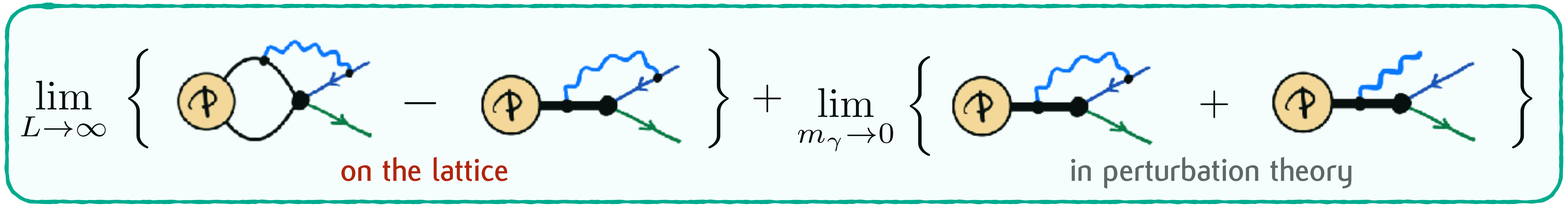
on the lattice

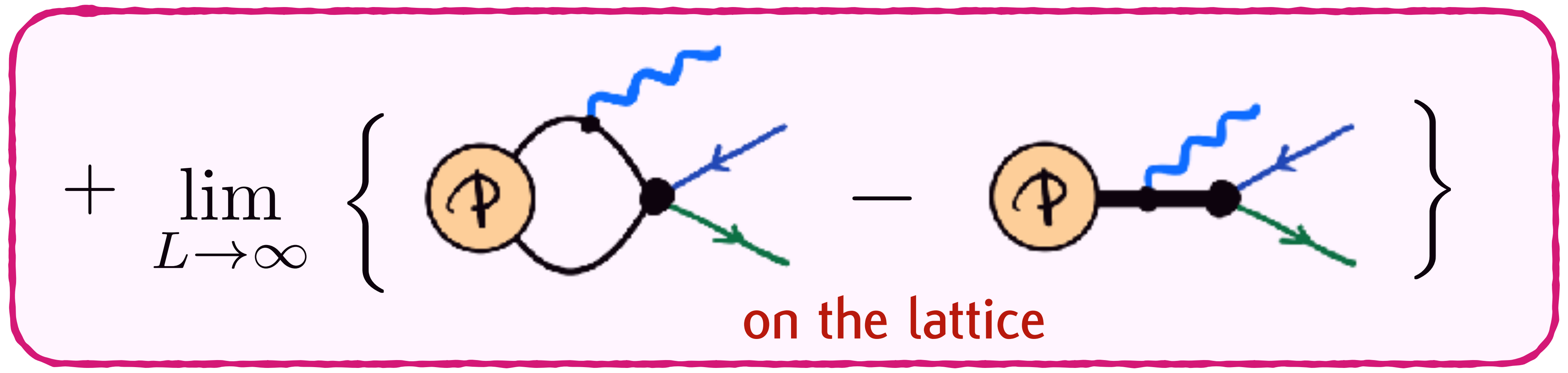
Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) =$





enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

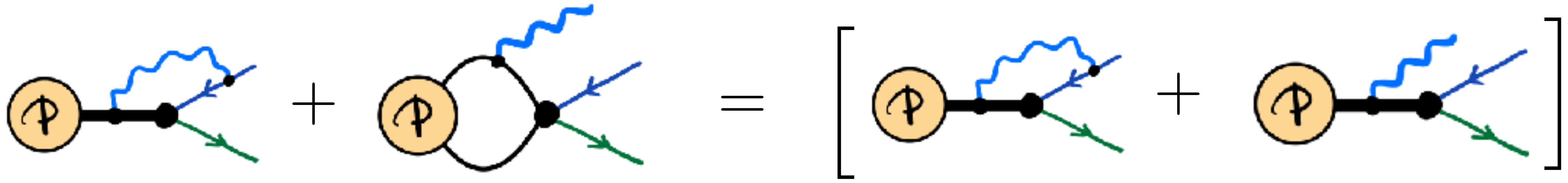
leading **finite-volume scaling** well studied

V.Lubicz et al., PRD **95** (2017)   N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

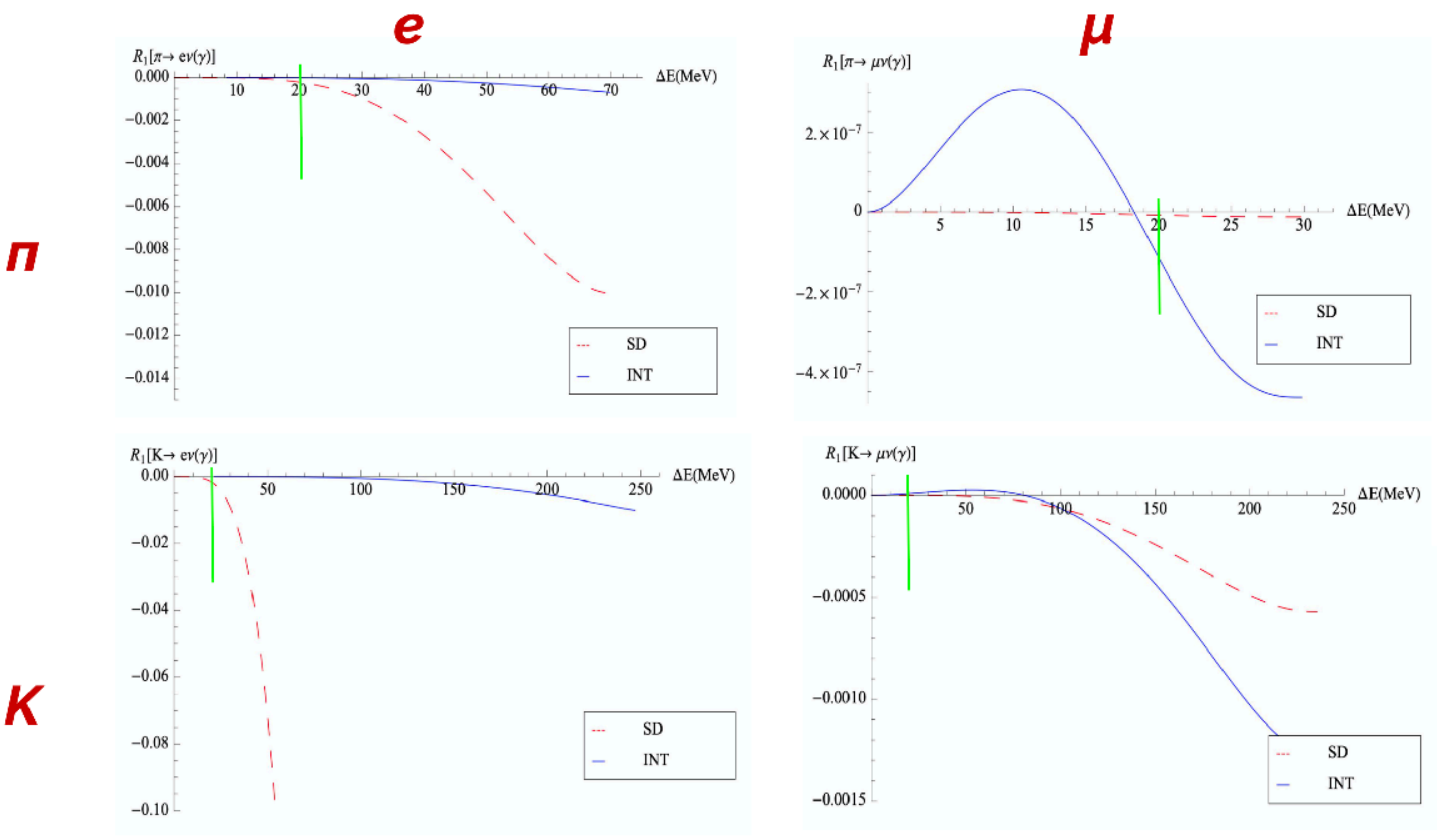
relevant for K_{e2} and π_{e2}
& decays of heavier mesons

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Real photon emission and structure dependence



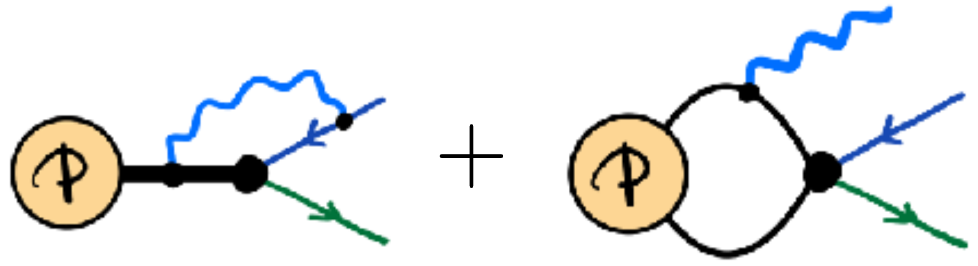
$$\left(1 + R_1^{\mathrm{SD}}(\Delta E) + R_1^{\mathrm{INT}}(\Delta E)\right)$$



Calculation at O(p4) in χPT

N. Carrasco et al., PRD 91 (2015)

Real photon emission and structure dependence



$$= \delta R_{\rm pt}(\Delta E) + \delta R_1^{\rm SD}(\Delta E) + \delta R_1^{\rm INT}(\Delta E)$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu 2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu 2[\gamma]}$
δR_0	$^{(*)}$	$0.0411\ (19)$	$^{(*)}$	$0.0341\ (10)$
$\delta R_{\rm pt}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\rm SD}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	$1.19\ (14)$	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\rm INT}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

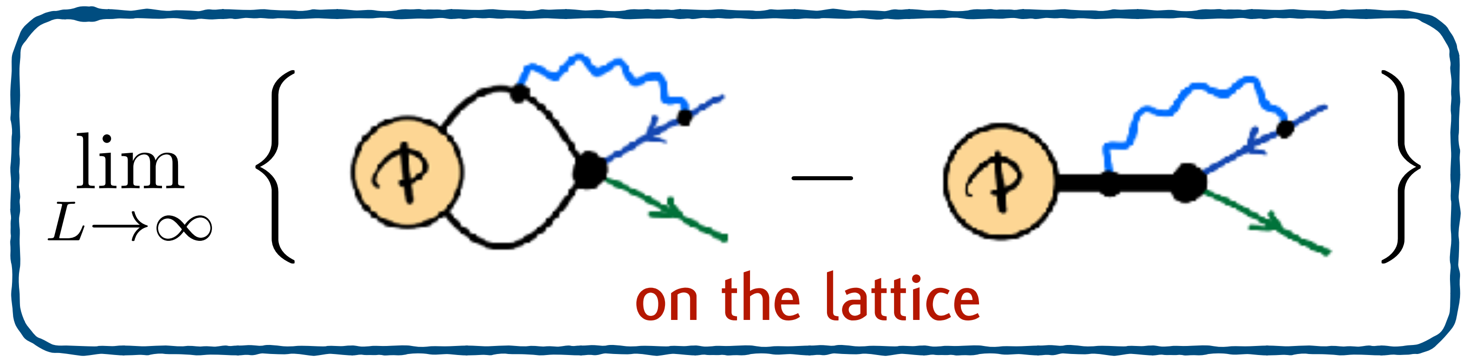
$^{(*)}$ *Not yet evaluated by numerical lattice QCD+QED simulations.*

Confirmed by numerical lattice calculation

A. Desiderio et al., PRD 102 (2021)
R. Frezzotti et al., PRD 103 (2021)

Leptonic decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate



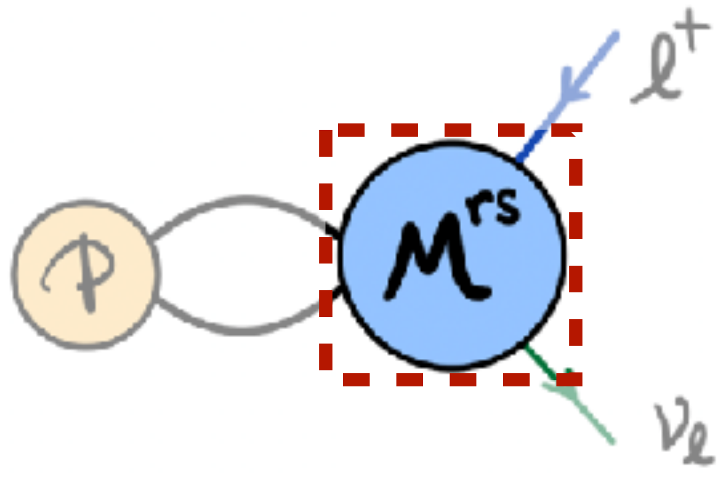
$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1+\delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1-\frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta \mathcal{Z}}{\mathcal{Z}_0}\right)$$

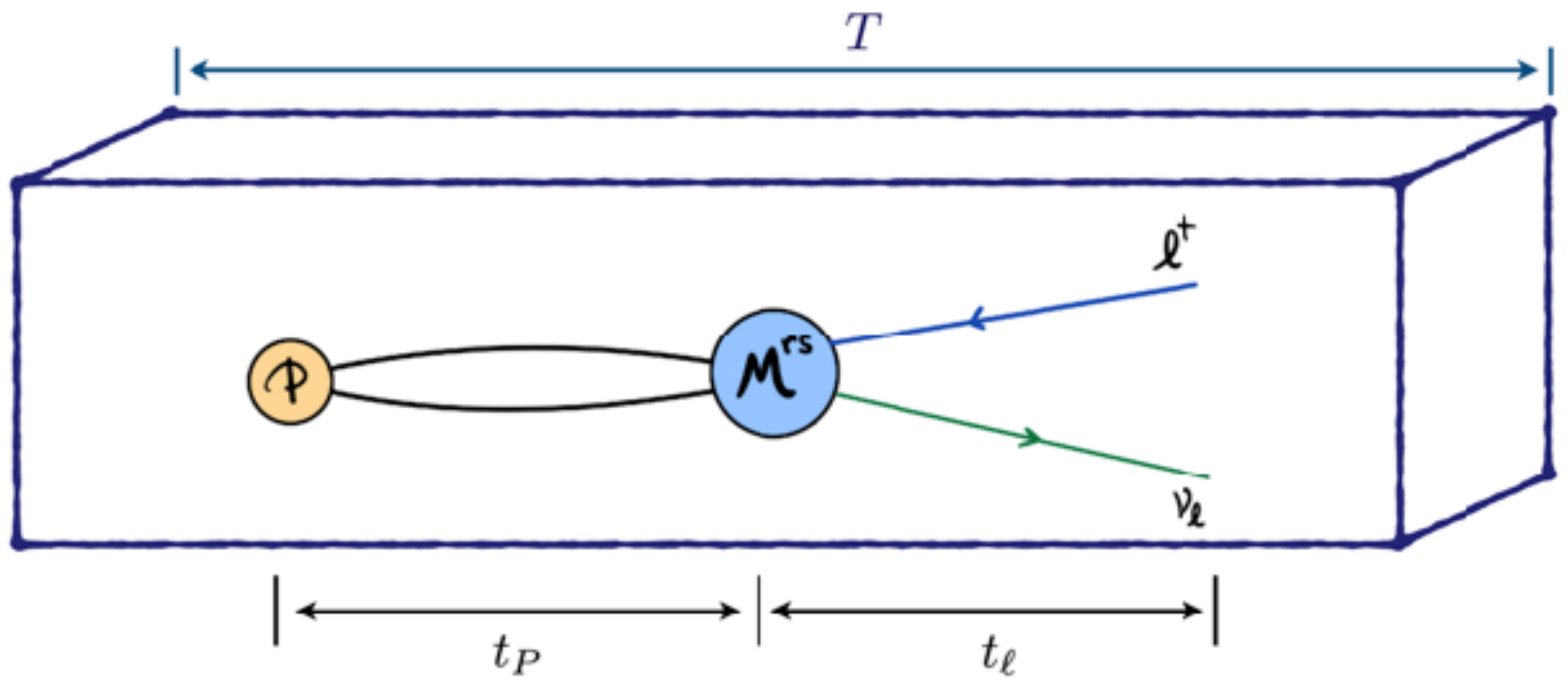
PDG convention

- $\boldsymbol{\delta \mathcal{A}_P}$ from the correction to the (bare) matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell;\, \nu_\ell, s, \mathbf{p}_\nu |\, O_W \,| P^+, \mathbf{0}\rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta \mathcal{Z}}$ correction to the renormalization of the weak operator O_W MDC et al., PRD 100 (2019)

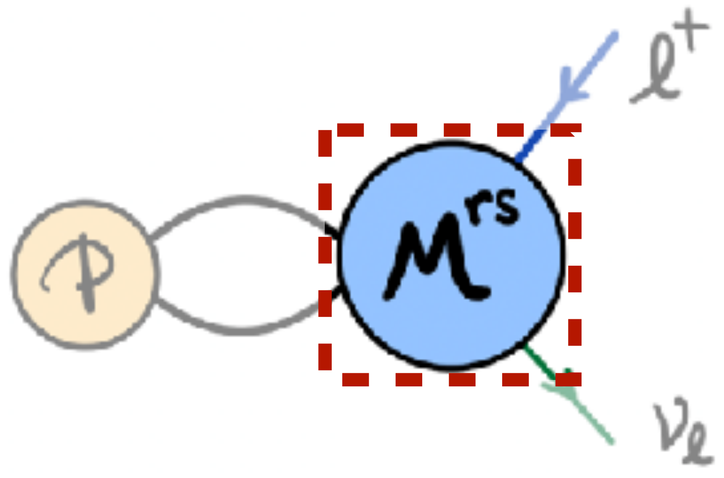
$$\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} \rightarrow \delta R_{K\pi} = 2\left(\frac{\delta \mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta \mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$$

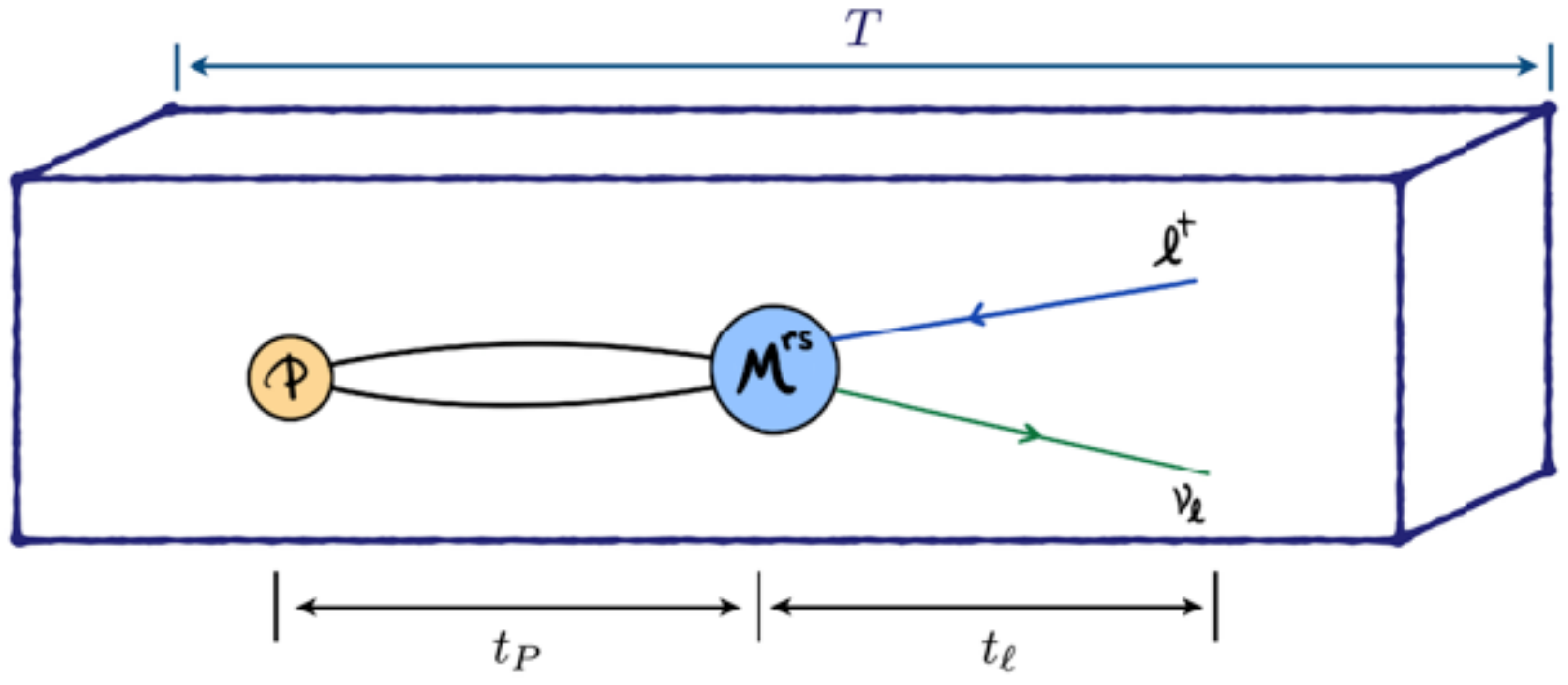
From correlators to matrix elements

Our goal: 

How we realise it: 

From correlators to matrix elements

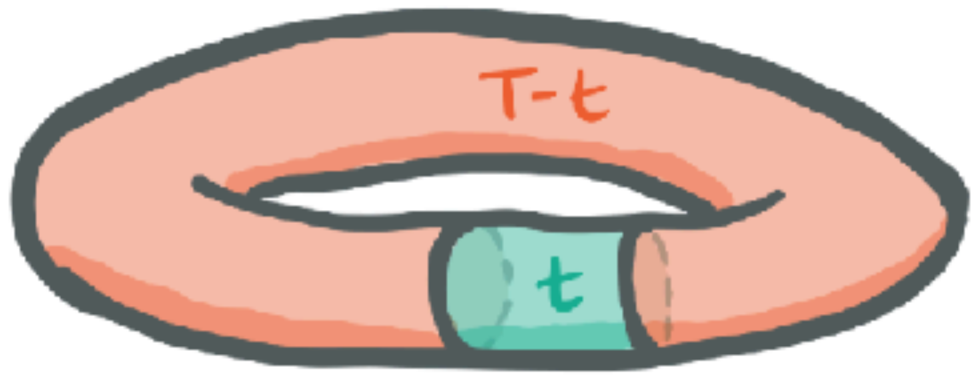
Our goal: 

How we realise it: 

Tree-level decay amplitude: $|\mathcal{M}_0(\mathbf{p}_\ell)|^2 = |\mathcal{A}_{P,0}|^2 |\mathcal{L}_0(\mathbf{p}_\ell)|^2 \quad \mathcal{A}_{P,0} = \langle 0|A^0|P\rangle_0 = i m_{P,0}\,[f_{P,0}]$

 $= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle_T \to \dfrac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\left\{ e^{-m_{P,0}t} - e^{-m_{P,0}(T-t)} \right\}$ $\quad Z_{P,0} = \langle P, \mathbf{p}=\mathbf{0}|\phi^\dagger|0\rangle_0$

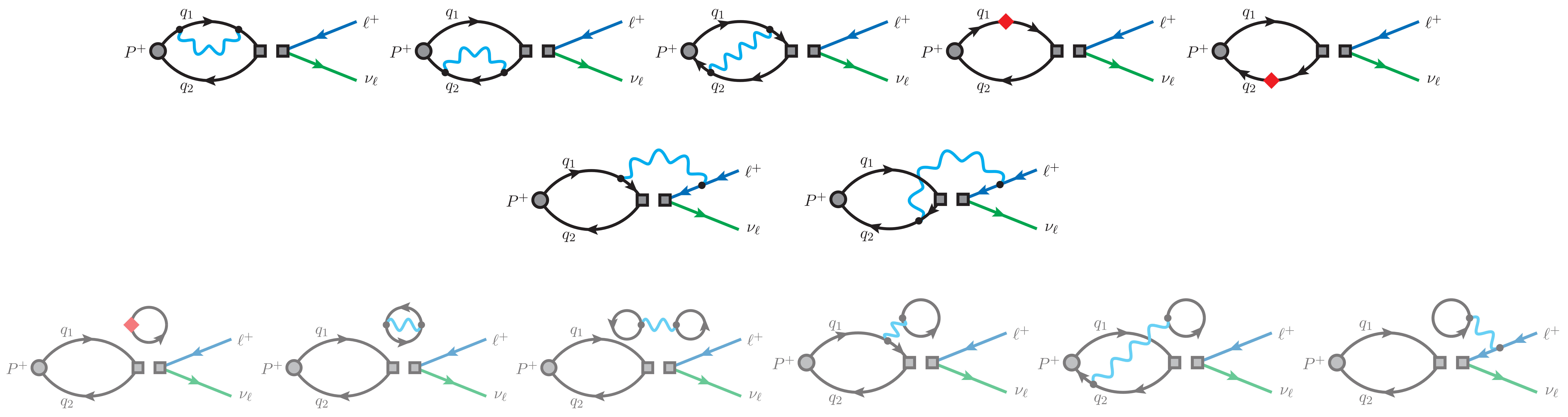
 $= \langle 0|\phi(0)\phi^\dagger(-t)|0\rangle_T \to \dfrac{Z_{P,0}^2}{2m_{P,0}}\left\{ e^{-m_{P,0}t} + e^{-m_{P,0}(T-t)} \right\}$



IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_{\mathrm{u}} - m_{\mathrm{d}} = 0$

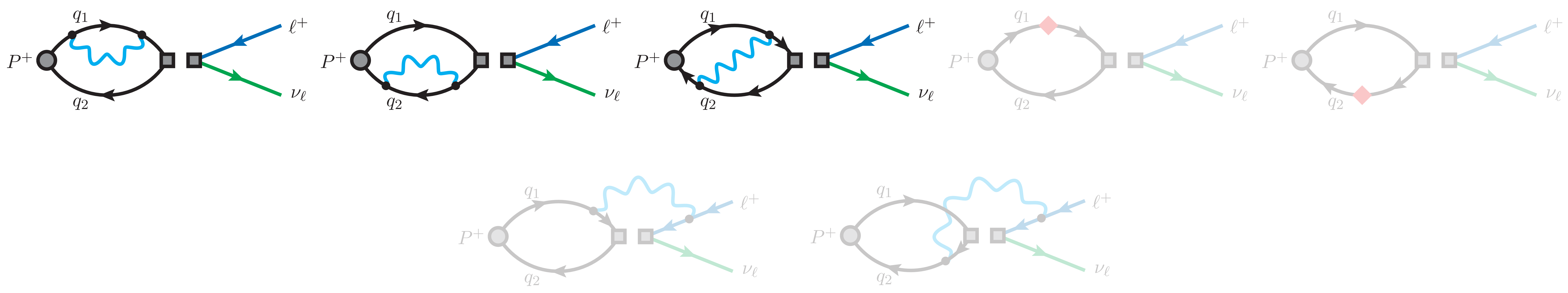


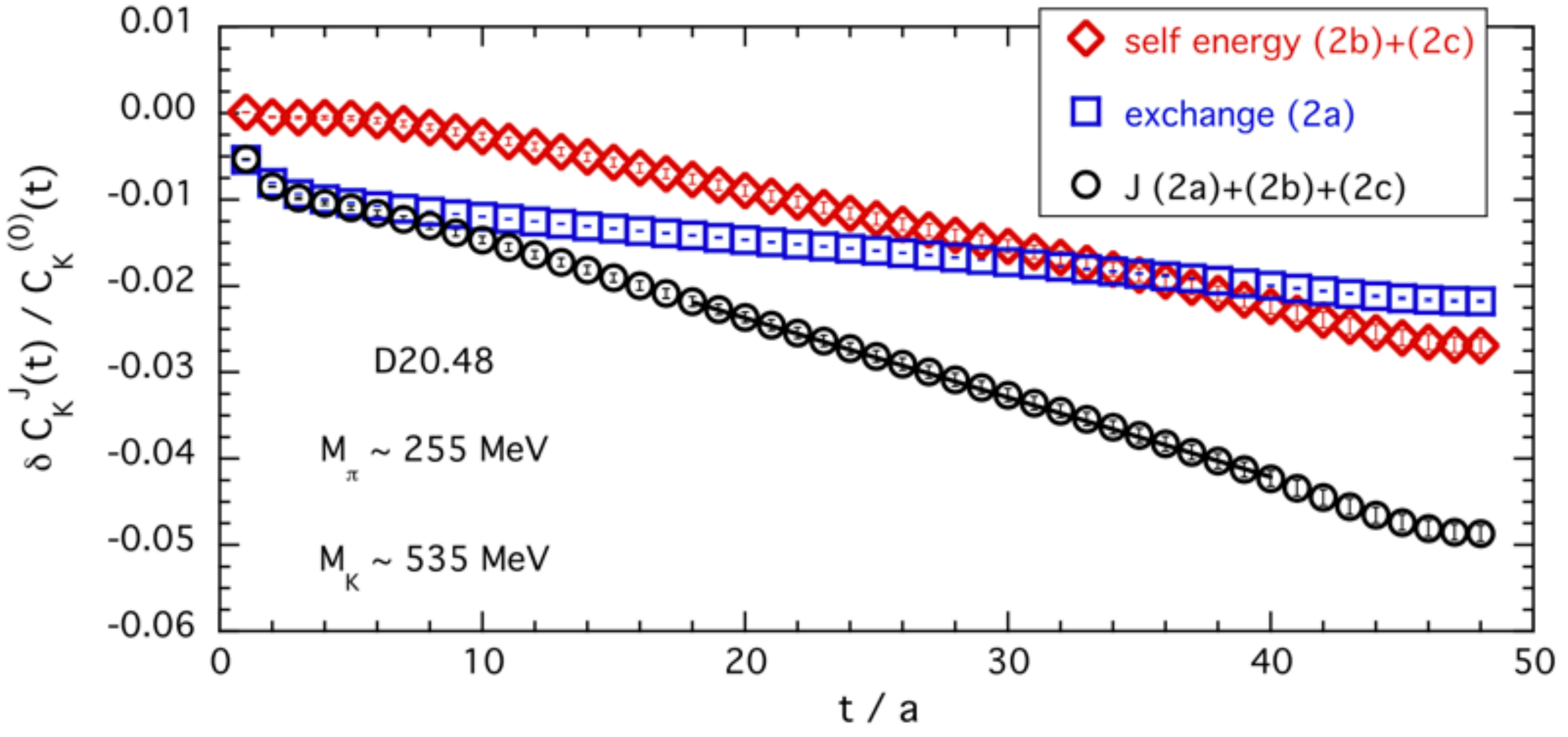
Both RM123S and RBC-UKQCD calculations are performed in the **electro-quenched approximation: sea quarks electrically neutral**

IB corrections to the decay amplitude

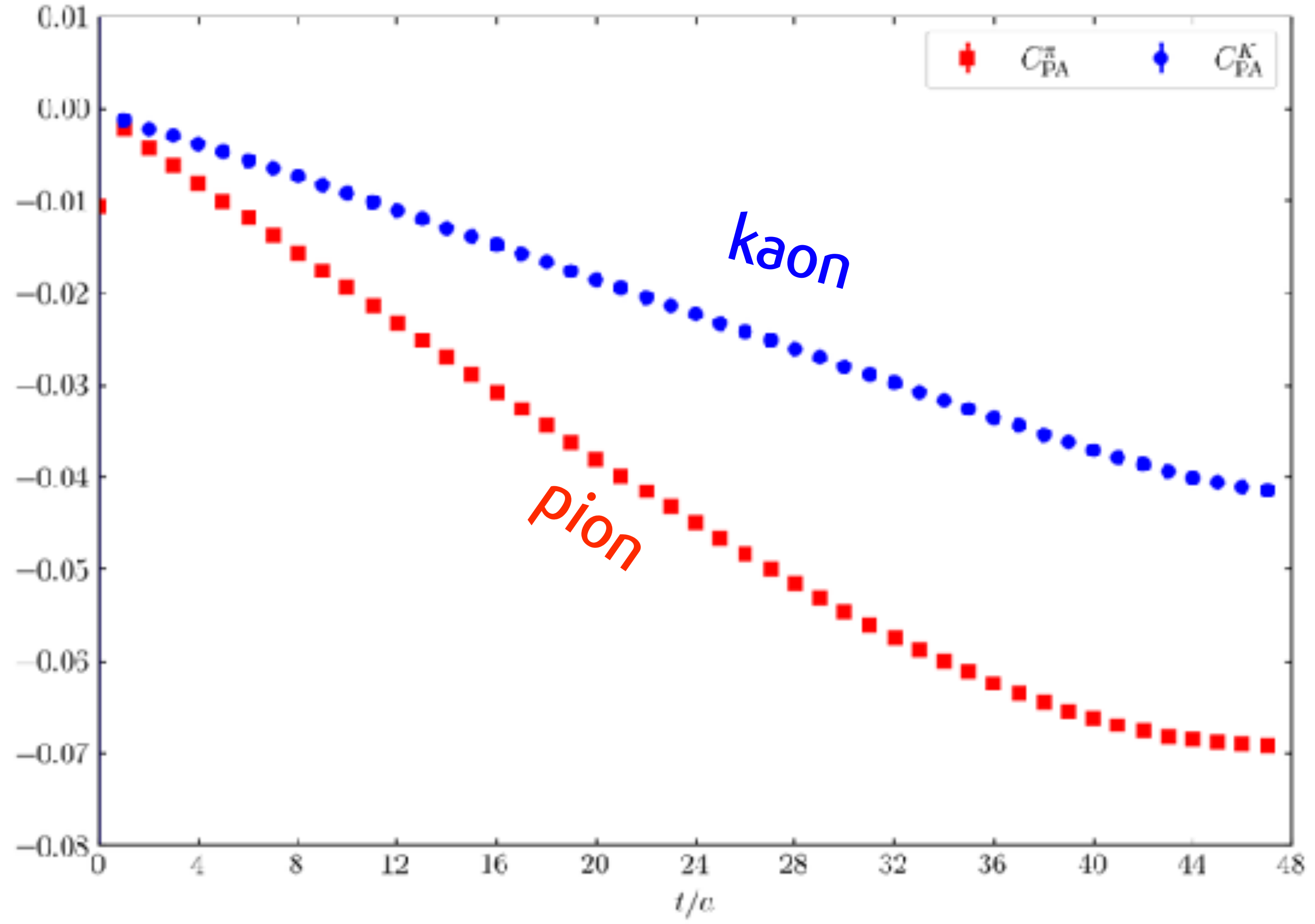
G.M.de Divitiis et al. [RM123], PRD 87 (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$





MDC et al., PRD 100 (2019)

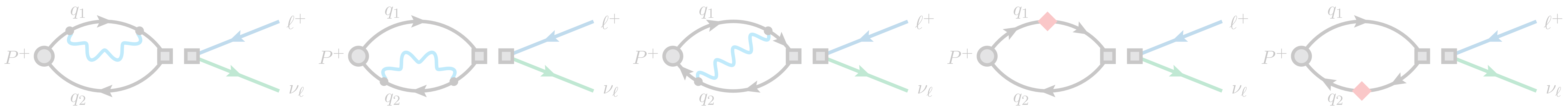


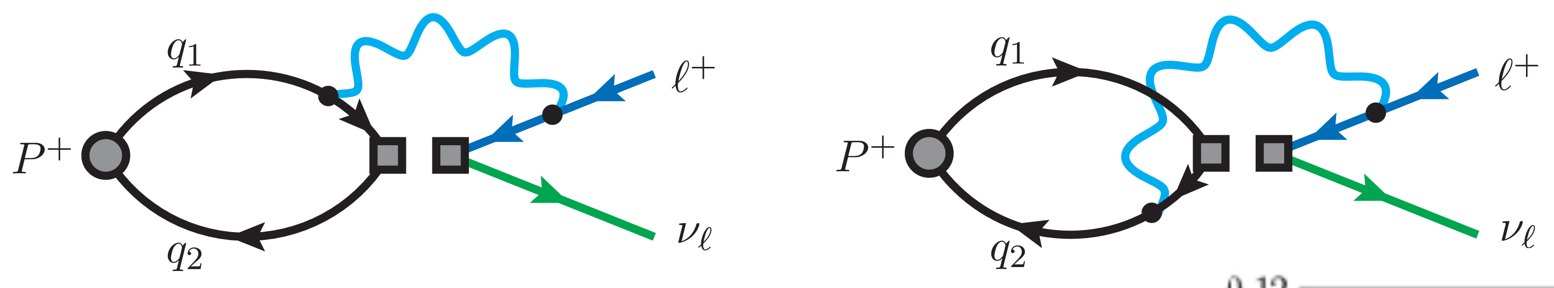
P.Boyle, MDC et al., JHEP 02 (2023)

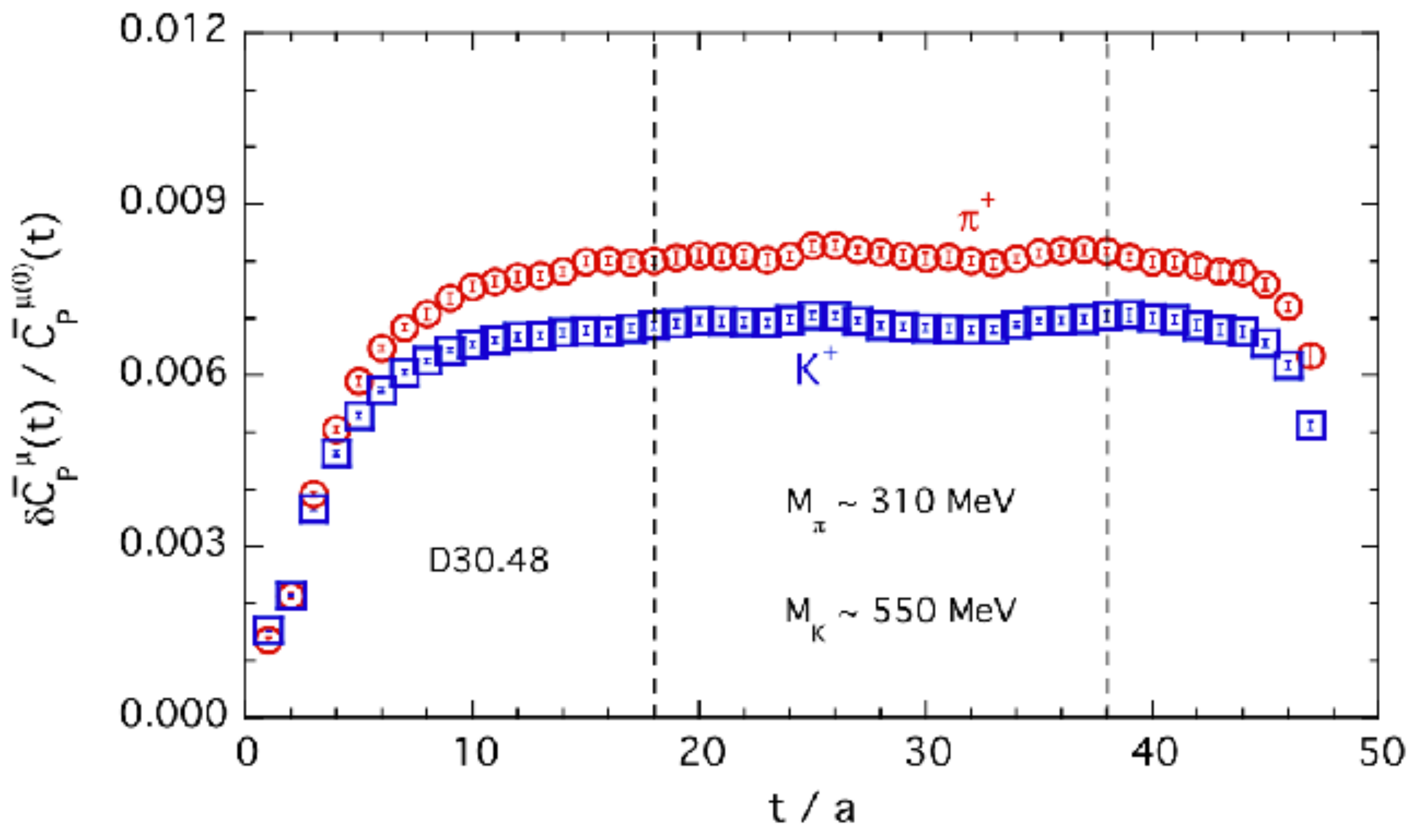
IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

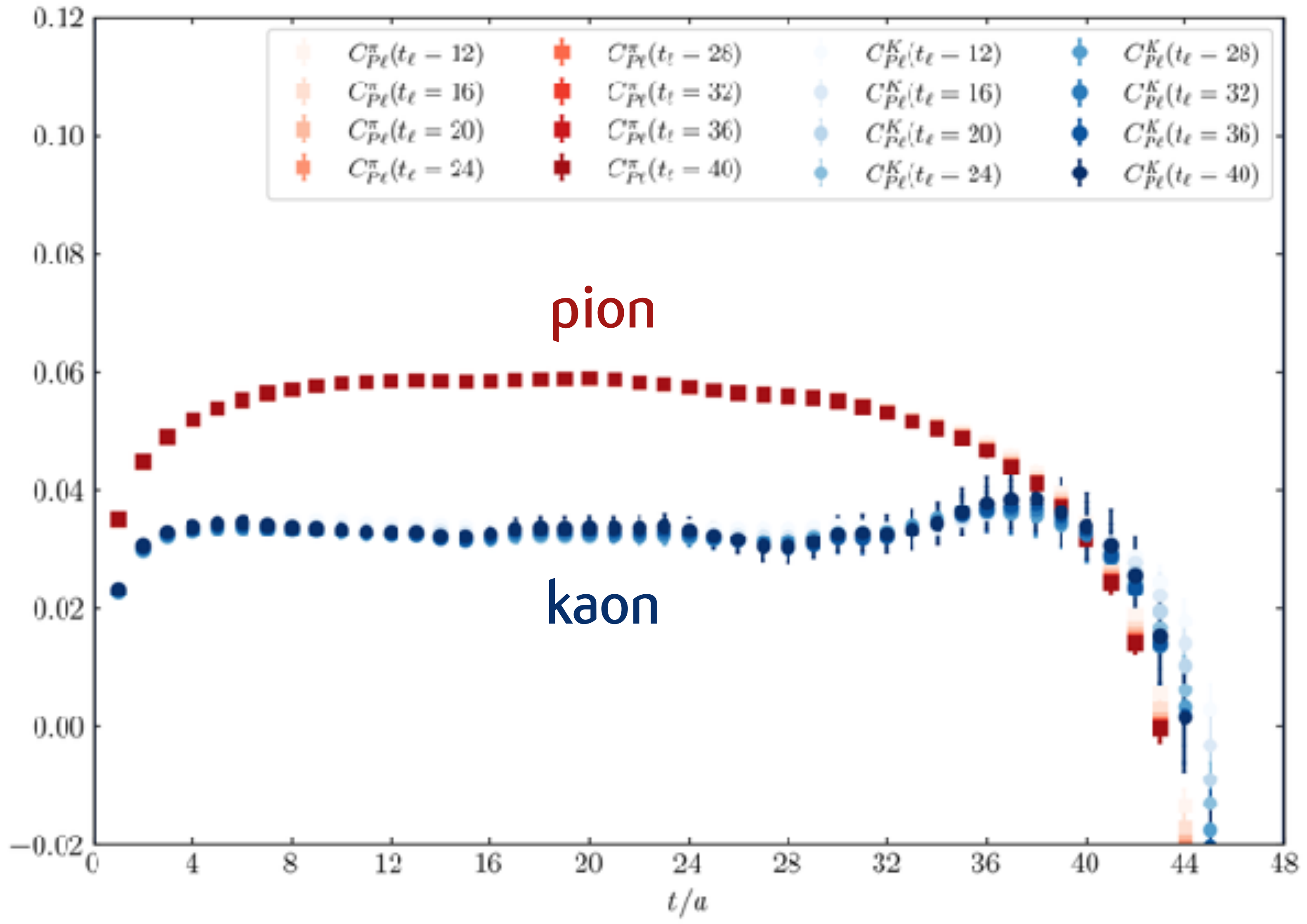
RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$







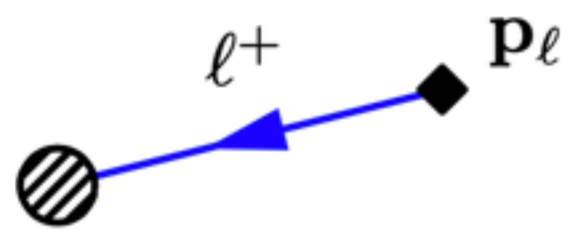
MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

Non-factorisable QED corrections

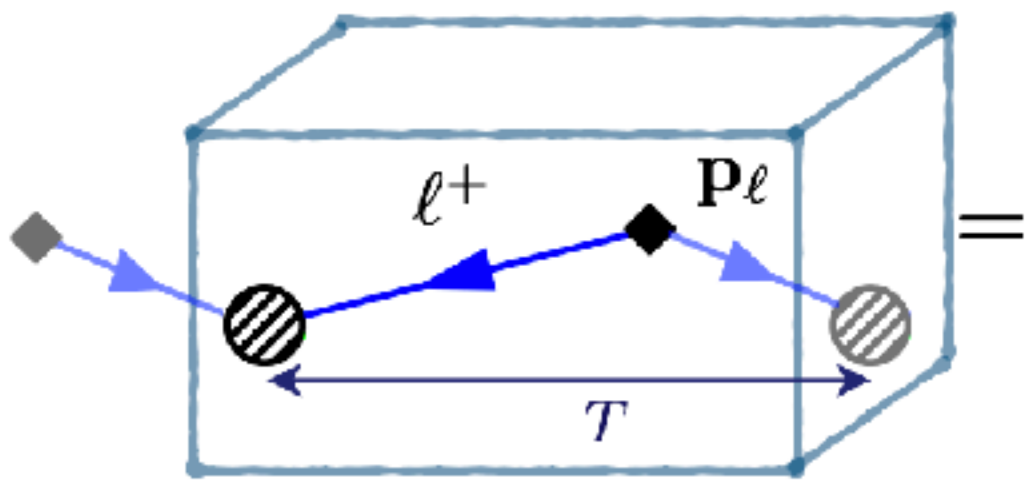
The lepton in a finite volume



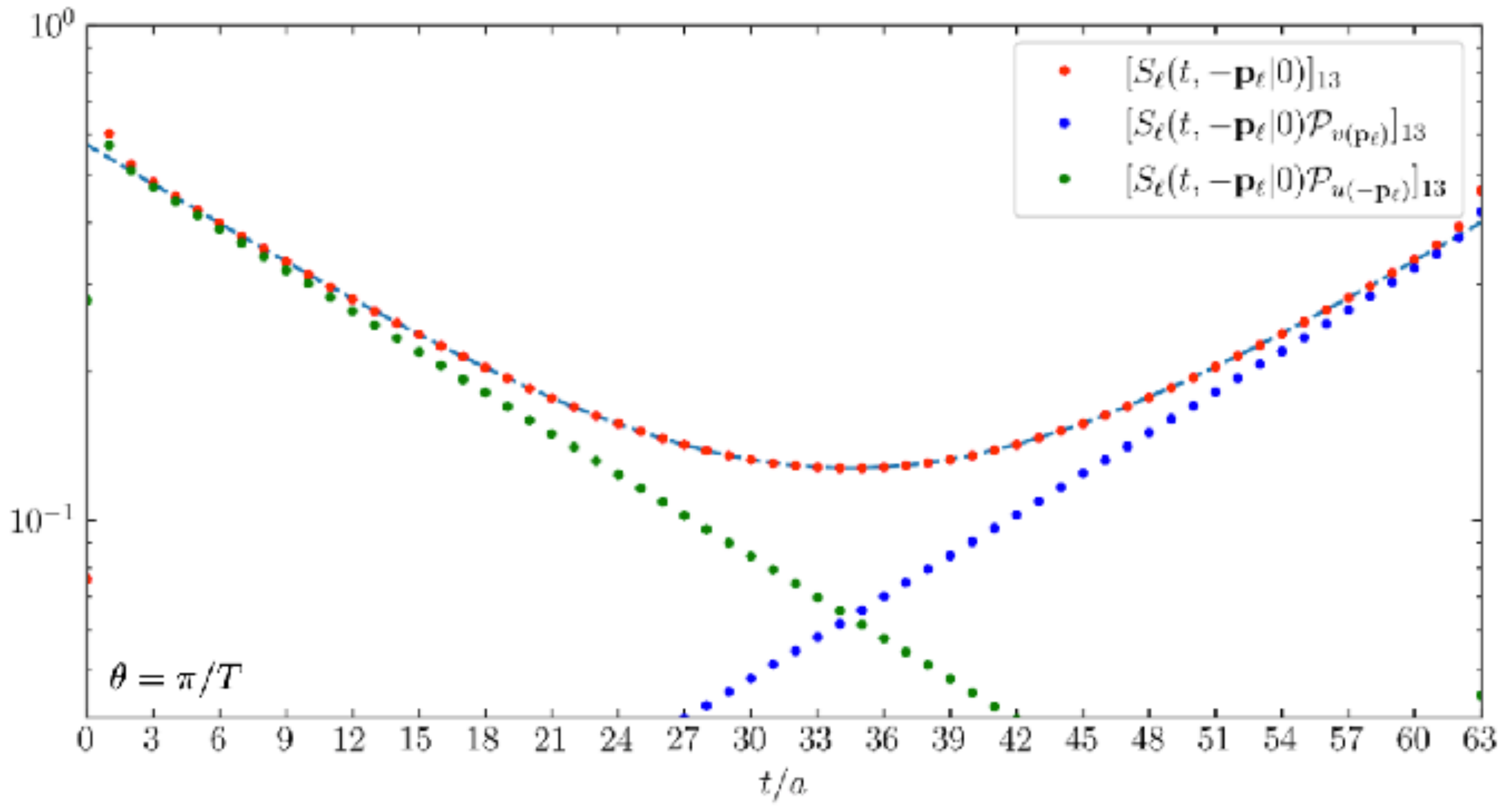
$$= \quad S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} \right\}$$

Non-factorisable QED corrections

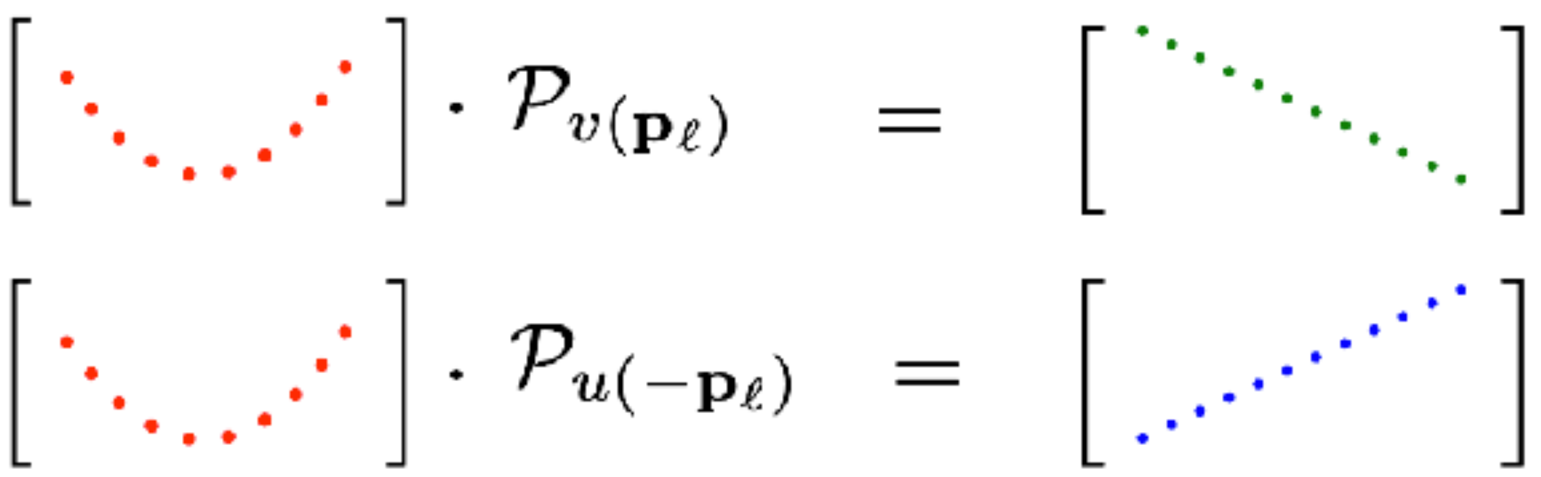
The lepton in a finite volume



$$= S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} + \mathrm{e}^{\mathrm{i}\theta T}\mathrm{e}^{-(T-t)E_\ell} \frac{u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)}{2\Omega_\ell} \right\} \times \frac{1}{1-\mathrm{e}^{-TE_\ell}\mathrm{e}^{\mathrm{i}\theta T}}$$

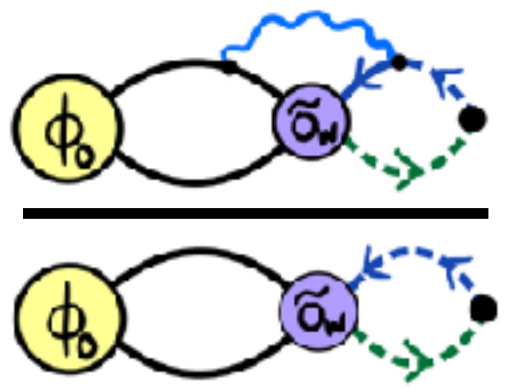


We can select specific components using projectors:

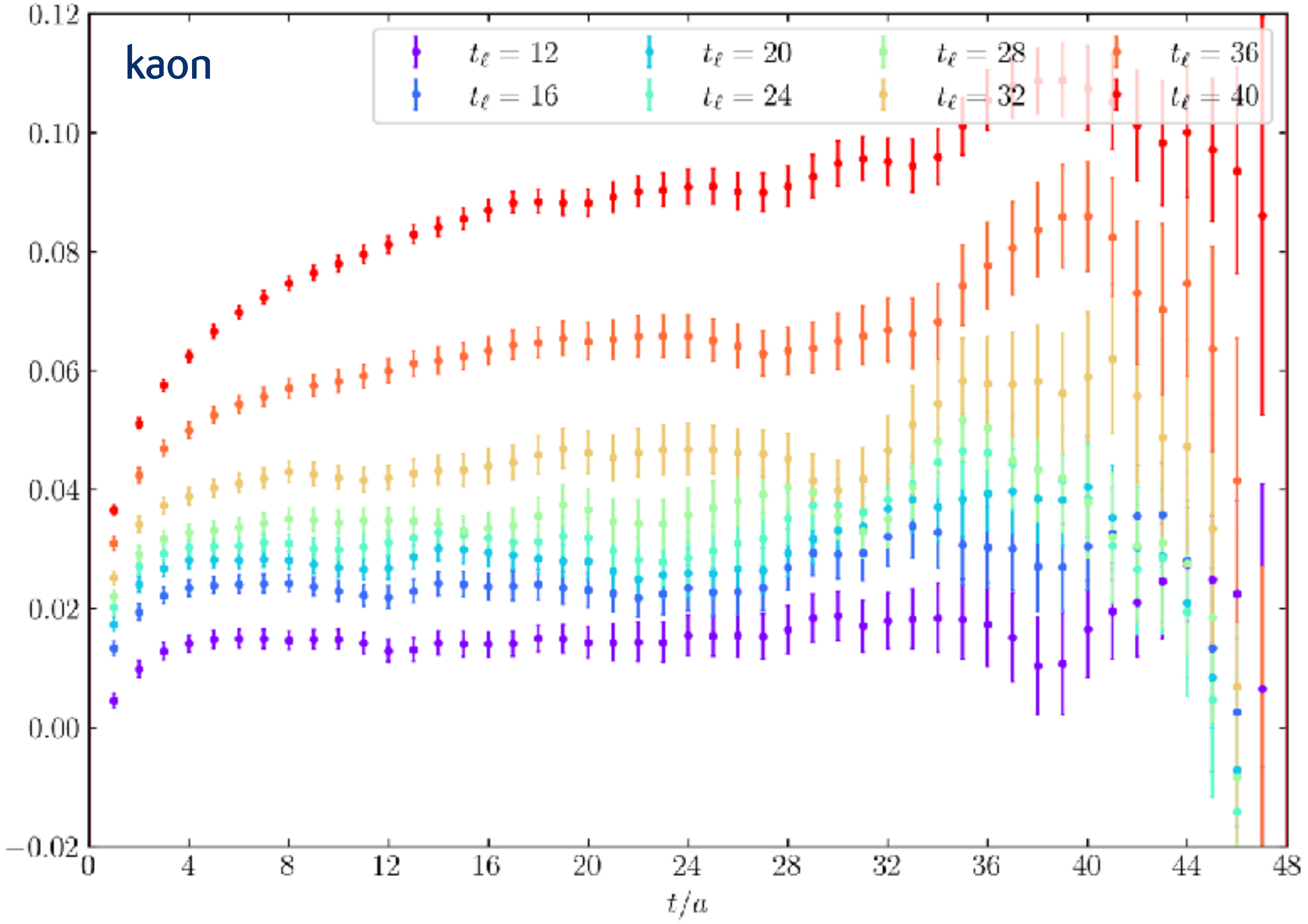


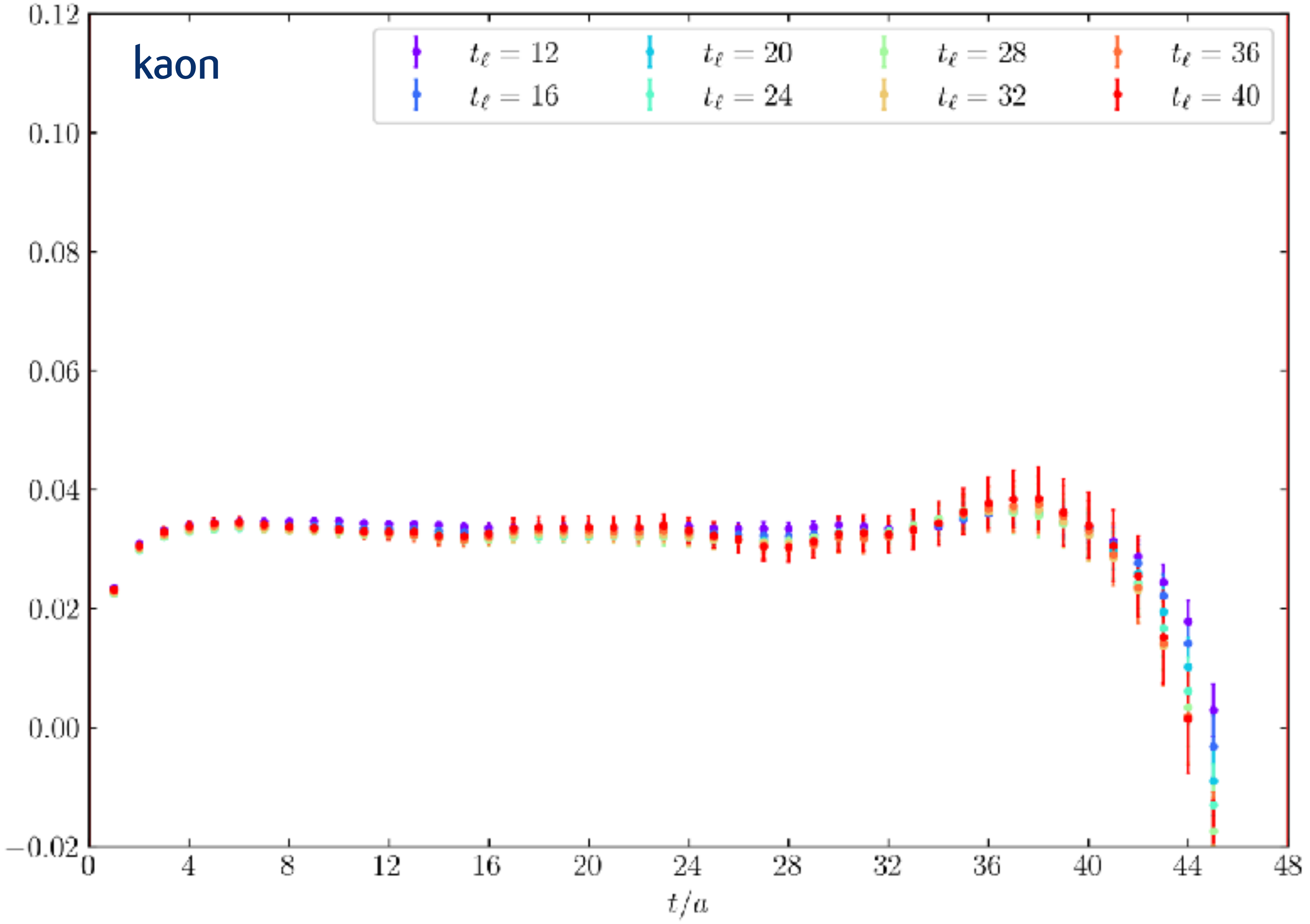
$$\mathcal{P}_{v(\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)]$$
$$\mathcal{P}_{u(-\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)]$$

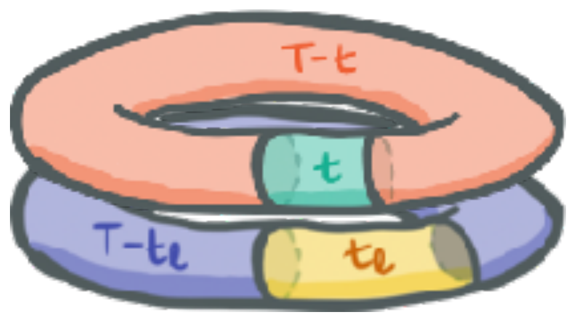
Non-factorisable QED corrections



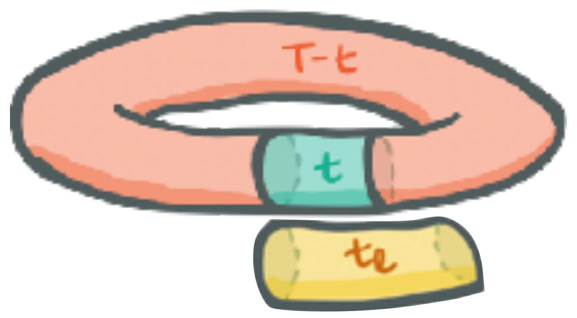
$$\rightarrow \frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \quad f_{\mathrm{P}\ell}(t,T)$$







without projection

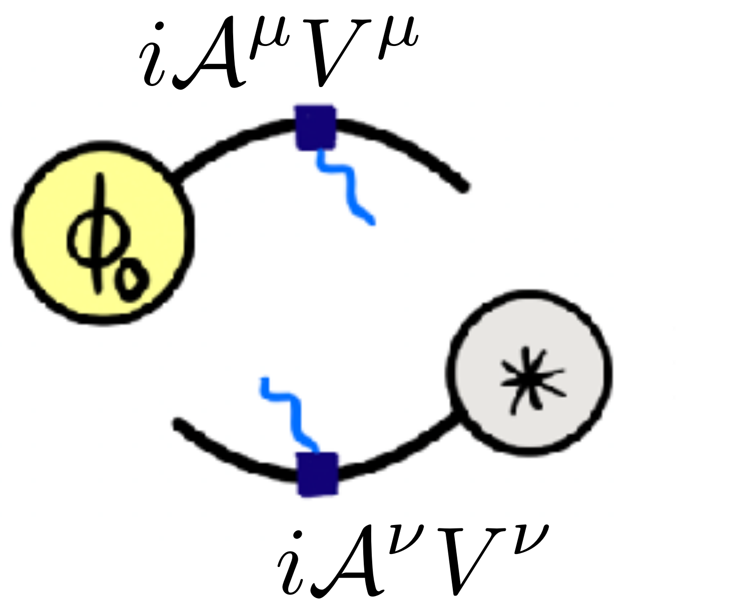


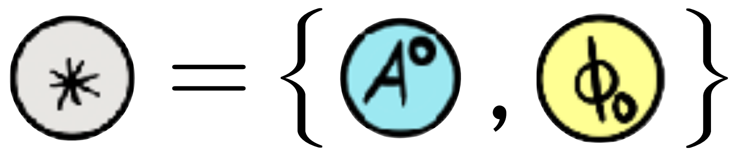
with projection

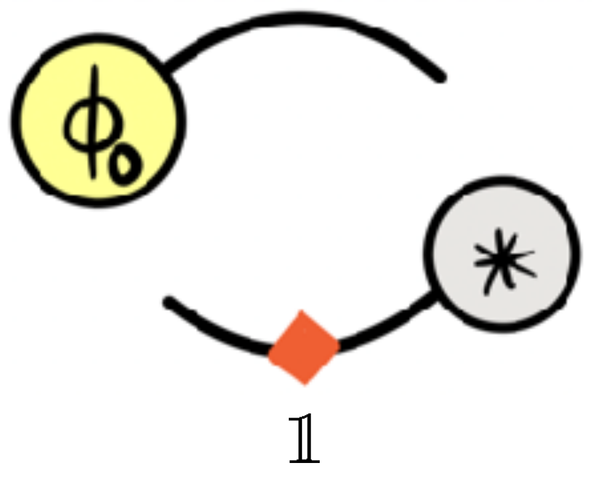
Numerical implementation of correlators

RBC-UKQCD work (2023)

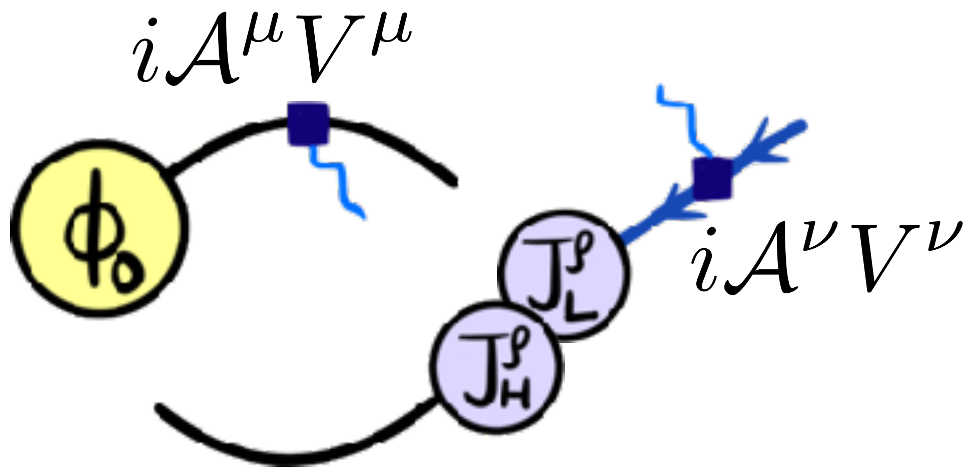
https://github.com/paboyle/Grid
https://github.com/aportelli/Hadrons







- Correlators created using **sequential propagators**
- $N_T = 96$ Coulomb **gauge-fixed wall sources** per configuration
- **Muon momentum** $\mathbf{p}_\ell \propto \{1,1,1\}$ fixed by energy conservation & injected via twisted boundary conditions
- **Muon propagator** evaluated for various source-sink separations
- **Photon fields** sampled from Gaussian distribution (QED_L)
- **Electromagnetic current:** renormalised local vector current



A general comparison of the calculations

	RBC/UKQCD	RM123+Soton
physical masses	✓ physical point simulations	extrapolation needed
chiral symmetry	✓ at finite lattice spacing	recovered in the continuum
fermionic action	Domain Wall	Twisted Mass
continuum limit	single lattice spacing	✓ continuum limit (3)
infinite volume limit	single volume	✓ multiple volumes
QED prescription	$\mathrm{QED_L}$	$\mathrm{QED_L}$
sea effects	electro-quenching	electro-quenching
(∗) IB scheme	BMW [a]	GRS [b]

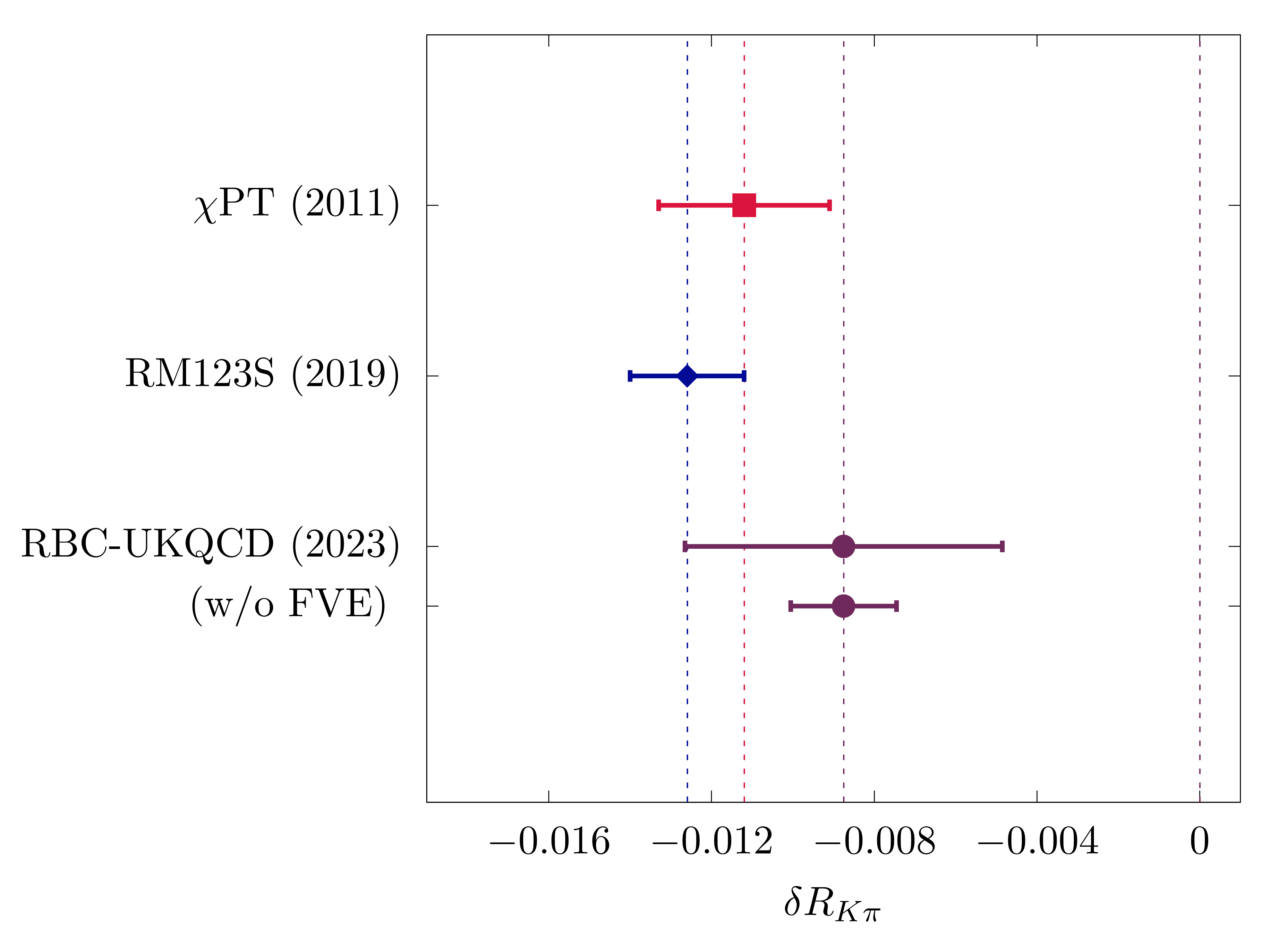
[a] BMW, PRL 111 (2013); BMW, PRL 117 (2016)

[b] Gasser, Rusetsky & Scimemi, EPJC 32 (2003); RM123, PRD 87 (2013)

Results for $\delta R_{K\pi}$

V. Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$



$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$

- Strong evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- **RBC-UKQCD error** dominated by a large systematic uncertainty related to **finite-volume effects**

Prospects for $|V_{us}/V_{ud}|$

An exercise on the error budget

$$\left|\frac{V_{us}}{V_{ud}}\right|^2 = \left[\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})}\frac{M^3_{K^+}}{M^3_{\pi^+}}\frac{(M^2_{K^+}-M^2_{\mu^+})^2}{(M^2_{\pi^+}-M^2_{\mu^+})^2}\right]_{\text{exp}} \cdot \left[\frac{f_{K,0}}{f_{\pi,0}}\right]^2 (1+\delta R_{K\pi})$$

- Using our new result $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
FLAG21 2+1 average	1.1930 (33)	$0.23154\ (28)_{\text{exp}}\,(15)_{\delta R}\,(45)_{\delta R,\text{vol.}}\,(65)_{f_P}$

- Using RM123S result $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
FLAG19 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\text{exp}}\,(17)_{\delta R}\,(35)_{f_P}$

Origin of the large systematic in RBC-UKQCD (2023)

- Main reason: calculation performed on a **single volume** ($m_\pi L \simeq 3.9$)
  - no $L \to \infty$ extrapolation

- **Partial knowledge of** finite-volume scaling of virtual decay rate in $\mathrm{QED_L}$

$$\Gamma_0(L) = \Gamma_0^{\rm tree}\left\{1 + 2\,\frac{\alpha}{4\pi}\,Y(L)\right\}$$

V. Lubicz et al., PRD **95** (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

$$Y(L) - Y(\infty) = Y_{\log}(L) + Y_0 + \frac{1}{m_P L}\,Y_1 + \frac{1}{(m_P L)^2}\,Y_2 + \frac{1}{(m_P L)^3}\,Y_3^{\rm pt} + \frac{1}{(m_P L)^3}\,Y_3^{\rm SD} + \mathrm{O}(1/L^4) + \mathrm{O}(e^{-\alpha L})$$

$m_\pi L \approx 3.9$: ≈ -3.96 | ≈ -2.24 | ≈ 3.37 | currently unknown

Possible way forward?

Finite volume effects produce large **systematic uncertainty**

$$\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$$

repeat the calculation on **multiple volumes** & take **infinite volume limit**

$$\frac{1}{(m_P L)^3}\left[\textbf{structure-dependent}\right]$$

compute **missing effects** at $\mathcal{O}(1/L^3)$

$$\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)$$

adopt or **develop QED formulations** with **reduced** finite volume effects

Possible way forward?

Finite volume effects produce large **systematic uncertainty**

$$\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$$

repeat the calculation on **multiple volumes** & take **infinite volume limit**

$$\frac{1}{(m_P L)^3}\left[\textbf{structure-dependent}\right]$$

compute **missing effects** at $\mathcal{O}(1/L^3)$

$$\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)$$

adopt or **develop QED formulations** with **reduced** finite volume effects

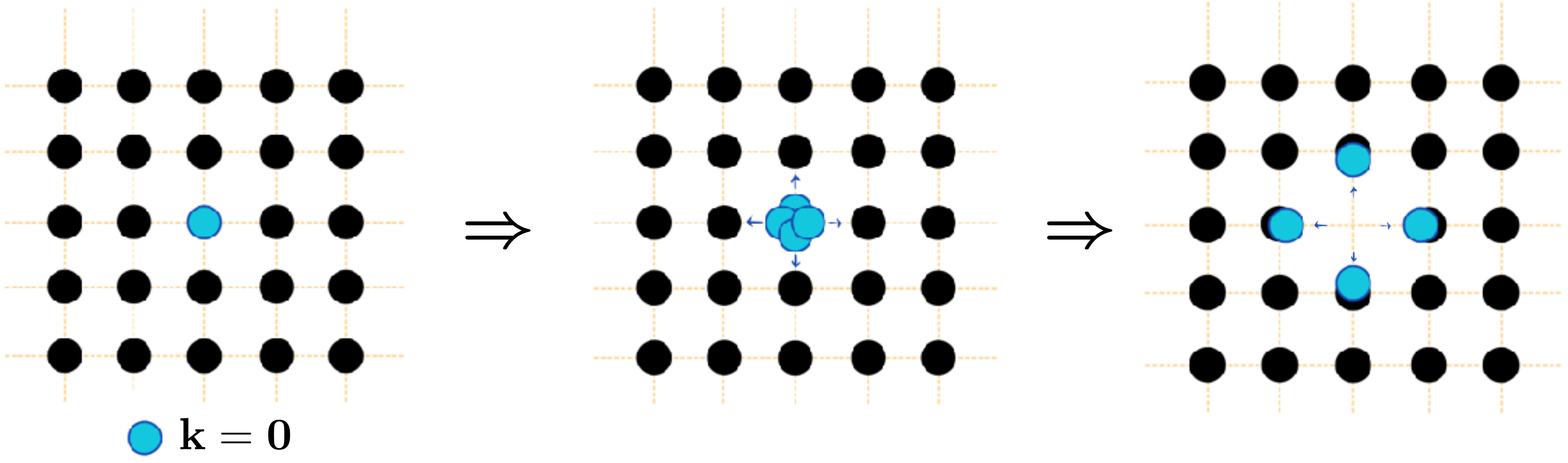
... can we formulate QED on a finite-volume without corrections at $O(1/L^3)$?

QEDr regularization

Special case of "IR-improvement"

Z.Davoudi et al., PRD **99** (2019)

MDC, PoS LATTICE2023 (2024) [2401.07666]



The spatial zero mode is not removed but **redistributed over the neighbouring modes** on a shell of radius $|\mathbf{p}| = \frac{2\pi}{L}|\mathbf{r}| \quad (\mathbf{r} \in \mathbb{Z}^3)$

QEDL : $D_L^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \dfrac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + \mathbf{k}^2}$ $\Rightarrow$ **QEDr :** $D_{\mathbf{p}}^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \dfrac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + \mathbf{k}^2} + \dfrac{\delta_{\mathbf{k}^2,\mathbf{p}^2}}{n(\mathbf{p}^2)} \dfrac{\delta^{\mu\nu}}{k_0^2 + \mathbf{p}^2}$

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}}\,\frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}} \frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}} \frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

universal terms fixed by Ward identities

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P} \Delta_{\mathbf{k}} \frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

$$\Delta m_P(L) = \frac{e^2}{4m_P} \left[c_2(\boldsymbol{\theta}) \frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta}) \frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta}) \frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty} \frac{(2\pi)^{\ell+1}}{L^{4+\ell}} \frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!} \mathcal{M}^{(\ell+2)}(0) \right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n} \in \Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n} \right) \frac{1}{|\mathbf{n}|^s}$$

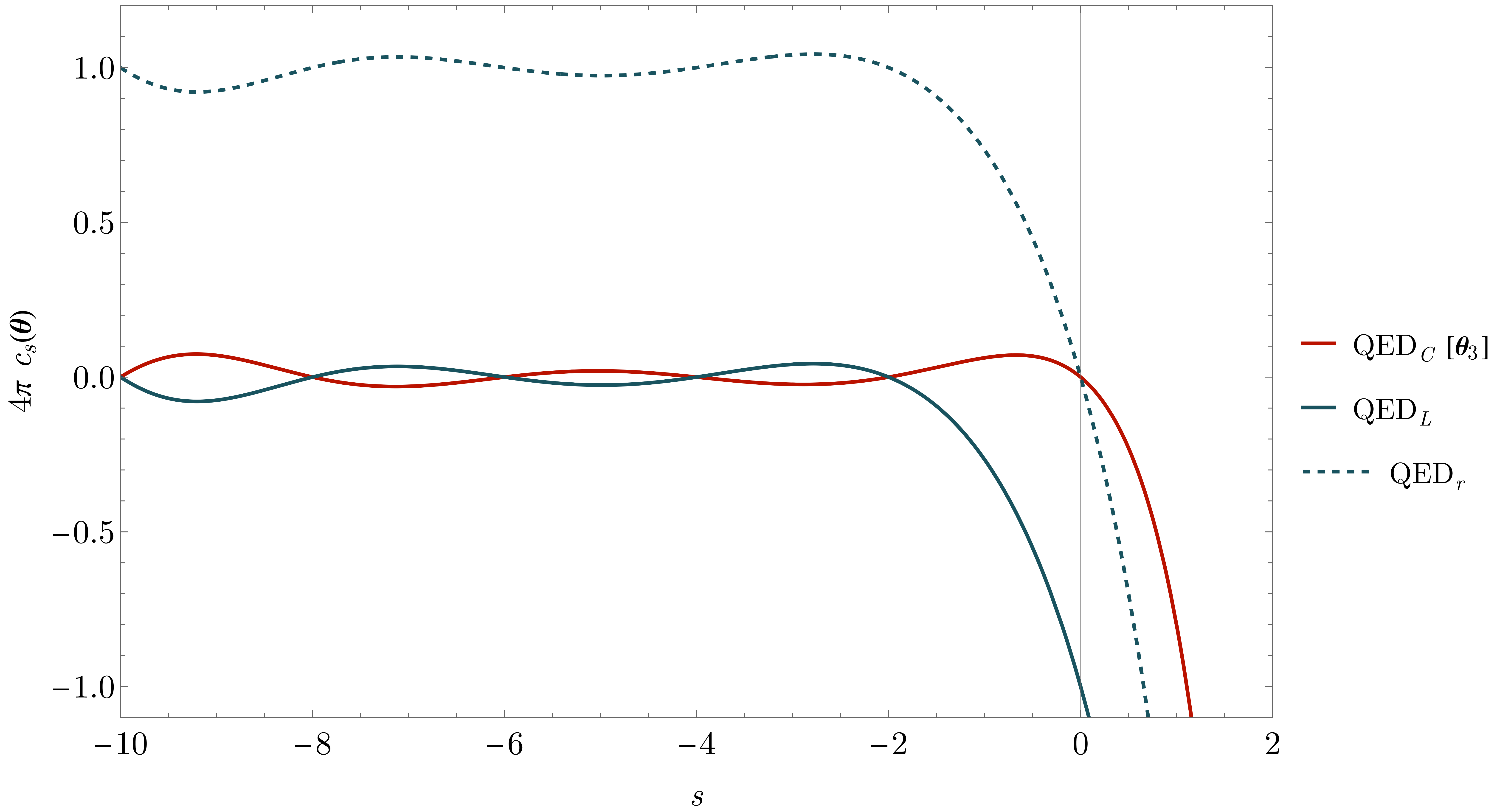
universal terms fixed by Ward identities — structure + multi-particle dependence

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$



QED finite-volume effects

Leptonic decay amplitude

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

$$\Delta Y_P(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

universal

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

QED finite-volume effects

Leptonic decay amplitude

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

$$\Delta Y_P(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

universal

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

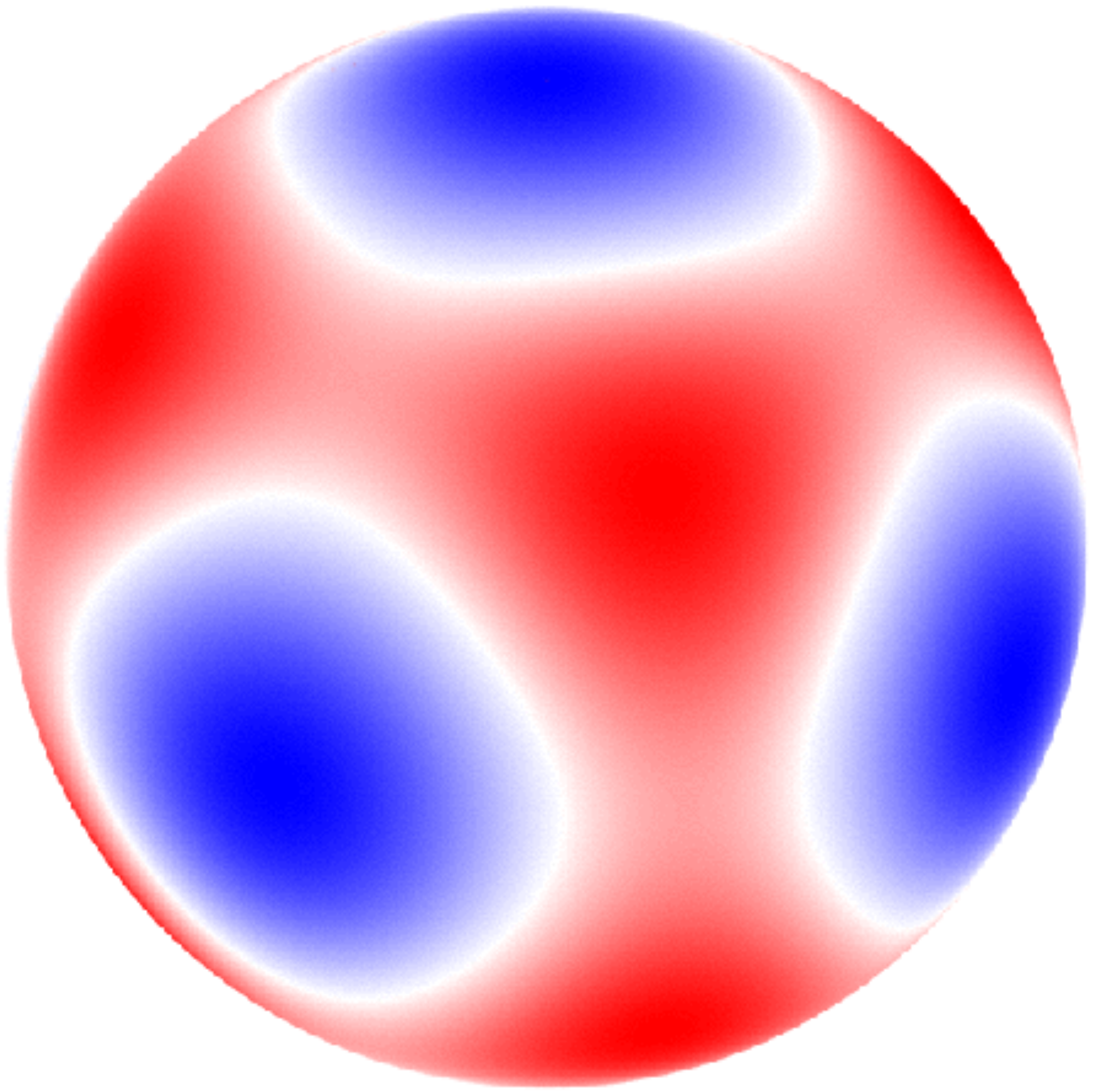
can $\mathrm{QED_r}$ help removing this term?

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

Velocity-dependent coefficients in QED_r

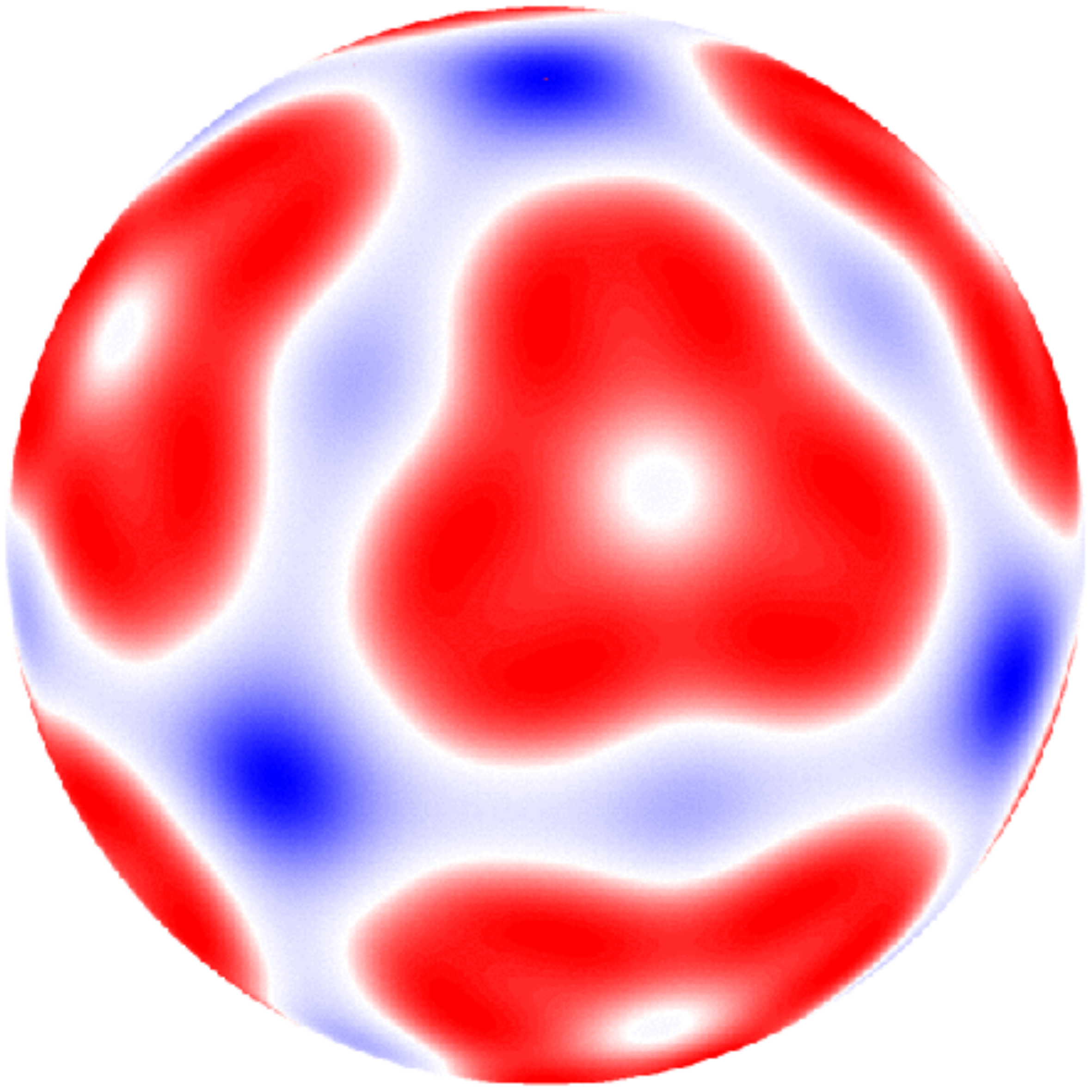
$|v| = 0.40$



$\max \bar{c}_0(\mathbf{v}) = 0.0171$

$\min \bar{c}_0(\mathbf{v}) = -0.0114$

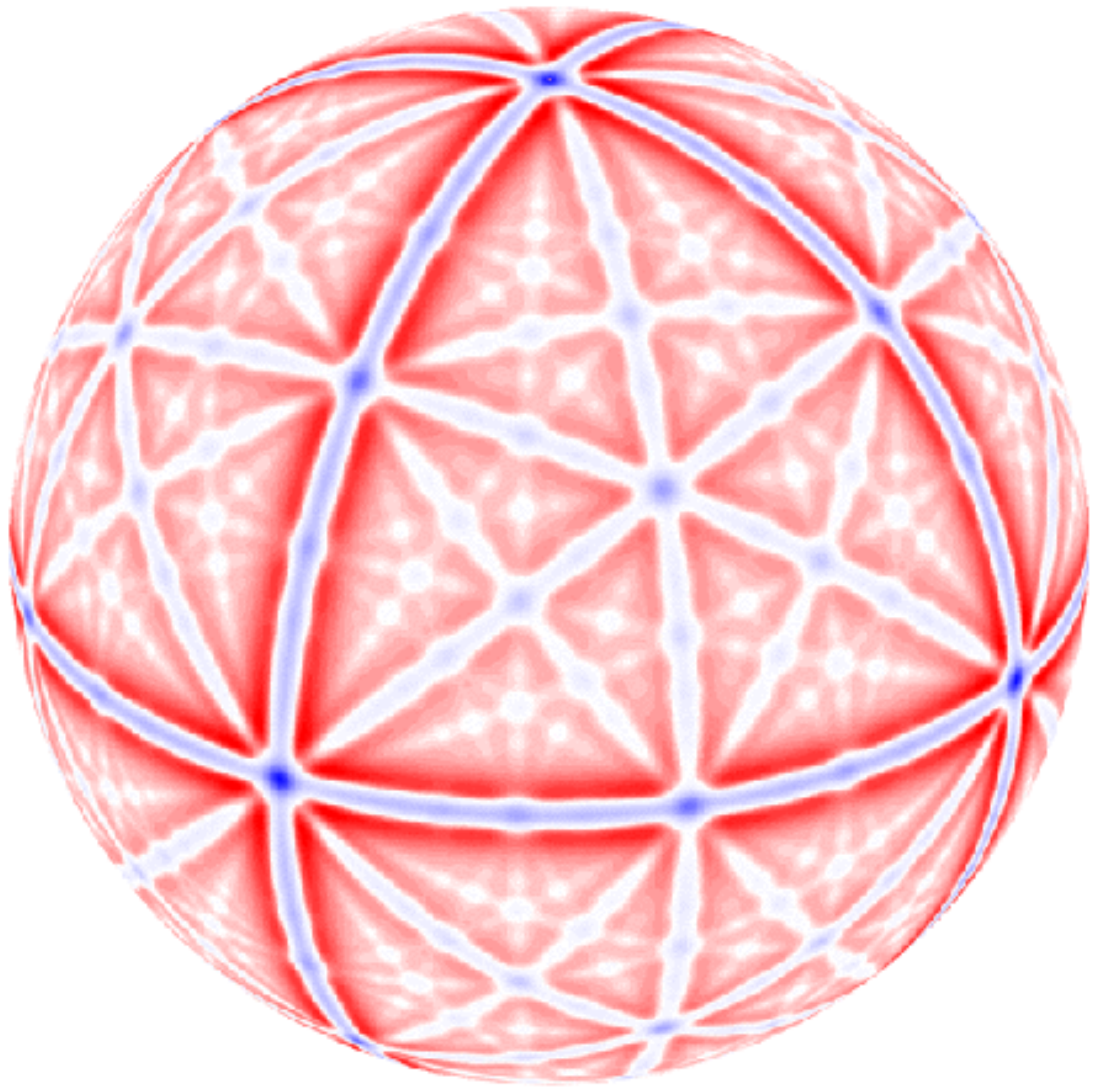
$|v| = 0.95$



$\max \bar{c}_0(\mathbf{v}) = 15.2832$

$\min \bar{c}_0(\mathbf{v}) = -2.8258$

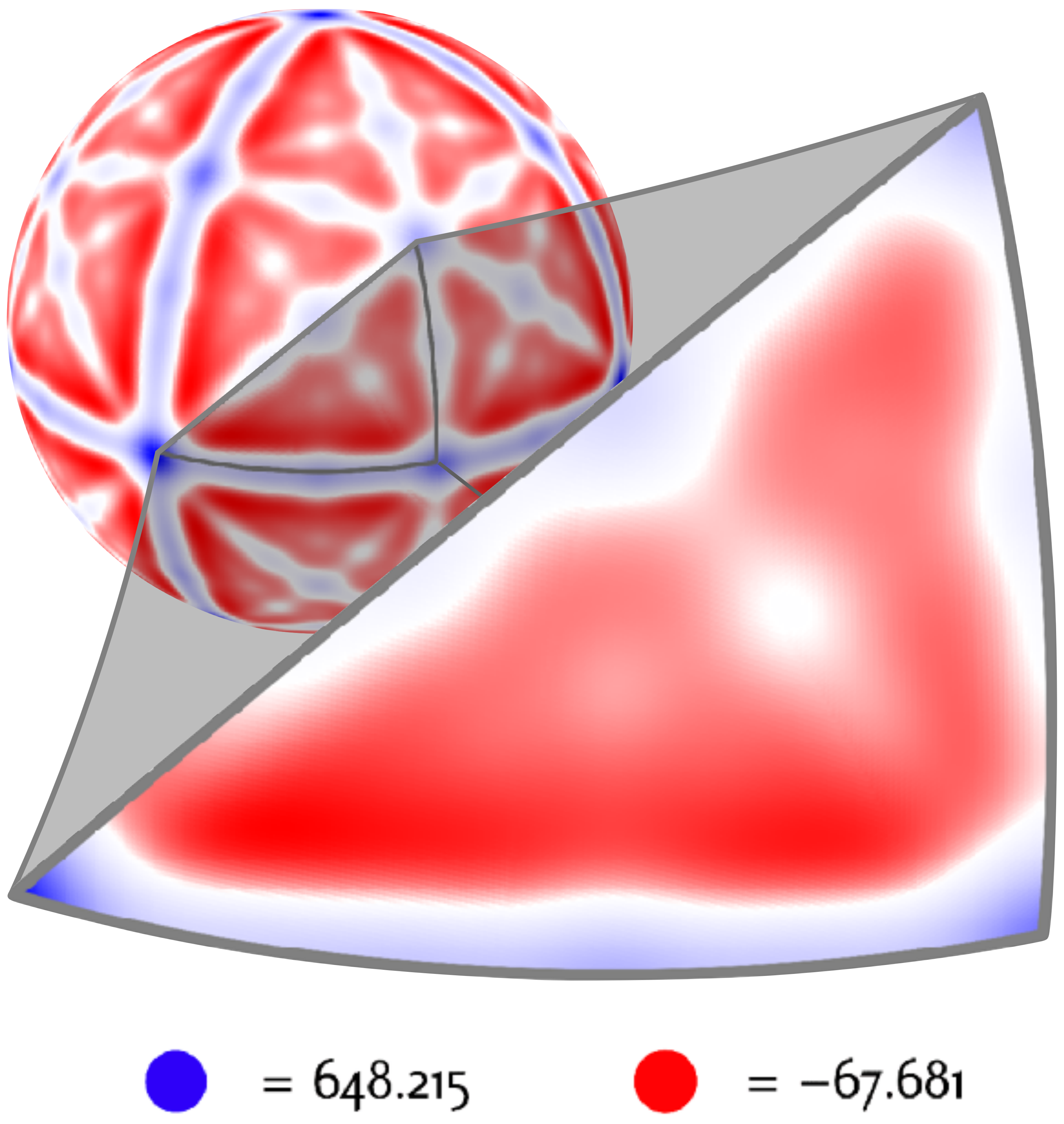
$|v| = 0.999$

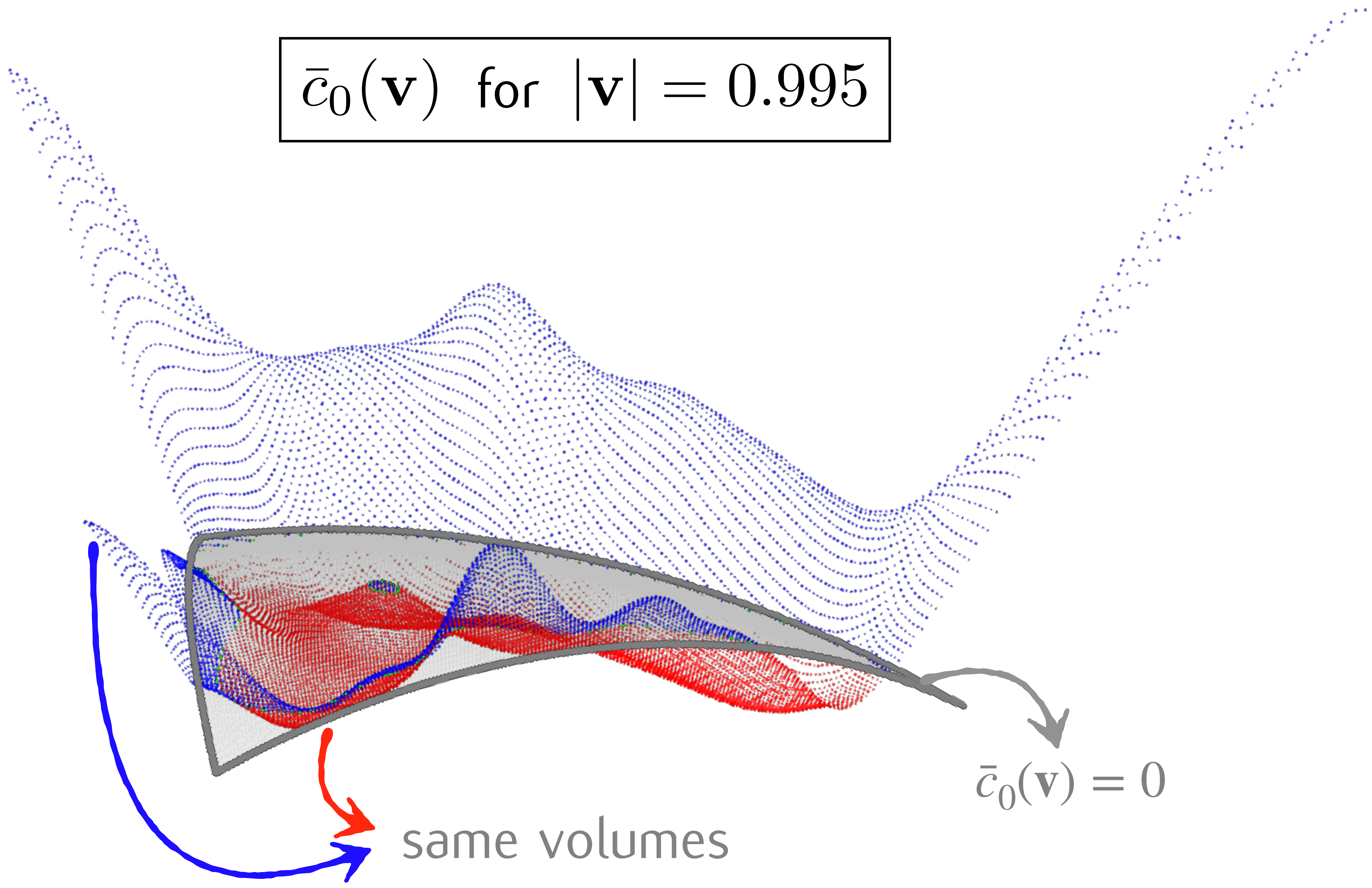


$\max \bar{c}_0(\mathbf{v}) = 9002.2317$

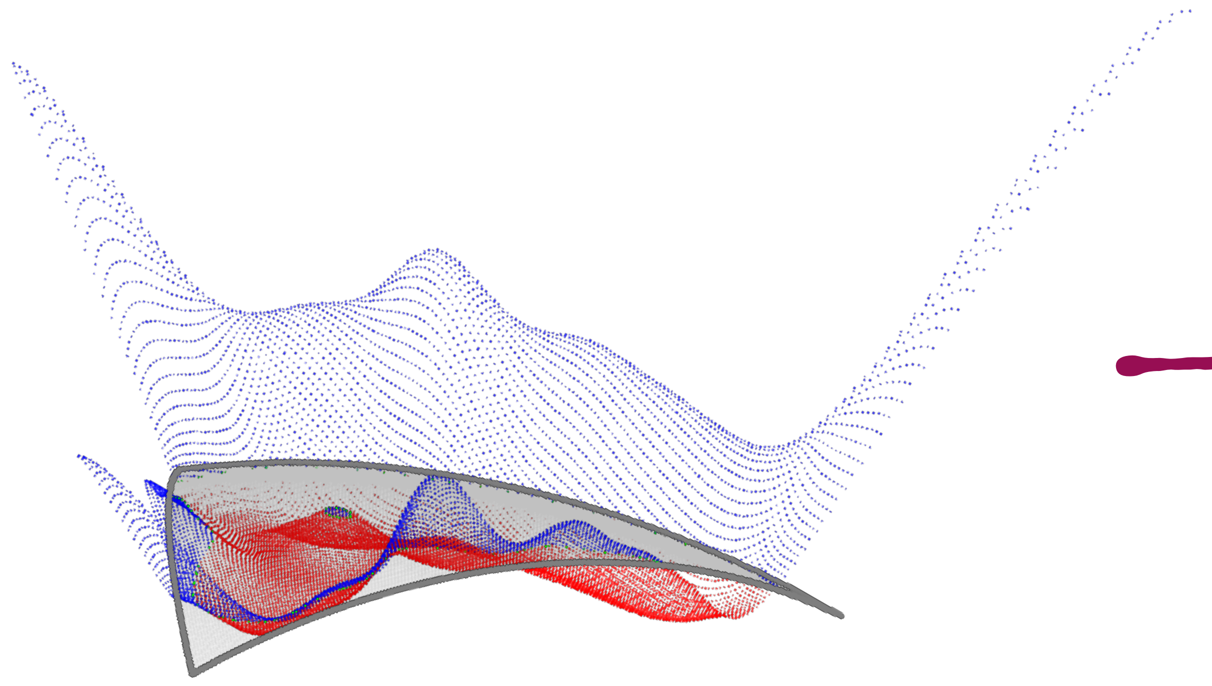
$\min \bar{c}_0(\mathbf{v}) = -807.4018$

Velocity-dependent coefficients in QED$_r$

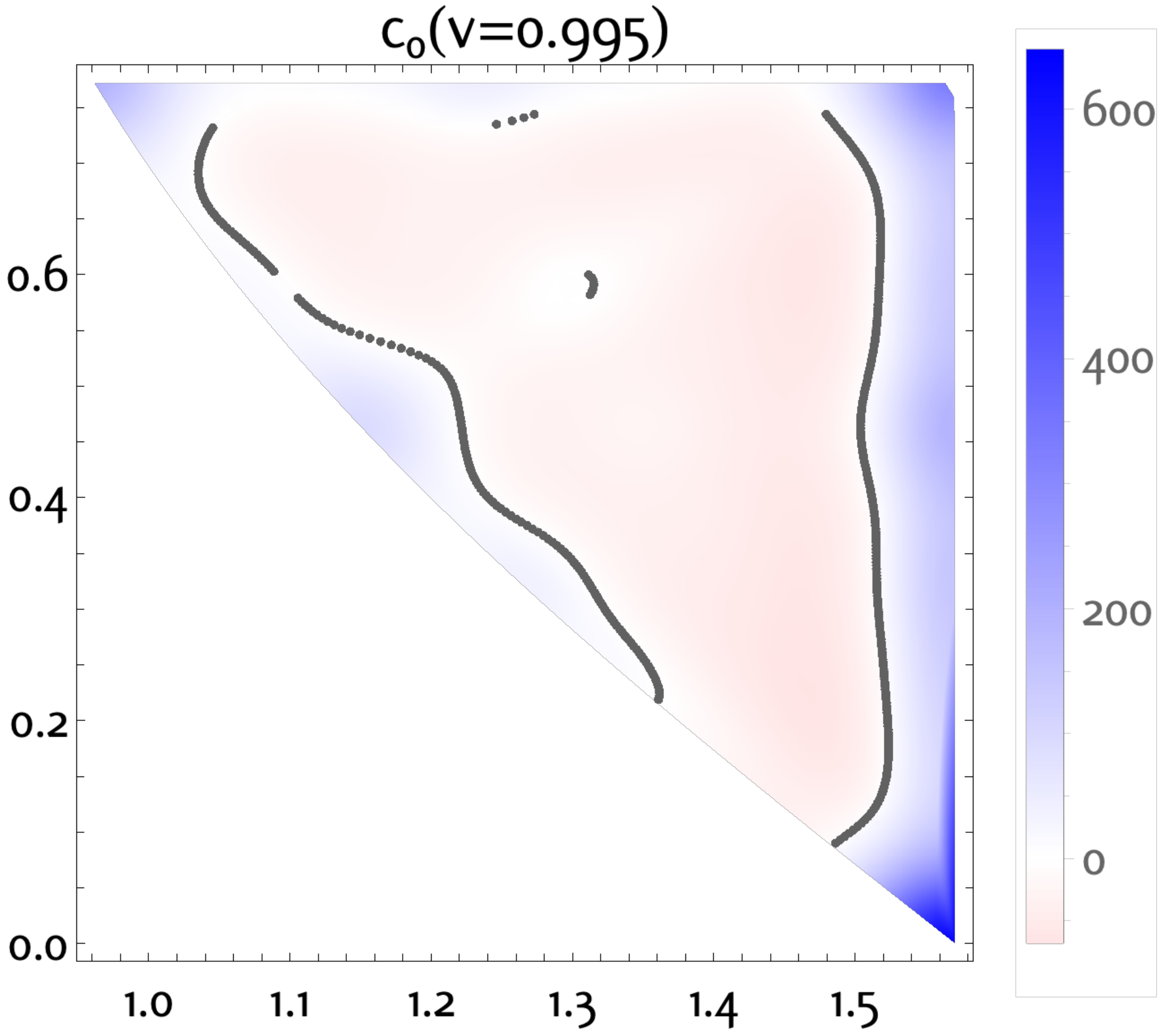




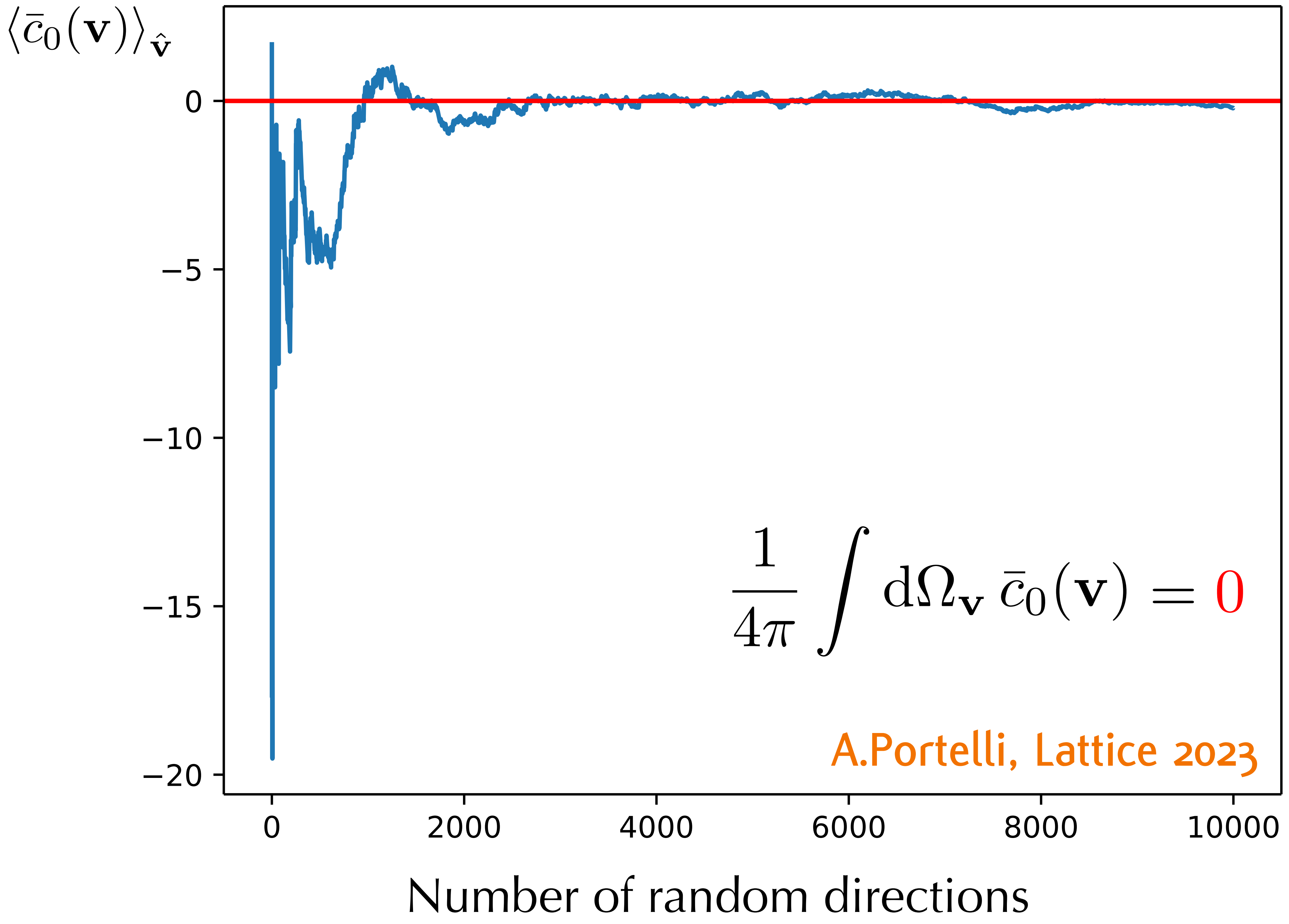
Velocity-dependent coefficients in QED_r



"magic angles"



Stochastic direction average

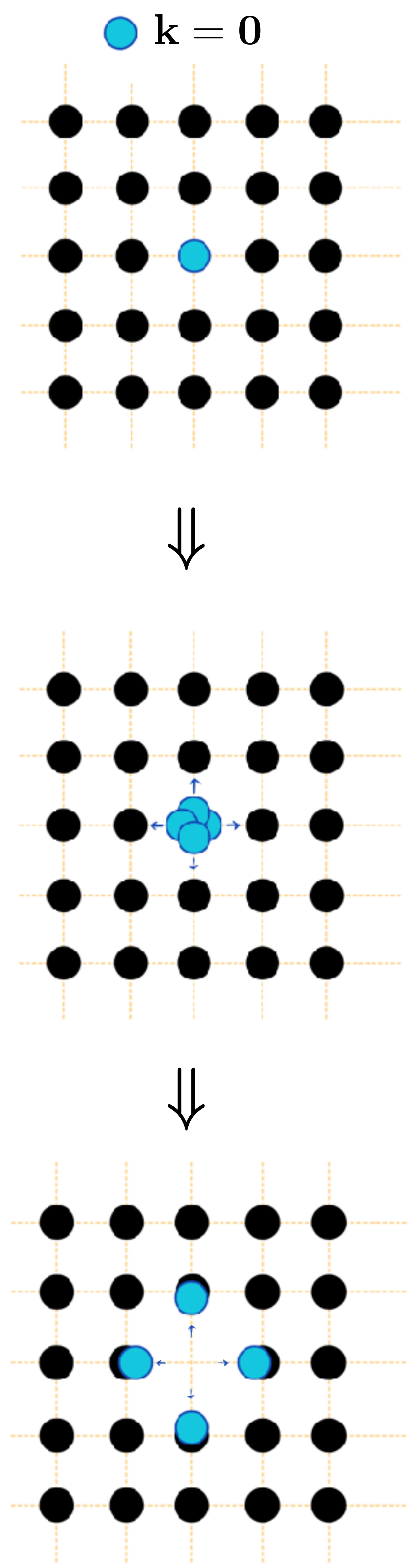


QED$_r$ summary

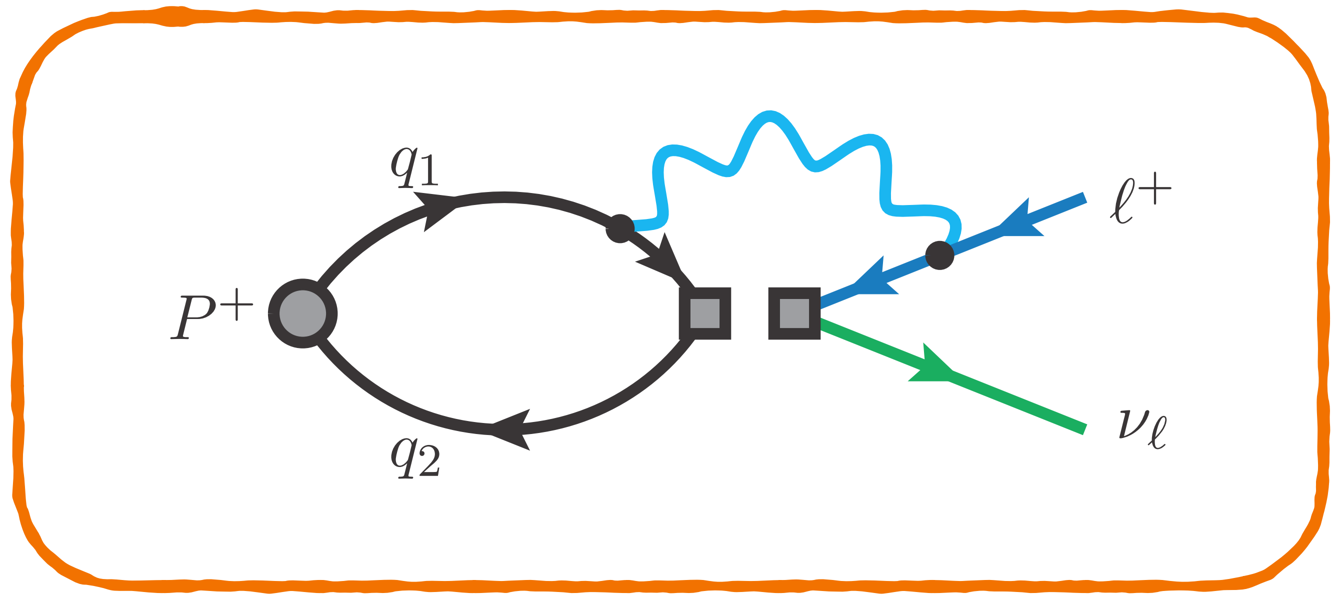
- **Infrared improvement of QED$_L$:** redistribution of the spatial zero-mode
- **Potentially free from** (problematic) **$O(1/L^3)$ effects**:
  - absent by construction for **zero-velocity** systems (masses, g-2 HVP, ...)
  - improvement less straightforward for **velocity-dependent observables**, due to non-trivial collinear divergences

Ongoing **numerical calculations** of QED$_r$

- finite-volume study on **new** (unphysical) **ensembles** with **4 volumes**
- investigation of π, $\boldsymbol{K}$, $\boldsymbol{D}$ **and** $\boldsymbol{D_s}$ **decays** at physical point

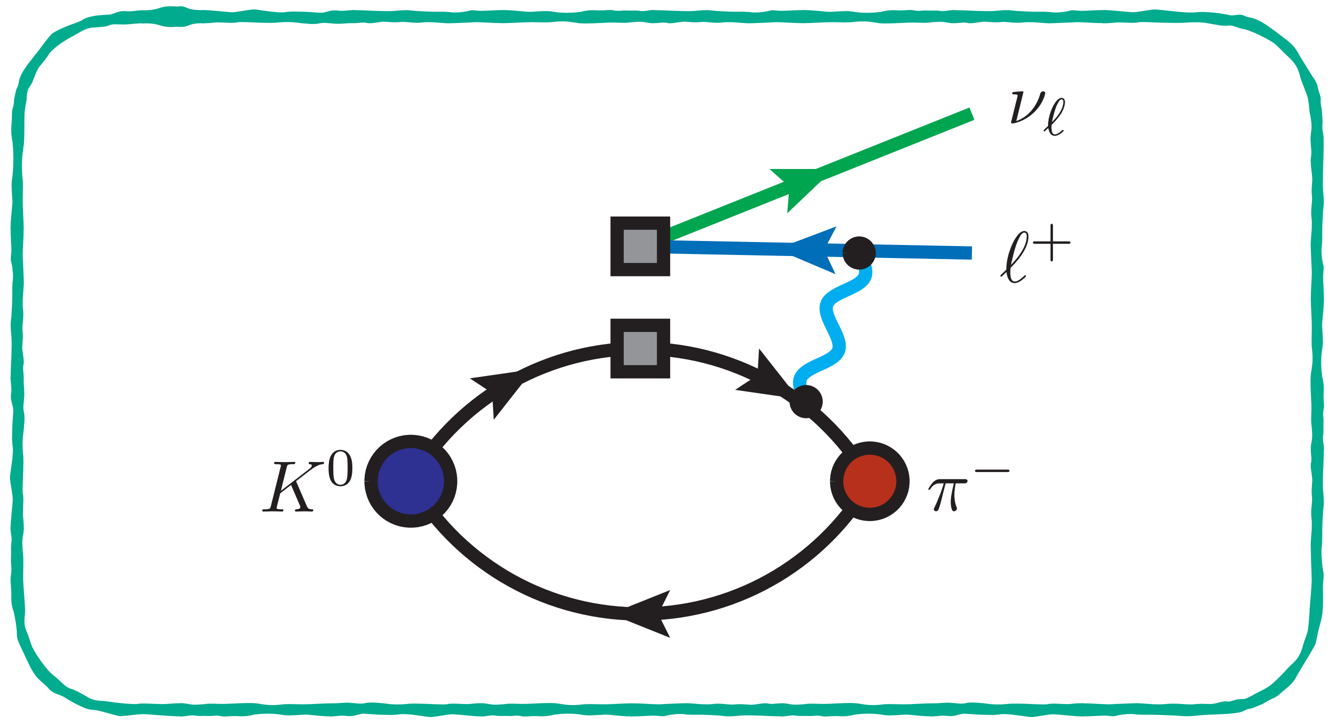


Future directions



leptonic decays of heavy pseudoscalar mesons

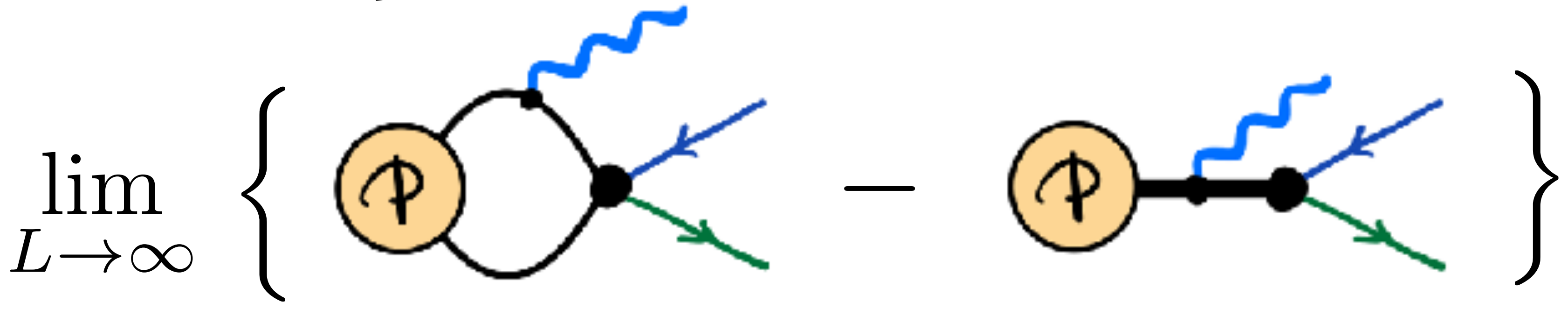
semileptonic kaon decays



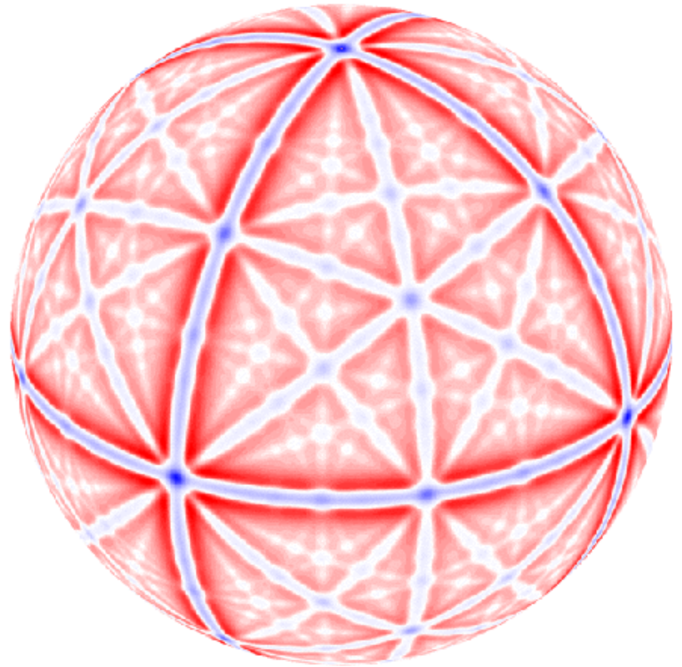
Leptonic decays of heavy mesons

- In principle, the **same method can be applied** to decays of heavy meson, e.g. $D_{(s)}$ or $B_{(s)}$
- Besides numerical cost, two main complications arise

1. Contribution of **structure-dependent** real photon emission is relevant and needs to be computed non-perturbatively



2. Lepton velocities are **ultra-relativistic**, yielding highly non-trivial angular dependence in **finite-volume effects:**

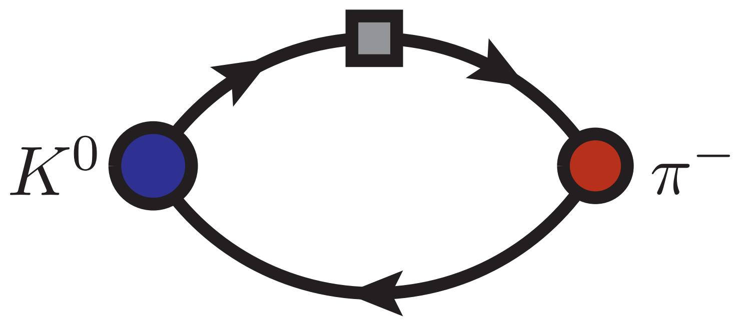


$D^+ \to \mu^+\nu_\mu$ $\quad |\mathbf{v}| \simeq 0.994$

$B^+ \to \mu^+\nu_\mu$ $\quad |\mathbf{v}| \simeq 0.999$

QED corrections to semileptonic decays

- Without QED corrections:



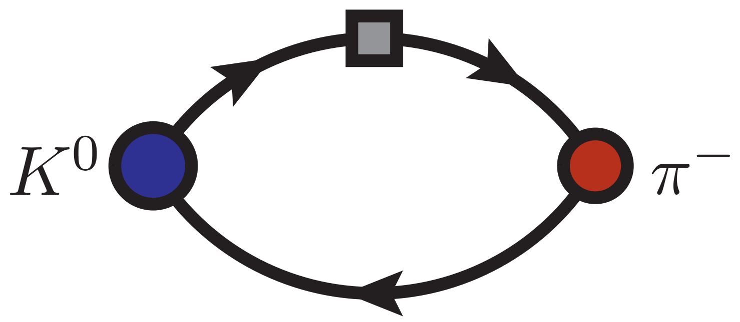
$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = \boldsymbol{f_+(q^2)}\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + \boldsymbol{f_0(q^2)}\, \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\ q^2 = (p_K - p_\pi)^2$

$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2|V_{us}|^2\left[a_1(q^2, s_{\pi\ell})\, |\boldsymbol{f_+(q^2)}|^2 + a_2(q^2, s_{\pi\ell})\, \boldsymbol{f_+(q^2)f_0(q^2)} + a_3(q^2, s_{\pi\ell})\, |\boldsymbol{f_0(q^2)}|^2\right]$$

QED corrections to semileptonic decays

- Without QED corrections:



$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = \boldsymbol{f_+(q^2)}\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + \boldsymbol{f_0(q^2)}\,\frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

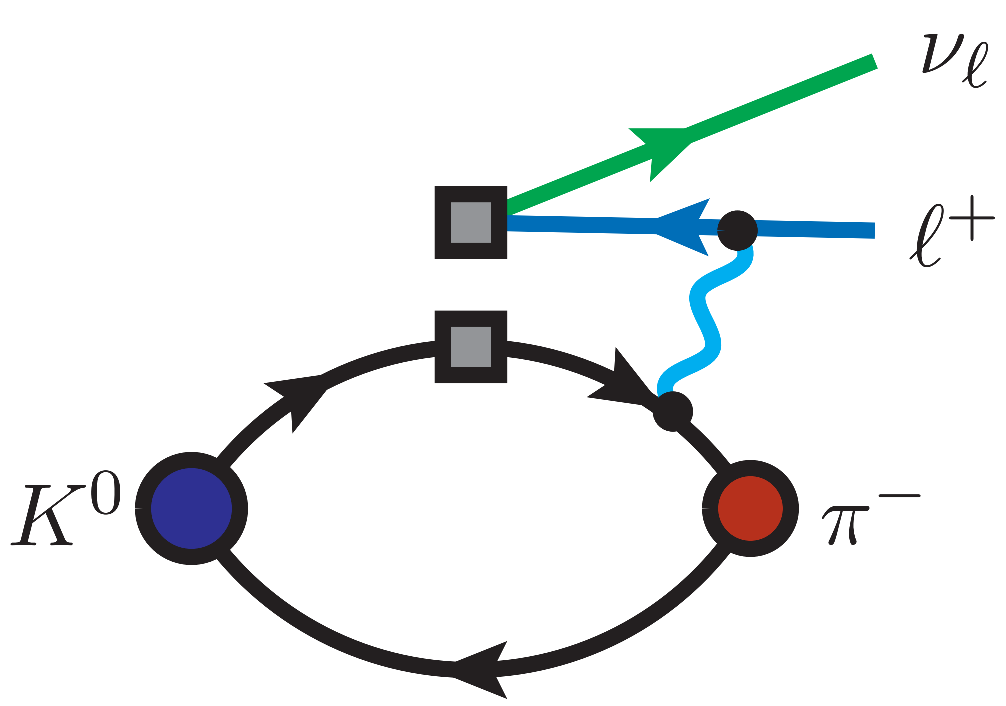
An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\ q^2 = (p_K - p_\pi)^2$

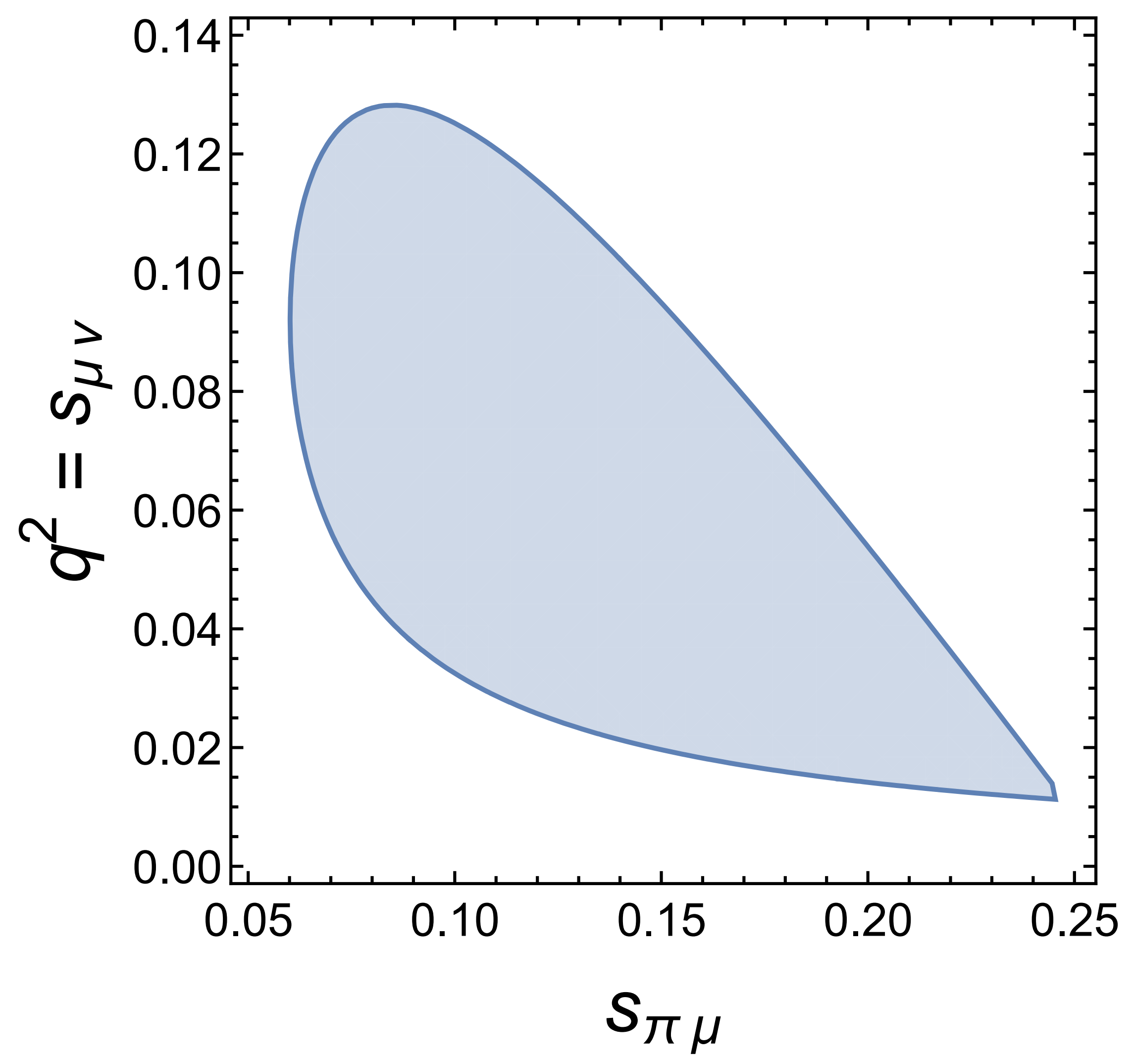
$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2|V_{us}|^2\left[a_1(q^2, s_{\pi\ell})\,|\boldsymbol{f_+(q^2)}|^2 + a_2(q^2, s_{\pi\ell})\,\boldsymbol{f_+(q^2)f_0(q^2)} + a_3(q^2, s_{\pi\ell})\,|\boldsymbol{f_0(q^2)}|^2\right]$$

- Including QED, we can treat IR divergences using the **RM123S method**: C.Sachrajda et al., [1910.07342]

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} - \frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right] + \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} + \frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right]$$

QED corrections to semileptonic decays





Although the RM123+Soton method could in principle be applied, additional **difficulties** arise compared to **leptonic decays**:

- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate on shell states are lighter than external ones
- evaluating **finite-volume corrections** potentially more complicated

A proper **finite-volume formalism is still missing**, but solutions are under study by different groups.

Conclusions and outlooks

- Current **tensions** in CKM unitarity require a combined effort of **theory** and **experiments**
- **Two lattice calculations** of IB and QED corrections to **light-meson leptonic decay rates**
- **Finite volume QED effects** have to be investigated to reach high precision on $|V_{us}/V_{ud}|$

- **QED_r regularisation** could help removing unknown $1/L^3$ structure-dependent contributions
- Extension of the calculation to **multiple lattice spacings and volumes** is under consideration
- Next important step: going **beyond electro-quenched approximation**

- Calculations of **heavy meson leptonic decays** are possible and investigations are ongoing
- A dream for the future: tackle **semileptonic kaon decays**

Thank you

This work has received funding from the European Union’s Horizon Europe research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101108006