Towards computational foundations of generalized symmetries

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1. Introduction
   - Symmetry and 't Hooft anomaly
   - Generalized global symmetry

2. Review on topology of lattice gauge fields [Lüscher]
   - Principal fiber bundle
   - Construction of topology of lattice gauge fields

3. Fractionality of topology in lattice $SU(N)/\mathbb{Z}_N$ gauge theory [Abe–OM–Suzuki, Abe–OM–Onoda–Suzuki–Tanizaki]

4. Axial $U(1)$ non-invertible symmetry [Honda–Onoda–OM–Suzuki]
   - $U(1) \times U(1)'$ lattice gauge theory with Ginsparg–Wilson fermion
     - Construction of chiral fermion integration measure [Lüscher]
   - Lattice realization of axial $U(1)$ non-invertible symmetry

5. Summary

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Symmetry and 't Hooft anomaly matching

- Symmetry: fundamental tool in physics
  - Universal applications to high energy physics, condensed matter physics and mathematics
- Noether theorem and conservation law

$$\text{Sym} : \phi \mapsto \phi'; S(\phi) = S(\phi'), \qquad \partial_\mu j^\mu = 0$$

$$\text{Charge } Q \equiv \int j^0 d^{D-1}x$$

- 't Hooft anomaly matching for strongly coupled theories ['79]
  - Assume global symmetry G in system
  - Introduce background gauge field A assoc. G (by gauging G)

$$\mathcal{Z}[A] = \int \mathcal{D}\{\phi\}\, e^{-S(\{\phi\},A)} \stackrel{?}{=} \mathcal{Z}[A^g] = \cdots = e^{\mathcal{A}[A,g]}\mathcal{Z}[A]$$

$e^{\mathcal{A}} \neq 1 \xrightarrow{\text{Anomalous}}$ 't Hooft anomaly which is invariant at any energy scale (renormalization group inv.)

  - Restriction on low-energy dynamics: SSB, phase structure, SPT

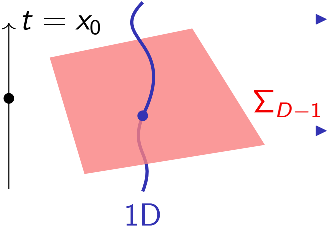
Recent generalization of symmetry

- Generalized global symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
  - Coupled with topological field theory
    - ⋆ Changing topological structure without changing local dynamics [Kapustin–Seiberg '14]:
    - ⋆ Nontrivial information by using generalized 't Hooft anomaly matching [E.g., OM–Wada–Yamaguchi '23]
- Basic property: fractionality of topological charge
  - Usually, topo. charge $Q \sim \int F\tilde{F} \in \mathbb{Z}$ under topo. sectors
    - ⋆ θ term: θQ (strong CP problem, axion physics, sign problem)
  - Discrete higher-form sym $\to$ background gauge field B: $Q \sim \int B\tilde{B} \in \frac{1}{N}\mathbb{Z}$, 't Hooft anomaly $\mathcal{Z}_{\theta+2\pi}[B] = e^{-2\pi i Q}\mathcal{Z}_\theta[B]$ [Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]

$\overset{\text{global desc.}}{\Rightarrow}$ 't Hooft twisted boundary condition $Q \sim \int F\tilde{F} \in \frac{1}{N}\mathbb{Z}$ [van Baal '82] cf. [Edwards-Heller-Narayanan, de Forcrand-Jahn, Fodor-Holland-Kuti-Nógrádi-Schroeder, Kitano-Suyama-Yamada, Itou, . . .]

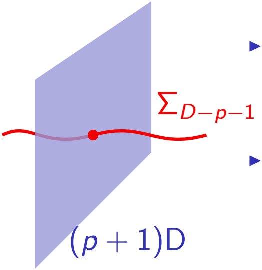
Generalization: higher-form symmetry

- (0-form) Symmetry



  - Charge (codim 1)
  $$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}$$
  - Symmetry operator
  $$U_\alpha(\Sigma_{D-1}) = e^{i\alpha Q}$$
  Topological under deformation of Σ_{D-1}

- Higher-form symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]



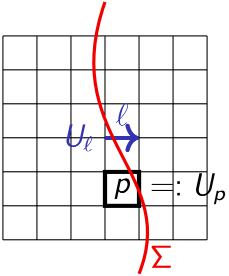
  - p-form symmetry $G^{[p]}$ (codim $p+1$)
  $$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$
  - Transforming a "loop" operator $W(C)$
  $$W(C) \mapsto U(\Sigma)W(C)$$
  $$= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \#$$

Center symmetry in YM theory

- Lattice $SU(N)$ YM theory
  - link variable $U_\ell \in SU(N)$
- Center symmetry: $\mathbb{Z}_N^{[1]}$

  $e^{\frac{2\pi i}{N}k} \in \mathbb{Z}_N \subset SU(N); \quad U_\ell \mapsto e^{\frac{2\pi i}{N}k\#(\Sigma,\ell)} U_\ell$

  Intersection # of Σ & link ℓ; $U_p \mapsto U_p$

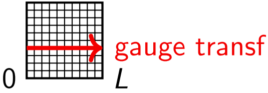


- Gauging the center symmetry

$$S \sim \sum \operatorname{Tr} e^{-\frac{2\pi i}{N} B_p} U_p \qquad B_p\text{: 2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - Recall 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

Our enterprise: Fully lattice regularized framework

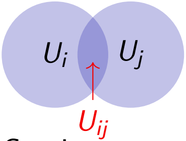
- Wise but *not transparent* understanding
  - Topological objects from lattice viewpoint as center sym
  - Formal discussion in continuum theory
    - ⋆ $\mathbb{Z}_N^{[q]}$ gauge field: $U(1)$ field $B^{(q)}$
      with constraint $NB^{(q)} = dB^{(q-1)}$ from charge-N Higgs
  - $Q \sim \frac{1}{N}\int B \wedge B? \xrightarrow[\text{cohomology op}]{\text{Swapping } \wedge \text{ w/}} \frac{1}{N}\int P_2(B) \sim -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N}$
    - ⋆ Global nature described by Čech cohomology (discrete group!)
  - Indicating mixed 't Hooft anomaly with chiral sym/θ-periodicity

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q}\mathcal{Z}_\theta[B_p] \quad \text{not } 2\pi \text{ periodic}$$

- Fully regularized framework: Lattice regularization
  - Lattice construction of fiber bundle and $Q \in \mathbb{Z}$ [Lüscher '84]
  - $Q \in \frac{1}{N}\mathbb{Z}$ [Abe–OM–Suzuki, Abe–OM–Onoda–Suzuki–Tanizaki]
    - ⋆ Higher-group [Kan–OM–Nagoya–Wada, Abe–OM–Onoda],
      Magnetic operators [Abe–OM–Onoda–Suzuki–Tanizaki]
- Application to non-invertible symmetry

Principal fiber bundle

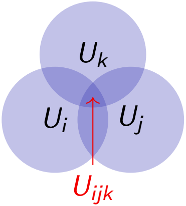
- Recall Dirac's discussion
  - Gauge fields cannot be defined globally in spacetime
  - Defined on each subspace: northern/southern hemisphere
- Manifold (spacetime) X; open covering (patches) $\{U_i\}$
  - Gauge group G, gauge field a_i on U_i
  - Relation between a_i and a_j?



Gauge transformation:

$$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} d g_{ij} \quad \text{at } U_{ij}$$

  - Consistency condition for transition function g_{ij}:

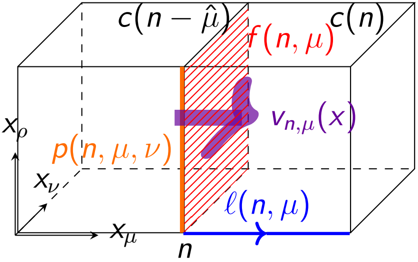


$g_{ii} = \mathrm{id}$, $g_{ji} = g_{ij}^{-1}$

Cocycle condition: $\quad g_{ij} g_{jk} g_{ki} = 1 \quad$ at U_{ijk}

Bundle structure on lattice?

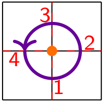
- No continuity for lattice fields?
  - Any configuration can be deformed continuously to others
  - We can observe topological structure even on lattice by Lüscher
- Setup/strategy: Lattice Λ divides X into 4D hypercubes



edge ↓

4D cell $c(n)$
3D face $f(n,\mu)$
2D plaquette $p(n,\mu,\nu)$
1D link $\ell(n,\mu)$

1. Regard $\{c(n)\}$ as patches
2. Define transition function $v_{n,\mu}(x)$ at $f(n,\mu)$ from data as U_ℓ
   - Difficult to define it at $x \neq n$ s.t. cocycle condition is kept intact

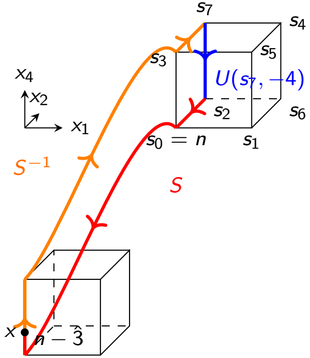


$v_1 v_2 v_3^{-1} v_4^{-1} = 1$
at $x \in p(n)$

Construction of topo. sectors on lattice [Lüscher]

- *Parallel transporter* (interpolation):
  $U_\ell \to v_f(n) \to v_f(x) \Leftarrow$ Cocycle

  $$v_{n,\mu}(n) = U(n-\hat{\mu},\mu)$$
  $$v_{n,\mu}(x) \equiv S^{n-\hat{\mu}}_{n,\mu}(x)^{-1} v_{n,\mu}(n) S^{n}_{n,\mu}(x)$$



- Topo. sectors on lattice so that $[Q$ in terms of $v_f(x)] \in \mathbb{Z}$

$$Q = \sum_{n\in\Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} \Big\{ 3 \int_{p_{n,\mu\nu}} d^2x \ \mathrm{Tr}\left[\left(v_{n,\mu}\partial_\rho v_{n,\mu}^{-1}\right) \left(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}\right)\right]$$
$$+ \int_{f_{n,\mu}} d^3x \ \mathrm{Tr}\left[\left(v_{n,\mu}^{-1}\partial_\nu v_{n,\mu}\right) \left(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu}\right) \left(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu}\right)\right] \Big\}$$

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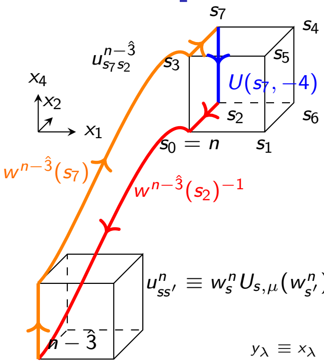

$$u^n_{ss'} \equiv w^n_s U_{s,\mu} (w^n_{s'})^{-1}$$

$$y_\lambda \equiv x_\lambda - n_\lambda$$

$$w^n(x) = U^{y_4}_{n,4} U^{y_3}_{n+y_4\hat{4},3} U^{y_2}_{n+y_4\hat{4}+y_3\hat{3},2} U^{y_1}_{n+y_4\hat{4}+y_3\hat{3}+y_2\hat{2},1}$$

$$f^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_7} u^m_{s_7 s_2} u^m_{s_2 s_0})^{y_\gamma} u^m_{s_0 s_2} (u^m_{s_2 s_7})^{y_\gamma},$$

$$g^m_{n,\mu}(x_\gamma) = (u^m_{s_5 s_1})^{y_\gamma} (u^m_{s_1 s_5} u^m_{s_5 s_4} u^m_{s_4 s_6} u^m_{s_6 s_1})^{y_\gamma} u^m_{s_1 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$h^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_5} u^m_{s_5 s_1} u^m_{s_1 s_0})^{y_\gamma} u^m_{s_0 s_1} (u^m_{s_1 s_5})^{y_\gamma},$$

$$k^m_{n,\mu}(x_\gamma) = (u^m_{s_7 s_2})^{y_\gamma} (u^m_{s_2 s_7} u^m_{s_7 s_4} u^m_{s_4 s_6} u^m_{s_6 s_2})^{y_\gamma} u^m_{s_2 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$l^m_{n,\mu}(x_\beta, x_\gamma) = [f^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} [f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} h^m_{n,\mu}(x_\gamma) [g^m_{n,\mu}(x_\gamma)]^{y_\beta},$$

$$S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) = (u^m_{s_0 s_3})^{y_\gamma} [f^m_{n,\mu}(x_\gamma)]^{y_\beta} [l^m_{n,\mu}(x_\beta, x_\gamma)]^{y_\alpha}.$$

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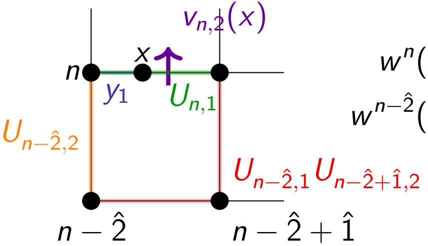
$$+ \int_{f_{n,\mu}} d^3x \, \mathrm{Tr}\left[\left(v^{-1}_{n,\mu}\partial_\nu v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\rho v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\sigma v_{n,\mu}\right)\right] \Big\}$$

Exercise: Lattice 2D $U(1)$ gauge theory

- $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$ [Lüscher '98, Fujiwara et al. '00]



$$w^n(x) = {U_{n,1}}^{y_1}$$

$$w^{n-\hat{2}}(x) = [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2}]^{y_1} {U_{n-\hat{2},2}}^{1-y_1}$$

- Explicit expression of v:

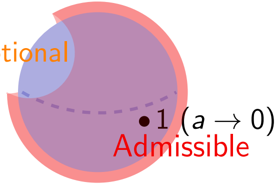
$$v_{n,1}(x) = U_{n-\hat{1},1} \quad v_{n,2}(x) = U_{n-\hat{2},2} [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2} U_{n,1}^{-1} U_{n-\hat{2},2}^{-1}]^{y_1}$$

$$= U_{n-\hat{2},2} \exp[i y_1 F_{12}(n-\hat{2})]$$

  - Field strength: $F_p \equiv \frac{1}{i} \ln U_p$ for $-\pi < F_p \leq \pi$
  - (nD) To ensure Bianchi identity $dF_p = 0$, we should impose $\sup_p |F_p| < \epsilon$, $0 < \epsilon < \frac{\pi}{3}$ →Admissibility condition (see also §6)

Admissibility condition

- In general, admissibility = well-defined-ness of u^y $(0 \leq y \leq 1)$
  - ★ $U(1)$: $F_p = \frac{1}{i} \ln U_p$ for plaquette U_p
  - ★ $S^m_{n,\mu}(x)$ is written in terms of $(u^n_{ss'})^y$ where u is a loop $n \to s \to s' \to n$
  - ▸ E.g., u^y is ill-defined at $u = -1$; ill-def regions separate sectors
- Admissibility condition $\mathrm{tr}(1 - U_p) < \epsilon$ [Lüscher '84]



- ▸ Admissible lattice gauge fields: well-defined conf space $\sim$ disk
- ▸ Exceptional region
  - ★ Topological freezing
  - ★ Monopole as lattice artifact

- Under the admissibility condition, we can prove that $Q \in \mathbb{Z}$; we observe topo. sectors even on lattice!
- How about index theorem for finite a?

$$\mathrm{Index}(D) = \underbrace{-\frac{a}{2} \mathrm{Tr}\, \gamma_5 D_{\mathrm{ov}}}_{\text{Admissibility } \epsilon_{\mathrm{ov}}} = \underbrace{n_+ - n_-}_{\in \mathbb{Z}} \stackrel{?}{=} \frac{1}{32\pi^2} \int_x \varepsilon_{\mu\nu\rho\sigma} \mathrm{tr}[F_{\mu\nu} F_{\rho\sigma}]$$

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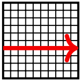
Cocycle condition relaxed by higher-form sym

- Lüscher's construction *Coupled* with higher-form gauge fields
  - $SU(N)$ YM theory coupled with $\mathbb{Z}_N$ 2-form gauge field

$$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N} B_p} U_p \qquad B_p\text{: 2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



0 gauge transf L

$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

- Cocycle condition can take a $\mathbb{Z}_N$ value:

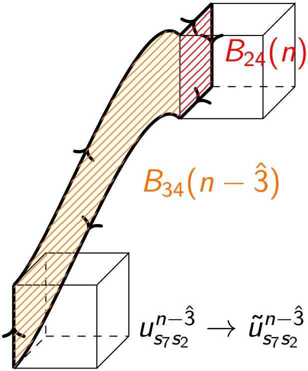
$$\tilde{v}_{n-\hat{\nu},\mu}(x)\tilde{v}_{n,\nu}(x)\tilde{v}_{n,\mu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}$$

  - $\mathbb{Z}_N$ blind matters: adjoint repr.
  - $\mathbb{Z}_N^{[1]}$ gauge inv. if $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_{n-\hat{\mu},\mu}} \tilde{v}_{n,\mu}(x)$
- What is definition of $\tilde{v}_{n,\mu}(x)$?

$\mathbb{Z}_N^{[1]}$ gauge invariant construction

- Gauge invariant plaquette
  $\tilde{U}_p \equiv e^{-\frac{2\pi i}{N} B_p} U_p$
  - Recall $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell$
  - Admissibility $\mathrm{tr}(1 - \tilde{U}_p) < \epsilon$
- u: product of plaquettes $\to \tilde{u}$

$$\tilde{u}^{n-\hat{3}}_{s_7 s_2} = e^{\frac{2\pi i}{N} B_{34}(n-\hat{3})} e^{\frac{2\pi i}{N} B_{24}(n)} u^{n-\hat{3}}_{s_7 s_2}$$



- Similarly, $\tilde{v}$ is defined in terms of $\tilde{u}$
  - Gauge covariance
    $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_\mu(n-\hat{\mu})} \tilde{v}_{n,\mu}(x)$
- Fractional topological charge $Q = \sum_n q(n) \in -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N} + \mathbb{Z}$

$$q(n) = -\frac{1}{8N} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} B_{\mu\nu}(n) B_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \check{q}(n)$$

  where $\sum_n \check{q}(n) \in \mathbb{Z}$ (each term in q is not inv. under $\mathbb{Z}_N^{[1]}$)
- We observe mixed 't Hooft anomaly with lattice regularization!

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Generalized symmetry: Non-invertibility

- Lie Algebras in Particle Physics by H. Georgi

  "Group theory is the study of symmetry."
- Based on recent developments of generalized symmetry, symmetry is not necessarily described by group!
  1. Associativity: $(ab)c = a(bc)$ for $\forall a,\ b,\ c$
  2. Identity element: $\exists \mathrm{e}$ s.t. $a\mathrm{e} = \mathrm{e}a = a$
  3. Inverse element: $\exists b$ s.t. $ab = ba = \mathrm{e}$ $(b = a^{-1})$

  So far, symmetry possesses invertibility.
- Now, for "naively unitary" symmetry operators $\{U_\alpha = e^{i\alpha Q}\}$, there can exist Non-invertible symmetry in some systems

$$\mathcal{D} \times \mathcal{D}^\dagger \neq 1$$

  [Many studies; See recent lectures by Schäfer-Nameki, Shao]
- Let's consider non-invertible symmetry for axial $U(1)$ transf, and then, in this talk, our aim is to realize it on lattice!

Axial $U(1)$ symmetry in continuum theory

- Quantum anomaly $\to$ loss of symmetry
  ABJ anomaly: no conserved current for axial $U(1)$ symmetry
- For fractional rotation angles, conserved, gauge-invariant and topological non-invertible symmetry exists
  [Córdova–Ohmori '22, Choi–Lam–Shao '22]
- Naive symmetry operator

$$d \star j = \frac{1}{4\pi^2} F \wedge F, \qquad U_\alpha(M) = \exp\left(\frac{i\alpha}{2}\int_M \star j\right)$$

  is not topological
- Let us consider topological modification by Chern–Simons:

$$\hat{U}_\alpha(M) = \exp\left[\frac{i\alpha}{2}\int\left(\star j - \frac{1}{4\pi^2} A \wedge dA\right)\right]$$

  But gauge non-invariant!!
- For $\alpha = \frac{2\pi}{N}$, $-i\int_M \frac{1}{4\pi N} A \wedge dA$ is still gauge non-invariant because of fractional CS level $1/N$

Fractional quantum Hall effect and non-invertibility

- TQFT (fractional quantum Hall state) as

$$i\int_M \left(\frac{N}{4\pi} a\wedge da + \frac{1}{2\pi} a\wedge dA\right)$$

  which is gauge-invariant with fractional CS level
  - a: additional dynamical $U(1)$ gauge field
  - Naively, this action is (level-N CS) + $(A_\mu J^\mu)$; $a = -A/N$
    "Substituting" this into action reproduces $-i\int \frac{1}{4\pi N} A\wedge dA$
- To modify naive symmetry operator $U_\alpha(M)$, we use this action at boundary M instead of naive CS term:

$$\mathcal{D}_{1/N}(M) = \exp\left[i\int\left(\frac{\pi}{N}\star j + \frac{N}{4\pi} a\wedge da + \frac{1}{2\pi} a\wedge dA\right)\right]$$

- Gauge invariant and topological; no inverse operator

$$\mathcal{D}_{1/N}\times\mathcal{D}^\dagger_{1/N} = \mathcal{C} \neq 1$$

  - $\mathcal{C}$: condensation operator

Lattice realization?

- We want to realize lattice axial $U(1)$ non-invertible symmetry
- Problems:
  - Lattice Chern–Simons theory?
    - cf. Villain formulation of $U(1)$ GT [Jacobson–Sulejmanpasic '23] $\not\Rightarrow$ non-Abelian generalization
  - (Anomaly-free) Chiral lattice gauge theory?
    - For $U(1)$ gauge theory, Lüscher's construction ['98]
    - See [Intensive lecture by Kikukawa-san @YITP, 2/19-21]
  - Anomalous gauge theory? (Continuation of studying it?)
    - E.g., [Forster–Nielsen–Ninomiya '80, Harada–Tsutsui '87, . . .]
    - Lattice framework: no-go as in [Kikukawa–Suzuki '07]
- Construct (fermion measure in) anomalous chiral gauge theory
  - Introduce external $U(1)'$ gauge group
    1. to manipulate axial rotation
    2. to define topological defect
    3. to keep locality for dynamical fields

$U(1) \times U(1)'$ lattice gauge theory: Gauge

- Vector $U(1) \ni u(x,\mu)$ (physical) and Chiral $U(1)' \ni U(x,\mu)$ (non-dynamical) gauge group
- Expectation value with magnetic flux m:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}u \, e^{-S_{\mathrm{G}}} \langle \mathcal{O} \rangle_{\mathrm{F}} \,, \quad \langle \mathcal{O} \rangle_{\mathrm{F}} = w[m] \int \mathrm{D}\psi \mathrm{D}\bar{\psi} \, e^{-S_{\mathrm{F}}} \mathcal{O}.$$

- (Unconventional) Gauge action S_{G} following [Lüscher],

$$S_{\mathrm{G}} = \frac{1}{4g_0^2} \sum_{x \in \Gamma} \sum_{\mu,\nu} \mathcal{L}_{\mu\nu}(x)$$

$$\mathcal{L}_{\mu\nu}(x) = \begin{cases} [f_{\mu\nu}(x)]^2 \left\{ 1 - \frac{[f_{\mu\nu}(x)]^2}{\epsilon^2} \right\}^{-1} & \text{if } |f_{\mu\nu}(x)| < \epsilon, \\ \infty & \text{otherwise.} \end{cases}$$

for $0 < \epsilon < \pi/3$, where the field strength of $u(x,\mu)$ is

$$f_{\mu\nu}(x) = \frac{1}{i} \ln u(x,\mu) u(x+\hat{\mu},\nu) u(x+\hat{\nu},\mu)^{-1} u(x,\nu)^{-1}.$$

$U(1) \times U(1)'$ lattice gauge theory: Fermion

- Two left-handed Weyl fermions $\psi_{1,2}$ possess $U(1)$ charges

$$(\mathrm{e}_1, \mathrm{e}_1') = (+1, -1), \qquad (\mathrm{e}_2, \mathrm{e}_2') = (-1, -1).$$

- Since overlap Dirac operator fulfills Ginsparg–Wilson relation

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D,$$

chirality projection operators are defined by

$$\hat{\gamma}_5 = \gamma_5(1 - D), \qquad \hat{P}_\pm = \tfrac{1}{2}(1 \pm \hat{\gamma}_5), \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5).$$

- Fermion action is

$$S_{\mathrm{F}} = \textstyle\sum_{x\in\Gamma} \bar{\psi}(x) D \psi(x) = \sum_{x\in\Gamma} \bar{\psi}(x) P_+ D \hat{P}_- \psi(x)$$

- Fermion integration measure is

$$\mathrm{D}\psi\mathrm{D}\bar{\psi} = \prod_j \mathrm{d}c_j \prod_k \mathrm{d}\bar{c}_k, \ \psi(x) = \sum_j v_j(x) c_j, \ \bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

Basis $v_j(x)$ are in projected space $\hat{P}_- v_j = v_j$, $(v_k, v_j) = \delta_{kj}$

Construction of fermion integration measure

- Basis vectors depend on gauge field due to $\hat{P}_-$
- Under variation δ of gauge field, δv_j is not determined...
  - Requirements: (single-valued, gauge-inv) smooth, & local
- Under infinitesimal variations $\delta_\eta u(x,\mu) = i\eta_\mu(x)u(x,\mu)$, $\delta_\eta U(x,\mu) = i\eta'_\mu(x)U(x,\mu)$,

$$\delta_\eta \ln\det \bar{v}Dv = (\text{term from } \delta_\eta D) + (\text{term from } \delta_\eta v).$$

Second term is called measure term (measure currents)

$$-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j) := -i \sum_{x\in\Gamma} \left[\eta_\mu(x) j_\mu(x) + \eta'_\mu(x) J_\mu(x)\right].$$

- Consistent/covariant anomaly: $\mathcal{A}_{\text{cons}} = \frac{1}{3}\mathcal{A}_{\text{cov}}$ for Abelian GT
- To remedy this issue, we need to include counterterm [E.g., see Fujikawa–Suzuki Chap. 6]

Construction of measure currents

- For $\eta_\mu(x) = -\partial_\mu \omega(x)$, $\eta'_\mu(x) = -\partial_\mu \Omega(x)$, $(\gamma = -\frac{1}{32\pi^2})$

$$\delta_\eta \langle \mathcal{O} \rangle_{\mathrm{F}} = \langle \delta_\eta \mathcal{O} \rangle_{\mathrm{F}} - 2i\gamma \sum_{x \in \Gamma} \Omega(x) \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) \langle \mathcal{O} \rangle_{\mathrm{F}}. \tag{$\star$}$$

- To prove this, measure currents fulfill the following conditions:
  1. depend smoothly and locally on gauge fields
  2. satisfy "integrability condition" $([\delta_\eta, \delta_\zeta])$
  3. satisfy "anomalous conservation law" $(\partial^* j,\ \partial^* J)$
- We find non-local dependence of $U(x,\mu)$; but $U(x,\mu)$ is external
- In infinite volume, one can construct the currents explicitly w.r.t.

$$u(x,\mu) = \mathrm{e}^{i\mathfrak{a}_\mu(x)}, \quad |\mathfrak{a}_\mu(x)| \le \pi(1 + 8\|x\|), \quad f(x) = d\mathfrak{a}(x),$$
$$U(x,\mu) = \mathrm{e}^{i\mathfrak{A}_\mu(x)}, \quad |\mathfrak{A}_\mu(x)| \le \pi(1 + 8\|x\|), \quad F(x) = d\mathfrak{A}(x),$$

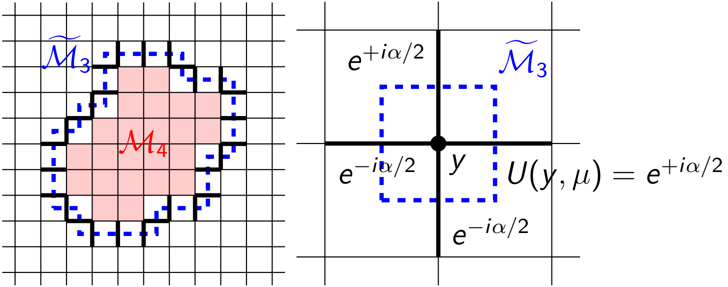
as follows:

Backup: Construction of measure currents $L \to \infty$

$$
\begin{aligned}
\mathfrak{L}_\eta^{\star\mathrm{inv}} &= i\int_0^1 ds\ \mathrm{Tr}(\hat{P}_-[\partial_s\hat{P}_-,\delta_\eta\hat{P}_-]) + \int_0^1 ds \sum_{x\in\mathbb{Z}^4}[\eta_\mu(x)k_\mu(x) + \mathfrak{a}_\mu(x)\delta_\eta k_\mu(x)] \\
&\quad + \int_0^1 ds \sum_{x\in\mathbb{Z}^4}\left[\eta'_\mu(x)K_\mu(x) + \mathfrak{A}_\mu(x)\delta_\eta K_\mu(x)\right] \\
&\quad - \frac{4}{3}\gamma\sum_{x\in\mathbb{Z}^4}\epsilon_{\mu\nu\rho\sigma}\{\eta'_\mu(x)\mathfrak{a}_\nu(x+\hat{\mu})f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\
&\qquad\qquad + \mathfrak{A}_\mu(x)\delta_\eta\left[\mathfrak{a}_\nu(x+\hat{\mu})f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]\} \\
&:= \sum_{x\in\mathbb{Z}^4}\left[\eta_\mu(x)j_\mu^{\star\mathrm{inv}}(x) + \eta'_\mu(x)J_\mu^{\star\mathrm{inv}}(x)\right], \\
\mathfrak{L}_\eta^{\star\text{non-inv}} &= 4\gamma\sum_{x\in\mathbb{Z}^4}\epsilon_{\mu\nu\rho\sigma}\{\eta'_\mu(x)\mathfrak{a}_\nu(x+\hat{\mu})f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\
&\qquad\qquad + \mathfrak{A}_\mu(x)\delta_\eta\left[\mathfrak{a}_\nu(x+\hat{\mu})f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]\} \\
&:= \sum_{x\in\mathbb{Z}^4}\left[\eta_\mu(x)j_\mu^{\star\text{non-inv}}(x) + \eta'_\mu(x)J_\mu^{\star\text{non-inv}}(x)\right].
\end{aligned}
$$

Introduction of topological defect

- $U(x,\mu) = e^{\pm i\alpha/2}$ for 3D closed surface $\widetilde{\mathcal{M}}_3$; otherwise 1



- From Eq. $(\star)$, anomalous Ward–Takahashi identity is

$$\langle \mathcal{O} \rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} = \exp\left[-i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right] \langle \mathcal{O}^{\alpha} \rangle_{\mathrm{F}},$$

where $\psi(x)^{\alpha} = e^{-i\alpha/2}\psi(x)$, $\bar{\psi}(x)^{\alpha} = \bar{\psi}(x) e^{i\alpha/2}$ for $x \in \mathcal{M}_4$

- Symmetry operator may be written by

$$\left\langle U_{\alpha}(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}} := \langle \mathcal{O} \rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \exp\left[i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]$$

Coupling with $\mathbb{Z}_N$ TQFT for $\alpha \sim 2\pi/N$

- Hard problem: lattice Chern–Simons with rotation angle α
- Instead of CS, lattice $\mathbb{Z}_N$ TQFT (thanks to Tanizaki-san)
- Set rotation angles α as $2\pi p/N$ (p, $N \in \mathbb{Z}$)
- It is natural to consider 3D level-N BF theory on $\mathcal{M}_3$

$$S_{\mathrm{BF}} = -\frac{ip\pi}{N} \sum_{x \in \mathcal{M}_3} \epsilon_{\mu\nu\rho} \Big\{ b_\mu(\tilde{x}) \left[\partial_\nu c(x+\hat{\mu}) - \frac{1}{2} z_{\nu\rho}(x+\hat{\mu}) \right] - \frac{1}{2} z_{\mu\nu}(x) c_\rho(x+\hat{\mu}+\hat{\nu}) \Big\}$$

  - Dual lattice $\tilde{x}$; $f = \delta a + 2\pi z$, $-\pi < a_\mu(x) \equiv \frac{1}{i} \ln u(x,\mu) \leq \pi$
- For simplicity, $S_{\mathrm{BF}} = \frac{-ip\pi}{N} \sum [b(\delta c - z) - z \cup c]$

Symmetry operator in terms of $\mathbb{Z}_N$ TQFT

- Consider $\mathcal{Z}_{\mathcal{M}_3}[z] = \frac{1}{N^s}\int \mathrm{D}b\mathrm{D}c\, e^{-S_{\mathrm{BF}}}$, s: # of sites $\in \mathcal{M}_3$
- From summation over b, $\mathcal{Z}_{\mathcal{M}_3}[z] = 0$ if $\sum_{\mathcal{M}_2} z \neq 0 \mod N$
  - If $z = 0$, $\mathcal{Z}_{\mathcal{M}_3}[0] = N^{b_2 - 1}$, where b_2: 2nd Betti number of $\mathcal{M}_3$
  - If $z = \delta\nu \mod N$,
$$\mathcal{Z}_{\mathcal{M}_3}[z] = \exp\left(-\frac{ip\pi}{N}\sum_{\mathcal{M}_3}\delta\nu \cup \nu\right)\mathcal{Z}_{\mathcal{M}_3}[0]$$
- Redefine symmetry operator on arbitrary 3-cycle $\widetilde{\mathcal{M}}_3$:

$$\left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}}_3)\mathcal{O}\right\rangle_{\mathrm{F}} = \langle\mathcal{O}\rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3}\exp\left[-\frac{ip}{4\pi N}\sum_{\mathcal{M}_3}(a\cup f + 2\pi z \cup a)\right]\mathcal{Z}_{\mathcal{M}_3}[z]$$

  - Then, topological: $\left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}'}_3)\mathcal{O}\right\rangle_{\mathrm{F}} = \left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}}_3)\mathcal{O}\right\rangle_{\mathrm{F}}$
  - Gauge invariant: cancellation between exp and $\mathcal{Z}_{\mathcal{M}_3}$ under $a \mapsto a - \delta\phi - 2\pi l$ and $z \mapsto z + \delta l$

Fusion rules

- From $\mathcal{Z}^{(p,N)}_{\mathcal{M}_3}[z]\mathcal{Z}^{(p,N)}_{-\mathcal{M}_3}[z] = \mathcal{Z}_{\mathcal{M}_3}[0]\mathcal{C}_{\mathcal{M}_3}[z]$, condensation operator:

$$\mathcal{C}_{\mathcal{M}_3}[z] = \frac{1}{N^s}\int \mathrm{D}b\mathrm{D}c\, e^{b(\delta c - z)}$$

- For different p_1 and p_2 (assuming $\gcd(p_1 + p_2, N) = 1$)

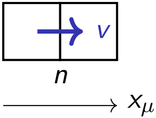
$$\tilde{U}_{2\pi p_1/N}\tilde{U}_{2\pi p_2/N} = \mathcal{Z}_{\mathcal{M}_3}[0]\,\tilde{U}_{2\pi(p_1+p_2)/N}$$

- More generally ($\gcd(p_1, N_1) = 1$, $\gcd(p_2, N_2) = 1$, $\gcd(N[p_1/N_1 + p_2/N_2], N) = 1$ with $N = \mathrm{lcm}(N_1, N_2)$),

$$\tilde{U}_{2\pi p_1/N_1}\tilde{U}_{2\pi p_2/N_2} = \frac{\mathcal{Z}^{(N_1)}_{\mathcal{M}_3}[0]\mathcal{Z}^{(N_2)}_{\mathcal{M}_3}[0]}{\mathcal{Z}^{(N)}_{\mathcal{M}_3}[0]}\tilde{U}_{2\pi(p_1/N_1+p_2/N_2)}$$

Summary [1/3]

- Generalized symmetries have been developed in this decade
  - Higher-form sym, higher-group sym, noninvertible sym, subsystem sym, . . .
  - Through the use of 't Hooft anomaly matching, new insights about *nontrivial dynamics & classification of phases*
- Standing on a fully regularized framework: lattice gauge theory
  - Generalization of Lüscher's construction of topology on lattice
  - Maintaining locality, $SU(N)$ gauge inv & higher-form gauge inv
  - There exists interpolation to smooth enough bundle structure
    - ⋆ Transition function $v_f(n) \to v_f(x)$
    - ⋆ Q is written in terms of $v_f(x)^{-1}\partial_\nu v_f(x)$



$$Q \in \mathbb{Z} \xrightarrow{\text{Gauging } \mathbb{Z}_N^{[1]}} \frac{1}{N}\mathbb{Z}$$

  - Mixed 't Hooft anomaly between $\mathbb{Z}_N^{[1]}$ & θ periodicity

Summary [2/3]

- Some simple applications
  - Higher-group symmetry from lattice (§7)
    - ⋆ Mixture of different higher-form symmetries
  - Lattice construction of magnetic operators and observation of Witten effect (§6)
    - ⋆ Admissibility condition implies absence of monopole
    - ⋆ Our proposal: Excision method (opening a hole in lattice)
    - ⋆ By arranging lattice/dual-lattice in 2D scalar theory, mixing magnetic charges with electric ones (Witten effect)
  - Future works
    - ⋆ Construction of monopole, 't Hooft line in gauge theory [See e.g., Honda–Onoda–Suzuki '24]
    - ⋆ Numerical simulation of $SU(N)$ gauge theory coupled with $\mathbb{Z}_N$ 2-form gauge fields [on-going Abe–OM]

Summary [3/3]

- (Again) Standing on a completely lattice regularized framework
  - Generalized Lüscher's construction of chiral lattice gauge theory
  - Construction of fermion measure: for dynamical fields, smooth and local; for external $U(x, \mu)$, non-local (unphysical)
- Level-N BF theory (thanks to Tanizaki-san)
  - We can construct symmetry operator for rational angles, and evaluate fusion rules!
  - Attempt to Karasik's prescription (§8)
    - ⋆ Consider "gauge average"; there are some subtleties in our lattice formulation...
    - ⋆ Under gauge non-invariant constraint and in terms of 4D bulk, we 'constructed' symmetry operator
- Questions
  - Physical phenomena (e.g., monopole) [on-going Honda–Kan–OM]
  - Generalization to non-Abelian gauge theory
  - Some aspects of anomalous/chiral lattice gauge theory (w/ boundary)

6. Magnetic operators and Witten effect on the lattice [Abe–OM–Onoda–Suzuki–Tanizaki (OM–Onoda–Suzuki)]
   - Admissibility condition and monopole
   - Magnetic operators in lattice 2D scalar theory
7. Higher-group structure in lattice gauge theory [Kan–OM–Nagoya–Wada, Abe–OM–Onoda]
   - Higher-group symmetry under modification of instanton sum
8. "Gauge average" prescription for non-invertibility [Honda–Onoda–OM–Suzuki]
   - Harada–Tsutsui formalism
   - Symmetry operator under gauge average

Mixed 't Hooft anomaly in $SU(N)$ or $U(1)$ GT

- 1-form inv. action: $S_{\mathrm{YM}} + S_{\mathrm{matter}} - i\theta Q$

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q[B_p]} \mathcal{Z}_\theta[B_p]$$

Anomaly between $\mathbb{Z}_N^{[1]}$ gauge inv. and θ periodicity

- In case of $U(1)$ gauge theory, gauging $\mathbb{Z}_q^{[1]} \subset U(1)_{\mathrm{E}}$ symmetry

$$Q \in \frac{1}{8q^2} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{4q} (\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu})_{\mu\nu\leftrightarrow\rho\sigma} \mathbb{Z} + \mathbb{Z}$$

Modified as $-iq\theta Q$ by Witten effect [Honda–Tanizaki '20]

  - If $-i\check{\theta} Q$, $\check{\theta} \sim \check{\theta} + 2\pi q$; $\check{\theta} = q\theta$, then $\theta \sim \theta + 2\pi$
  - Electric charge of dyon $= qe + \frac{1}{2\pi}\check{\theta}$ (matters with charge $q\mathbb{Z}$)
  - 't Hooft anomaly $= e^{-2\pi i q Q}$ with $qQ \in \frac{1}{q}\mathbb{Z}$
  - Under some kind of modification (see next §), there exist $\mathbb{Z}_N^{[1]}$ if multiple of q

Admissibility $\to$ absence of monopole?

- Magnetic monopole
  - Magnetic defect operators provide nontrivial topology
  - Quite heavy but significant in nonperturbative dynamics
  - Maxwell equation w/ monopole current j_{m}: $d \star F = j_{\mathrm{e}}$, $dF = j_{\mathrm{m}}$.
- Admissibility $dF = 0$ to reinstate topo. structure on lattice fields
  - Exercise: $0 < \epsilon < \pi/3$?
    $F_p \equiv \frac{1}{i} \ln U_p$, $a_\ell \equiv \frac{1}{i} \ln U_\ell$ in $(-\pi, \pi]$; $F_p = (da)_p + 2\pi N_p$ since

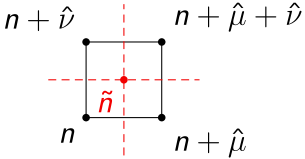
$$(da)_p = \Delta_\mu a_{n,\nu} - \Delta_\nu a_{n,\mu} = [a_{n+\hat{\mu},\nu} - a_{n,\nu}] - [a_{n+\hat{\nu},\mu} - a_{n,\mu}]$$

$$|da| \leq 4|a| \leq 4\pi; \qquad \text{Introduce } N \text{ as } |da + 2\pi N| \leq \pi.$$

    Imposing $|F_p| < \epsilon$, $6\epsilon > |\underbrace{\varepsilon_{\mu\nu\rho\sigma}\Delta_\nu F_{\rho\sigma}}_{\to 0}| = 2\pi|\underbrace{\varepsilon_{\mu\nu\rho\sigma}\Delta_\nu N_{\rho\sigma}}_{<1}|$;
    therefore $6\epsilon < 2\pi$
- How to discuss monopole properties on lattice w/ admissibility?
  - 2D compact bosons: θ term and magnetic operators
  - Observation of analogue of Witten effect
  - Future study: 4D gauge theory

Lattice formulation of θ angle and Witten effect

- Compact scalar fields: $\phi_1(n)$, $\phi_2(\tilde{n})$
  - Dual lattice $\tilde{n} = n + \frac{1}{2}\hat{1} + \frac{1}{2}\hat{2}$
  - $\partial\phi_a(s,\mu) \equiv \frac{1}{i} \ln e^{-i\phi_a(s)} e^{i\phi_a(s+\hat{\mu})}$
  - $\sup_\ell |\partial\phi_{a,\ell}| < \epsilon$, $0 < \epsilon < \pi/2$; then Bianchi identity $d\partial\phi = 0$



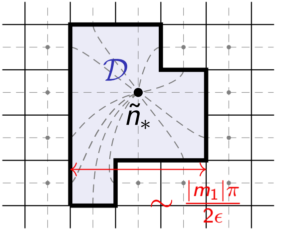
- Lattice action with θ angle

$$S = \frac{R^2}{2\pi} \sum [1 - \cos\partial\phi_{a,\ell}] - i\theta Q, \quad Q = -\frac{1}{4\pi^2} \sum \varepsilon_{\mu\nu} \partial\phi_{2,(\tilde{n},\mu)} \partial\phi_{1,(n+\hat{\mu},\nu)}$$

- Usually, $\mathcal{Q}_{\mathrm{m1}}(C) \equiv \frac{1}{2\pi} \sum_{\ell \in C} \partial\phi_{1,\ell} = 0$
- Excision method: sites&links eliminated inside $\mathcal{D}$; put one dual site $\tilde{n}_*$ in $\mathcal{D}$

$$m_1 = \mathcal{Q}_{\mathrm{m1}}(\partial\mathcal{D}) \in \mathbb{Z}$$

- $\langle M_1(\tilde{n}_*) \ldots \rangle_{\theta+2\pi} = \langle M_1(\tilde{n}_*) e^{i m_1 \phi_2(\tilde{n}_*)} \ldots \rangle_\theta$ for magnetic defect operator $M_1(x)$



EM symmetry and 't Hooft anomaly

- $U(1)$ electric/magnetic global symmetries: $j_{\mathrm{e}} = \star d\phi$, $j_{\mathrm{m}} = d\phi$
  - Electric charged object $e^{i\phi}$;
    Monopole $M_1(x)$ with $(0, m_1) \overset{\text{Witten}}{\to}$ dyon with (m_1, m_1)
- Background gauging $U^{\mathrm{em},a}_{p,\tilde{p}}$:
  $F^{(\mathrm{e},1)}_{p}$ and $F^{(\mathrm{m},1)}_{\tilde{p}}$ for ϕ_1, $F^{(\mathrm{e},2)}_{\tilde{p}}$ and $F^{(\mathrm{m},2)}_{p}$ for ϕ_2
  - $\sup|F| < \delta$ with $0 < \delta < \min(\pi, 2\pi - 4\epsilon)$
  - $F^{\mathrm{e}}_{\mu\nu} = \Delta_\mu D\phi_{n,\nu} - \Delta_\nu D\phi_{n,\mu}$
- Analogue of θ shift is given by
  $\theta \to \theta + 2\pi, \quad F^{(\mathrm{m},1)}_{\tilde{p}} \to F^{(\mathrm{m},1)}_{\tilde{p}} - F^{(\mathrm{e},2)}_{\tilde{p}}, \quad F^{(\mathrm{m},2)}_{p} \to F^{(\mathrm{m},2)}_{p} + F^{(\mathrm{e},1)}_{p}$
  - Mixing $\mathcal{Q}_{\mathrm{m}1}$ with $\mathcal{Q}_{\mathrm{e}2}$ on $\tilde{\Lambda}$; $\mathcal{Q}_{\mathrm{m}2}$ with $\mathcal{Q}_{\mathrm{e}1}$ on Λ (Witten effect)
- Under above shift, we observe mixed 't Hooft anomaly

$$\mathcal{Z}_{\theta+2\pi}[A^{(\mathrm{e},a)}, A^{(\mathrm{m},a)} - \varepsilon_{ab}A^{(\mathrm{e},b)}] = e^{-\frac{i}{2\pi}\sum \varepsilon_{\mu\nu} A^{(\mathrm{e},2)}_\mu(\tilde{n}) A^{(\mathrm{e},1)}_\nu(n+\hat{\mu})} \mathcal{Z}_\theta[A]$$

- Magnetic charge in bosonization of 2D $U(1)$ chiral gauge theory [OM–Onoda–Suzuki '24]

6. Magnetic operators and Witten effect on the lattice [Abe–OM–Onoda–Suzuki–Tanizaki (OM–Onoda–Suzuki)]
   - Admissibility condition and monopole
   - Magnetic operators in lattice 2D scalar theory

7. Higher-group structure in lattice gauge theory [Kan–OM–Nagoya–Wada, Abe–OM–Onoda]
   - Higher-group symmetry under modification of instanton sum

8. "Gauge average" prescription for non-invertibility [Honda–Onoda–OM–Suzuki]
   - Harada–Tsutsui formalism
   - Symmetry operator under gauge average

Generalization: higher-group structure

- In general suppose $\otimes_{i,p} G_i^{[p]}$ global symmetry
- After gauging, a naive direct product of symmetry groups?
  - Can each symmetry be gauged *individually*?
- Gauging $G^{[0]} \times H^{[1]}$ global symmetry, then gauge transf.:

$$A \mapsto A + d\lambda^{(0)}, \qquad B \mapsto B + d\lambda^{(1)} + A d\lambda^{(0)}$$

2-group symmetry (cf. superstring theory [Green–Schwarz '84])

  - p-group symmetry: $G_0^{[0]} \tilde{\times} \cdots \tilde{\times} G_{p-1}^{[p-1]}$
- E.g., 4D $SU(N)$ gauge theory with instanton number $p\mathbb{Z}$
  - For any $p \in \mathbb{Z}$, local and unitary [Seiberg '10]
  - Global symmetry: $\underbrace{\mathbb{Z}_N^{[1]} \text{ center sym}}_{\text{~~gauging~~}} \times \mathbb{Z}_p^{[3]}$ sym $\overset{\text{gauging}}{\to}$ 4-group

[Tanizaki–Ünsal '19]

cf. [Hidaka–Nitta–Yokokura '21]

- How to modify instanton sum & realize higher-group on lattice?

Modified instanton-sum: higher-group symmetry

- For $Q = \sum_n q_n$, insert the delta function
$$\delta\left(q_n - pc_n\right) \quad \to \quad Q = p\sum_n c_n \in p\mathbb{Z}$$
where $U(1)$ 4-form field strength c_n
- Obviously no nontrivial configurations for B_p
- Introducing new field Ω_n ($\Omega_n \in \mathbb{R}$ and $\sum_n \Omega_n \in \mathbb{Z}$)
  - Replacement: $c_n \to c_n - \frac{1}{Np}\Omega_n$: 3-form gauge inv
$$q_n - pc_n + \frac{1}{N}\Omega_n = 0 \qquad : \text{fractionality allowed}$$
  - Redefine Ω_n as $\tilde{\Omega}_n \equiv \frac{1}{N}\Omega_n - \underbrace{\frac{1}{8N}\varepsilon_{\mu\nu\rho\sigma}B_{n,\mu\nu}B_{n+\hat{\mu}+\hat{\nu},\rho\sigma}}_{\text{fractional part of } Q}$
  Again $\sum_n \tilde{\Omega}_n \in \mathbb{Z}$
$$\check{q}_n - pc_n + \tilde{\Omega}_n = 0 \quad \text{where } \check{q}_n\text{: integral part of } Q$$
- 1-form and 3-form gauge transf with $\Omega_n^{(3)} \in \mathbb{R}$:
$$B_p \mapsto B_p + (d\lambda)_p, \quad c_n \mapsto c_n + \frac{1}{p}d\Omega_n^{(3)}\,(+\mathbb{Z}),$$
$$\tilde{\Omega}_n \mapsto \tilde{\Omega}_n + d\Omega_n^{(3)}\,(+p\mathbb{Z}) + \text{“}\frac{2}{N}B \wedge d\lambda + \frac{1}{N}d\lambda \wedge d\lambda\text{”}\,(+\mathbb{Z})$$
- Mixture of 1-form and 3-form gauge transf

6 Magnetic operators and Witten effect on the lattice [Abe–OM–Onoda–Suzuki–Tanizaki (OM–Onoda–Suzuki)]
- Admissibility condition and monopole
- Magnetic operators in lattice 2D scalar theory

7 Higher-group structure in lattice gauge theory [Kan–OM–Nagoya–Wada, Abe–OM–Onoda]
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8 "Gauge average" prescription for non-invertibility [Honda–Onoda–OM–Suzuki]
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“Gauge average” prescription in anomalous GT

- Prescription by Karasik ['22] instead of [Choi–Lam–Shao '22]
  - “Gauge average” (similar to Harada–Tsutsui formalism)

$$\tilde{U}_\alpha(M) = \int \mathrm{D}\phi \, \hat{U}_\alpha(M)|_{A \to A - d\phi}$$

- What is anomalous gauge theory [Kikukawa–Suzuki]?

  In other words, do we have to know an anomaly-free fundamental theory very precisely to study dynamics of chiral gauge theories at an (low) energy scale of our concern?
  - [Faddeev–Shatashvili 84', 86', ...]
- Harada–Tsutsui formalism ['87]
  - Faddeev–Popov determinant as (g denotes a gauge transf)

$$\Delta_f[A] \int \mathcal{D}g \, \delta(f[A^g]) = 1,$$

  - Integration of Wess–Zumino action over g
    $\to$ gauge-invariant effective action with non-local terms.
- There are some subtleties $\to$ an other method by Tanizaki-san

Gauge average and projection

- U_α is not invariant under gauge transf on boundary variables
- An operator defined by average over gauge transf

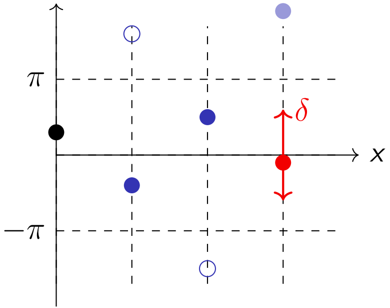
$$\left\langle \tilde{U}_\alpha(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}} := \langle \mathcal{O} \rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \int \mathrm{D}\lambda \, e^{\left[i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \right]^\lambda}$$

$[\ \ ]^\lambda$ indicates

$u(x,\mu) \to \lambda(x) u(x,\mu) \lambda(x+\hat{\mu})^{-1}$

- Smoothness condition

$\frac{1}{i} \ln \lambda(x)\lambda(x+\hat{\mu})^{-1}$



- Winding k_ν can be defined under this gauge non-inv condition
- $|k_\nu| < \frac{\delta}{2\pi} L$ with lattice size L; if $L \to \infty$,
  $\sum_{k=-\infty}^{\infty} e^{ikx} \propto \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n)$
- Gauge average implies (S_μ^1)
  $\delta\left(\frac{\alpha}{2\pi} \frac{1}{4\pi} \sum_{x\in\mathcal{M}_2^\nu} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}(x) - \mathbb{Z} \right)$
  - $\frac{\alpha}{2\pi}$ irrational: no magnetic flux
  - $\frac{\alpha}{2\pi} = \frac{p}{N}$: $\frac{1}{2\pi} \int_{\mathcal{M}_2} da = N\mathbb{Z}$
- $\tilde{U}(\widetilde{\mathcal{M}}_3) = U(\widetilde{\mathcal{M}}_3) P(\widetilde{\mathcal{M}}_3)$
  - P for allowed magnetic fluxes

Some subtleties in Karasik prescription on lattice

- We constructed non-invertible symmetry operator $\tilde{U}$ following Karasik in lattice gauge theory
  - Non-locality of $U(x,\mu)$ in fermion integration measure
  - Gauge non-invariant constraint for gauge transf; but irrelevant in continuum limit?
  - Non-intrinsically 3D construction in reference to auxiliary 4D bulk
  - (Are relative weights correct? Really topological?)
- Instead of CS, lattice $\mathbb{Z}_N$ TQFT (thanks to Tanizaki-san)
  - Set rotation angles α as $2\pi p/N$ (p, $N \in \mathbb{Z}$)
  - It is natural to consider 3D level-N BF theory

$$S_{\mathrm{BF}} = -\frac{ip\pi}{N} \sum_{x\in\mathcal{M}_3} \epsilon_{\mu\nu\rho} \left\{ b_\mu(\tilde{x}) \left[\partial_\nu c(x+\hat{\mu}) - \frac{1}{2} z_{\nu\rho}(x+\hat{\mu}) \right] - \frac{1}{2} z_{\mu\nu}(x) c_\rho(x+\hat{\mu}+\hat{\nu}) \right\}$$

  - ⋆ Dual lattice $\tilde{x}$; $f = \delta a + 2\pi z$, $-\pi < a_\mu(x) \equiv \frac{1}{i} \ln u(x,\mu) \leq \pi$
  - For simplicity, $S_{\mathrm{BF}} = \frac{-ip\pi}{N} \sum [b(\delta c - z) - z \cup c]$

Backup: Gauge average and *smooth* gauge transf

- U_α is not invariant under gauge transf on boundary variables
- An operator defined by average over gauge transf

$$\left\langle \tilde{U}_\alpha(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}}$$
$$:= \langle \mathcal{O} \rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \int \mathrm{D}\lambda \, \exp\left[i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \right]^{\lambda}$$

- $[\quad]^\lambda$ indicates gauge transf on boundary variables

$$u(x,\mu) \to \lambda(x) u(x,\mu) \lambda(x+\hat{\mu})^{-1}, \; \lambda(x) = e^{-i\phi(x)}, \; -\pi < \phi(x) \le \pi.$$

- Impose smoothness (gauge non-inv) condition on possible λ:
  for $-\pi < \frac{1}{i} \ln\left[e^{-i\phi(x)} e^{i\phi(x+\hat{\mu})} \right] = \partial_\mu \phi(x) + 2\pi l_\mu(x) \le \pi$,

$$\sup_{x,\mu} |\partial_\mu \phi(x) + 2\pi l_\mu(x)| < \delta, \qquad 0 < \delta < \pi/6.$$

  Then $f_{\mu\nu}(x) \to f_{\mu\nu}(x) + \partial_\nu \phi(x+\hat{\mu}) + 2\pi l_\nu(x+\hat{\mu})$.
- Gauge-inv of CS term is realized by this condition on the lattice.

Backup: Sum over winding number and projection

- Assume $\widetilde{\mathcal{M}}_3 = T^3$ is perpendicular to $\hat{\mu}$; $\widetilde{\mathcal{M}}_3 = S^1 \times \widetilde{\mathcal{M}}_2^\nu$
- Introduce "scalar potential" $\varphi(x)$ as $l_\nu(x) = \partial_\nu \varphi(x)$
- Winding on a cycle (ν direction) provides additional integer k_ν to φ; $|k_\nu| < \frac{\delta}{2\pi} L$ with lattice size L
- $\sum ff$ acquires $4\pi k_\nu \sum_{x+\hat{\mu}\in\mathcal{M}_3, x_\nu=0} \epsilon_{\mu\nu\rho\sigma} \left[f_{\rho\sigma}(x+\hat{\mu}) + f_{\rho\sigma}(x) \right]$
- Gauge average implies that

$$\int \mathrm{D}\lambda\, e^{8\pi i \alpha\gamma k_\nu \sum_{x\in\mathcal{M}_2^\nu} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}(x)} \text{ and } \sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n),$$

and then $$\delta\left(\frac{\alpha}{2\pi} \frac{1}{4\pi} \sum_{x\in\mathcal{M}_2^\nu} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}(x) - \mathbb{Z} \right) \quad \text{if } L \to \infty.$$

  - $\alpha/(2\pi)$ is irrational: no magnetic flux
  - $\alpha/(2\pi)$ is rational (p/N): $\frac{1}{2\pi}\int_{\mathcal{M}_2} da = N\mathbb{Z}$
- $\tilde{U}(\widetilde{\mathcal{M}}_3) = U(\widetilde{\mathcal{M}}_3) P(\widetilde{\mathcal{M}}_3)$, P is a projection operator for allowed magnetic fluxes