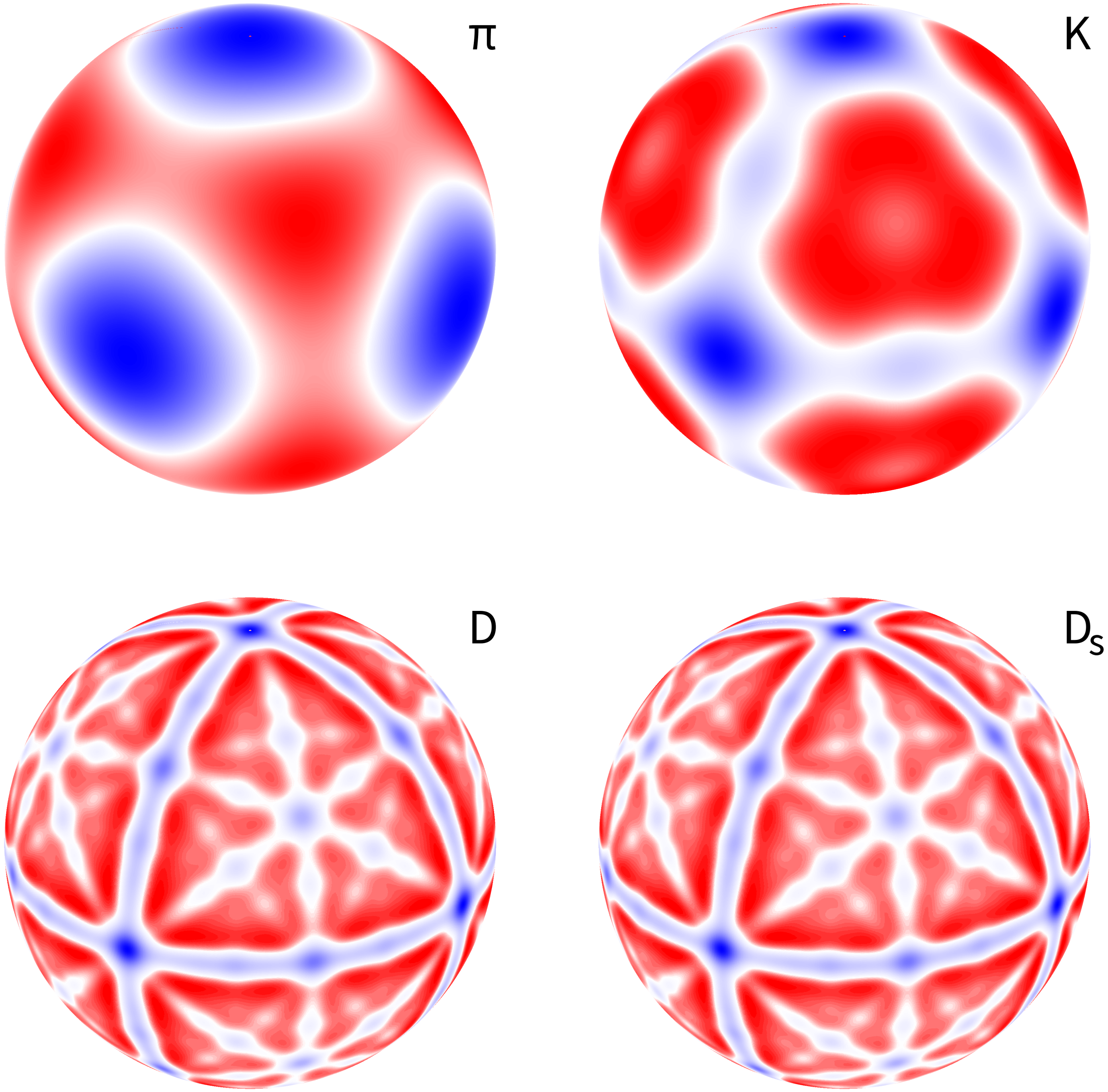

Challenges in high-precision determinations of CKM matrix elements using lattice QCD

Antonin Portelli — 10/05/2024
R-CCS FTRT Seminar

erc
CENTRE FOR THEORETICAL PHYSICS Higgs
THE UNIVERSITY OF EDINBURGH
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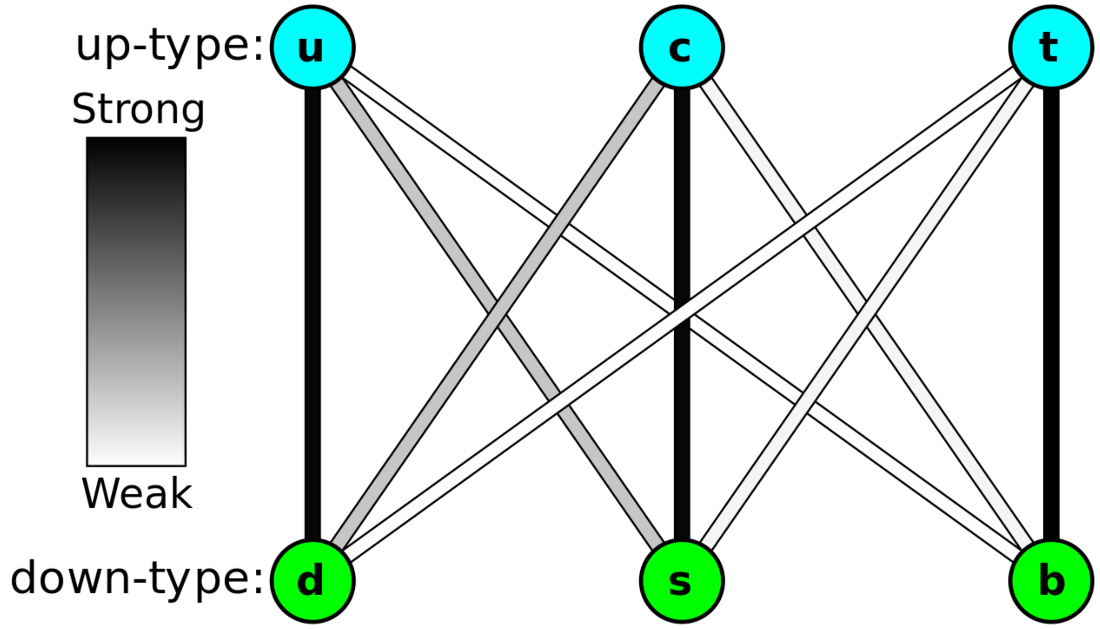
Image source: A Portelli & M Di Carlo

General context

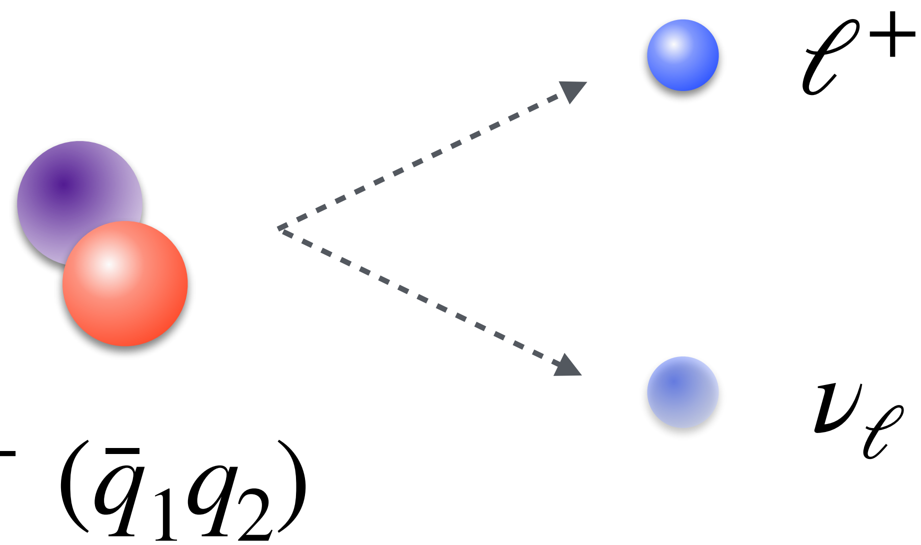
Flavour structure of the Standard Model



$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}$$

- The flavour structure of the SM is largely unexplained
- CKM matrix elements are inferred from measurements
- Non-unitary of the CKM matrix is still a good target for searching new physics

CKM matrix elements from leptonic decays



$$\propto |V_{q_1q_2}|^2$$

- Leptonic decays: W-boson quark pair annihilation
- Radiation inclusive decay rate

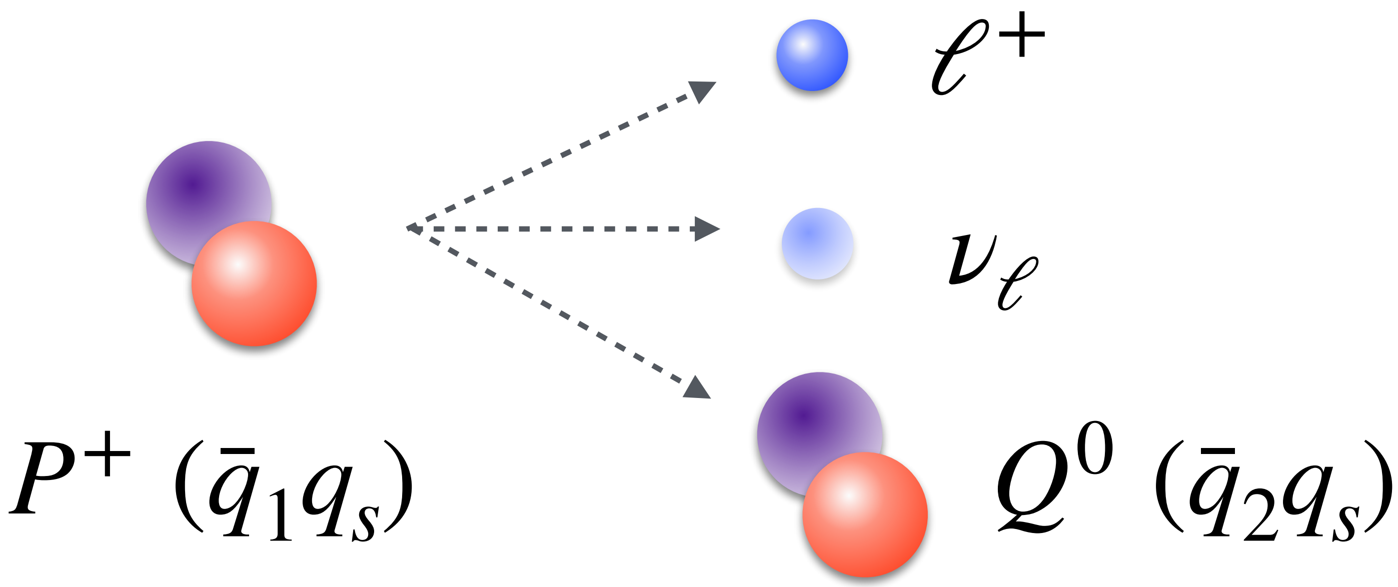
$$\Gamma(P^+ \to \ell^+\nu_\ell[\gamma]) = \frac{G_F^2}{8\pi} f_P^2 m_\ell^2 M_P \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2 |V_{q_1q_2}|^2 (1 + \delta R_{\mathrm{P}})$$

CKM matrix elements from leptonic decays

$$\Gamma(P^+ \to \ell^+ \nu_\ell[\gamma]) = \frac{G_F^2}{8\pi} f_P^2 m_\ell^2 M_P \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2 (1 + \delta_{\mathrm{IB}}) |V_{q_1 q_2}|^2$$

- from experiment/PDG
- isospin-symmetric QCD component
- isospin-breaking QCD+QED component

CKM matrix elements from semi-leptonic decays

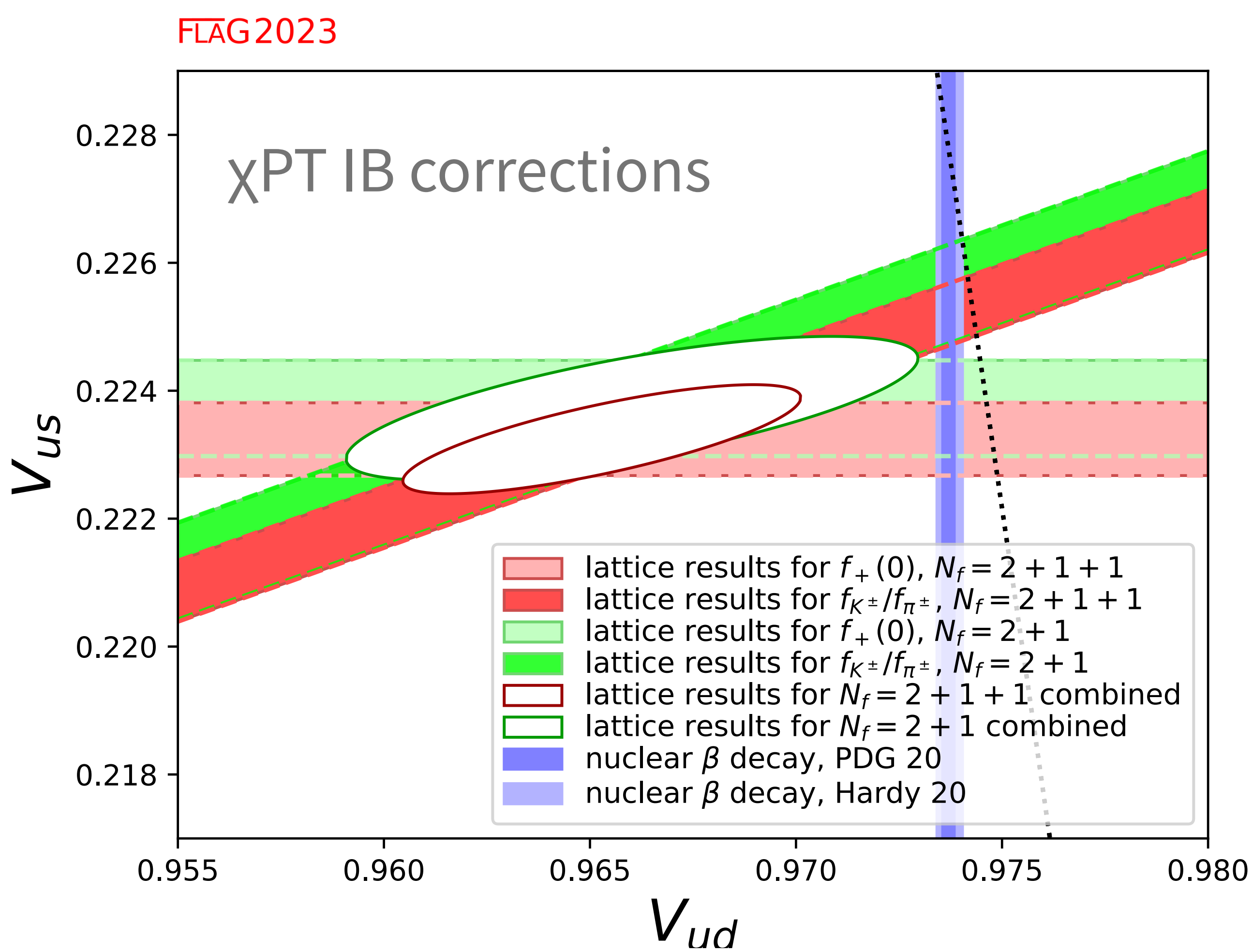


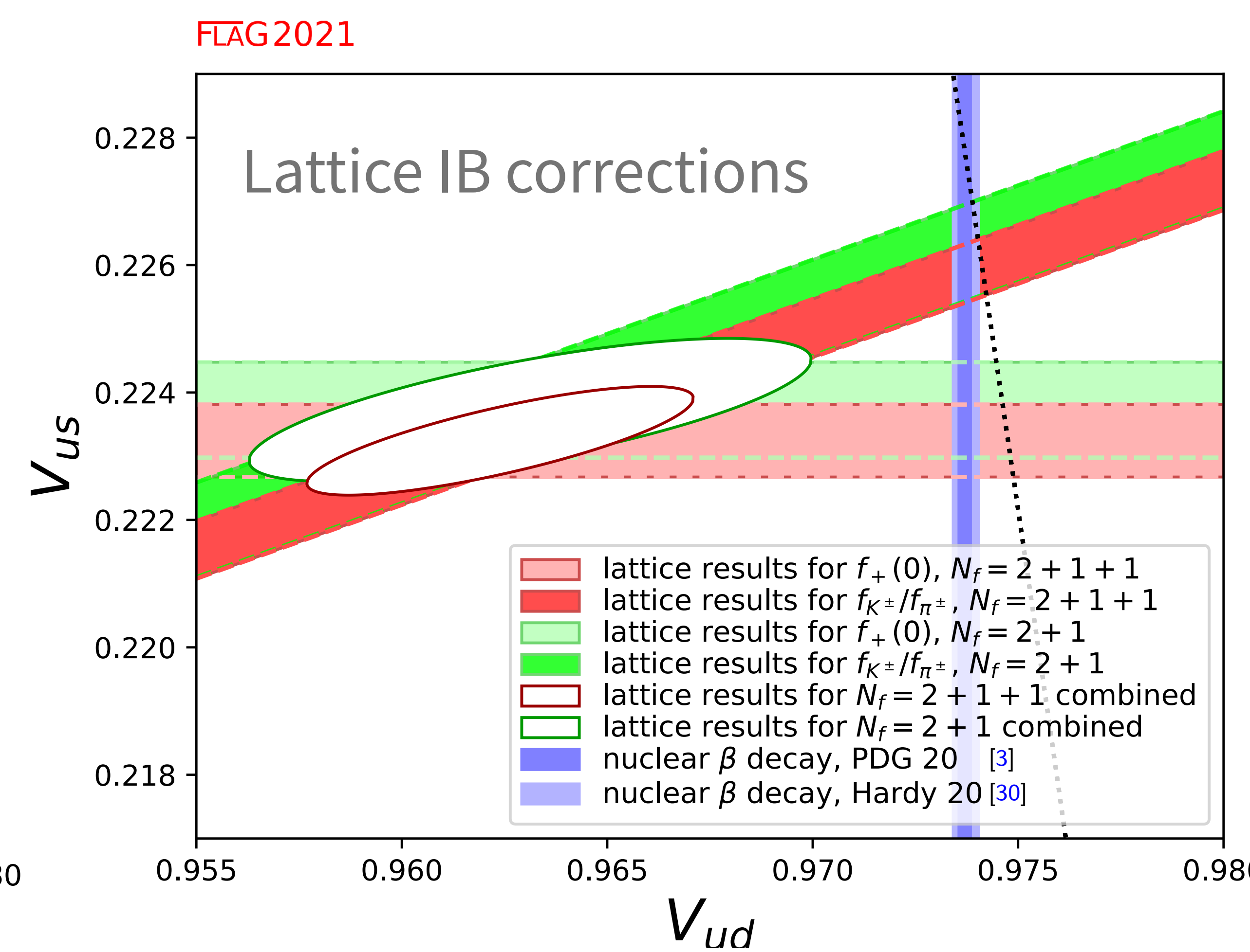
$$\propto |V_{q_1 q_2}|^2$$

- Semi-leptonic decays: flavour changing charged current
- Radiation inclusive decay rate

$$\Gamma(P^+ \to Q^0 \ell^+ \nu_\ell[\gamma]) = G_F^2 |V_{q_1 q_2}|^2 \mathcal{I} (1 + \delta_{\mathrm{IB}})$$

$|V_{us}|$ & $|V_{ud}|$ anomalies



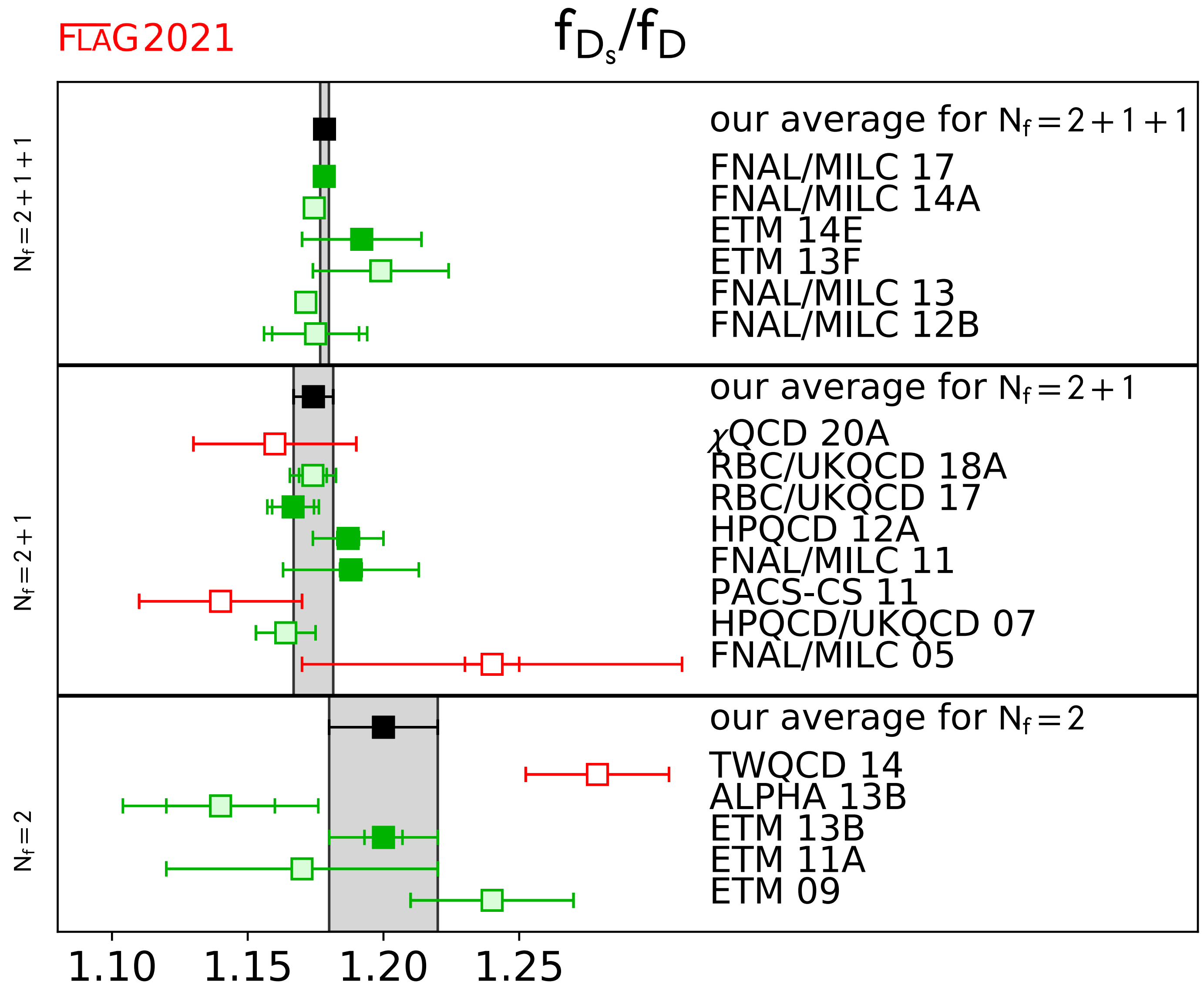


Significant tensions from

β decays $|V_{ud}|$ measurements & radiative corrections input

FLAG 2021 + web update

f_D/f_{D_s} accuracy



$N_f = 2+1+1$ FLAG average $f_{D_s}/f_D = 1.1783(0.0016)$

0.1% accuracy, however QED corrections are not known…

Bottom sector, f_B and f_{B_s}

Table 30. Lattice inputs for decay constants $f_{B_{(s)}}$ and bag parameters $B_{B_{(s)}}$ in the SM. The current average of $f_{B_{(s)}}$ for $N_f = 2+1$ and $2+1+1$ are obtained from Refs. [150,213–216] and Refs. [212,217], respectively. The average of $B_{B_{(s)}}$ is obtained from Refs. [148,150,151]. $f_{B_{(s)}}\sqrt{B_{B_{(s)}}}$ is in units of MeV.

N_f	Input	f_B [MeV]	f_{B_s} [MeV]	f_{B_s}/f_B
	Current	188(3)	227(4)	1.203(0.007)
	5 yr w/o EM	188(1.5)	227(2.0)	1.203(0.0035)
2+1+1	5 yr with EM	188(2.4)	227(3.0)	1.203(0.013)
	10 yr w/o EM	188(0.60)	227(0.80)	1.203(0.0014)
	10 yr with EM	188(2.0)	227(2.4)	1.203(0.012)
	Current	192.0(4.3)	228.4(3.7)	1.201(0.016)
	5 yr w/o EM	192.0(2.2)	228.4(1.9)	1.201(0.0080)
2+1	5 yr with EM	192.0(2.9)	228.4(2.9)	1.201(0.014)
	10 yr w/o EM	192.0(0.86)	228.4(0.74)	1.201(0.0032)
	10 yr with EM	192.0(2.1)	228.4(2.4)	1.201(0.012)
N_f	Input	$f_B\sqrt{B_B}$	$f_{B_s}\sqrt{B_{B_s}}$	ξ
	Current	225(9)	274(8)	1.206(0.017)
	5 yr w/o EM	225(4.5)	274(4.0)	1.206(0.0085)
2+1	5 yr with EM	225(5.0)	274(4.8)	1.206(0.015)
	10 yr w/o EM	225(1.8)	274(1.6)	1.206(0.0034)
	10 yr with EM	225(2.9)	274(3.2)	1.206(0.013)
N_f	Input	B_B	B_{B_s}	B_{B_s}/B_B
	Current	1.30(0.09)	1.35(0.06)	1.032(0.036)
	5 yr w/o EM	1.30(0.045)	1.35(0.030)	1.032(0.018)
2+1	5 yr with EM	1.30(0.047)	1.35(0.033)	1.032(0.021)
	10 yr w/o EM	1.30(0.018)	1.35(0.012)	1.032(0.0072)
	10 yr with EM	1.30(0.022)	1.35(0.018)	1.032(0.013)

- Suggests lattice will start including EM corrections within 5 years…
- … 5 years ago

Belle II Physics Book (2019)

General issues regarding isospin breaking effects

- Isospin-breaking (IB) effects are a **small perturbation of hadronic quantities**, generally $\mathcal{O}(1\%)$

- **Two components required**
  1) distinct up and down masses
  2) electromagnetic interactions between quarks

- Required for **precision hadronic physics**

- **Including QED is challenging**. Computing IB effects might not be required for lower precision targets.

Conventions defining pure QCD

- For an observable X one ideally wants an **expansion**

$$X^{\phi} = \bar{X} + X_{\gamma} + X_{\mathrm{SU}(2)}$$

strong isospin-breaking
EM isospin-breaking
iso-symmetric

- A complete set of hadron masses defines X^{ϕ} **unambiguously**

- The separation in 3 contributions requires additional conditions, and is **scheme-dependent**

Radiative corrections to leptonic decays

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Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

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JHEP02(2

Lattice QCD

- Quantum field theory on a **discrete Euclidean space-time**
- Enable **Monte-Carlo estimations** of the path integral
- It is free from weak-coupling approximations
- **Systematic way to compute non-perturbative hadronic quantities**

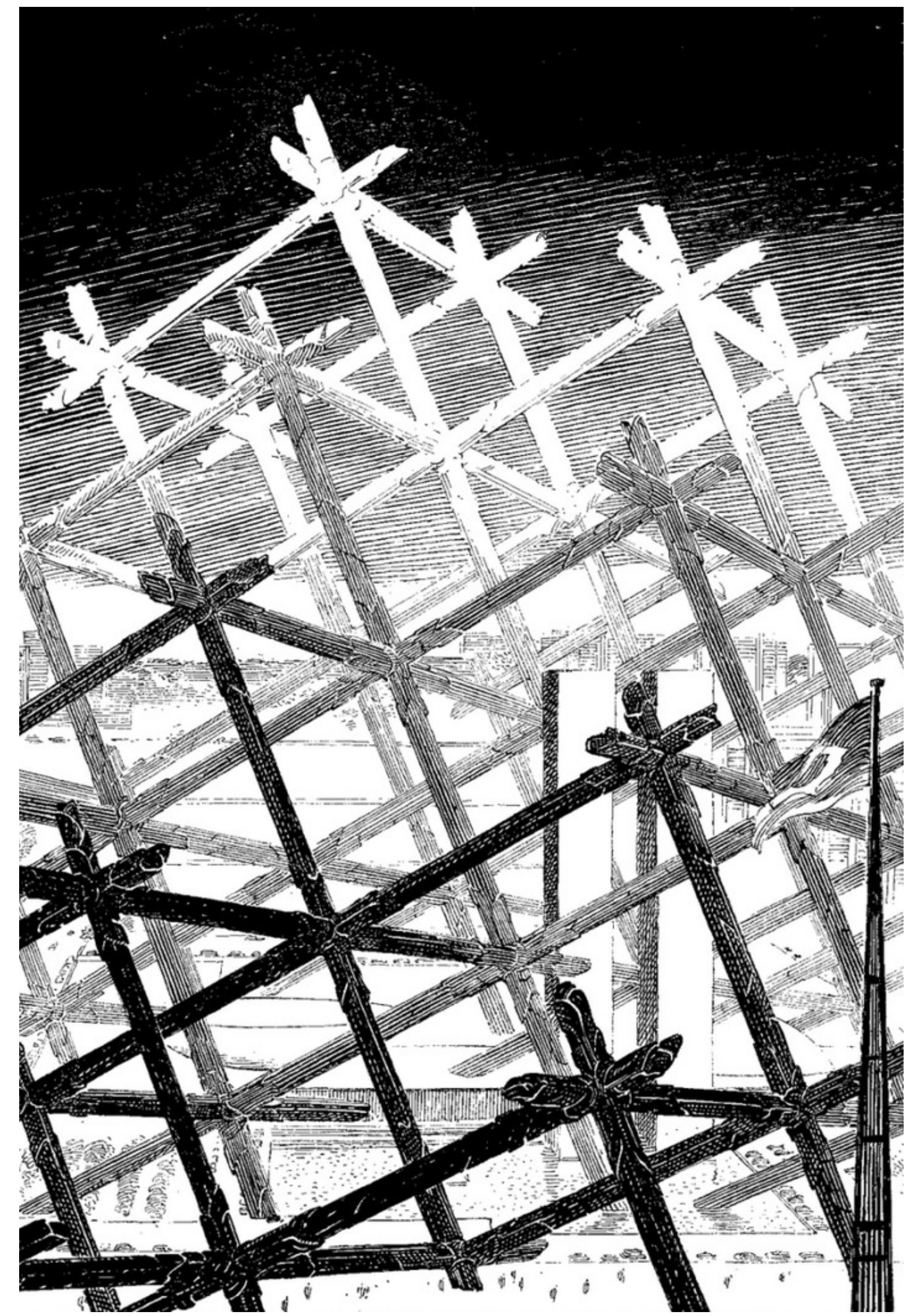
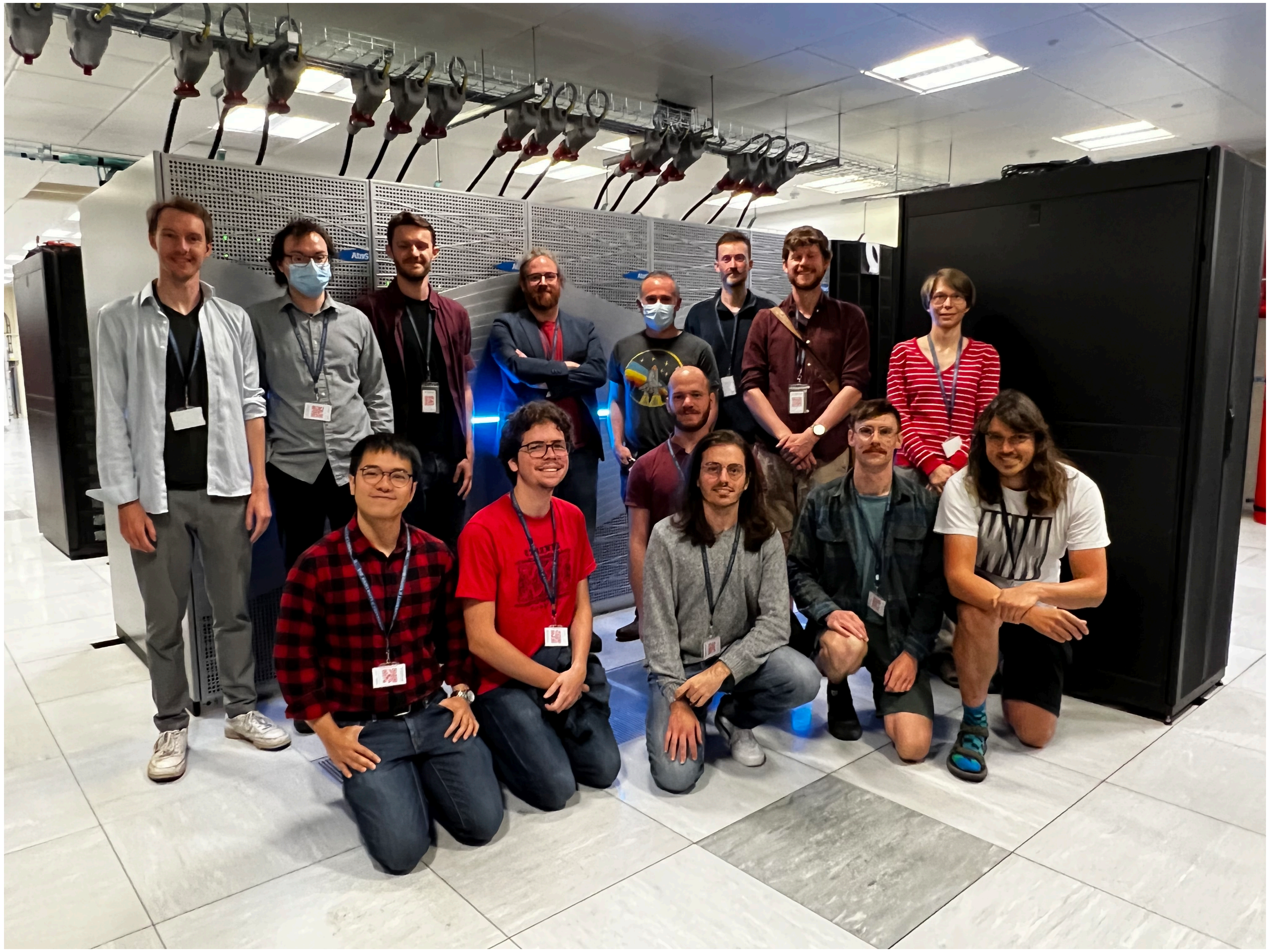


Image credit: Schuiten & Peeters, 1985, Casterman

Our "particle accelerator"

Edinburgh lattice team & Tursa, July 2022



Science and Technology Facilities Council | epcc | DiRAC

RBC/UKQCD physical point ensemble C0

- Möbius domain-wall fermions
- 2+1 flavours at the physical point
- $a \simeq 0.12\ \mathrm{fm}$ and $L^3 \times T = 48^3 \times 96$

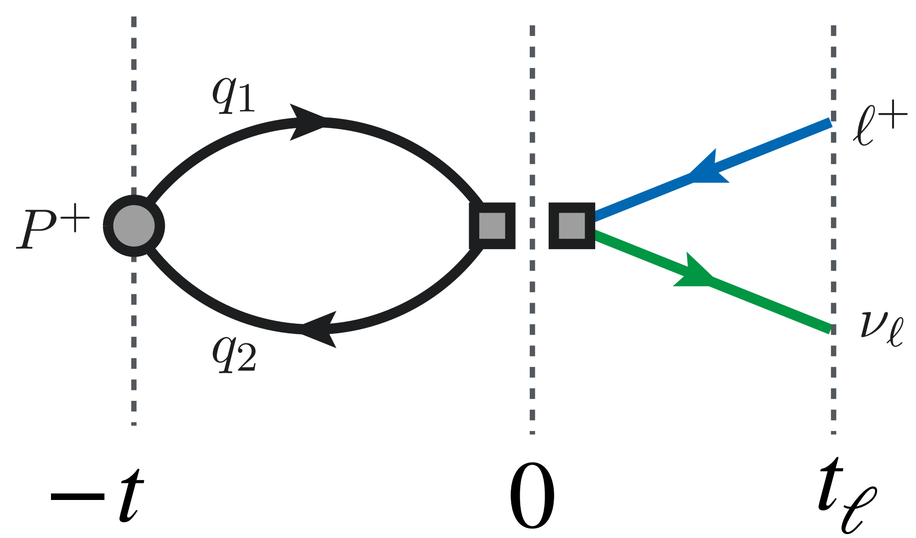
  RBC-UKQCD PRD 93(7), 074505 (2016)

- 60 independent configurations
- 96 measurements per configuration

Euclidean correlation functions

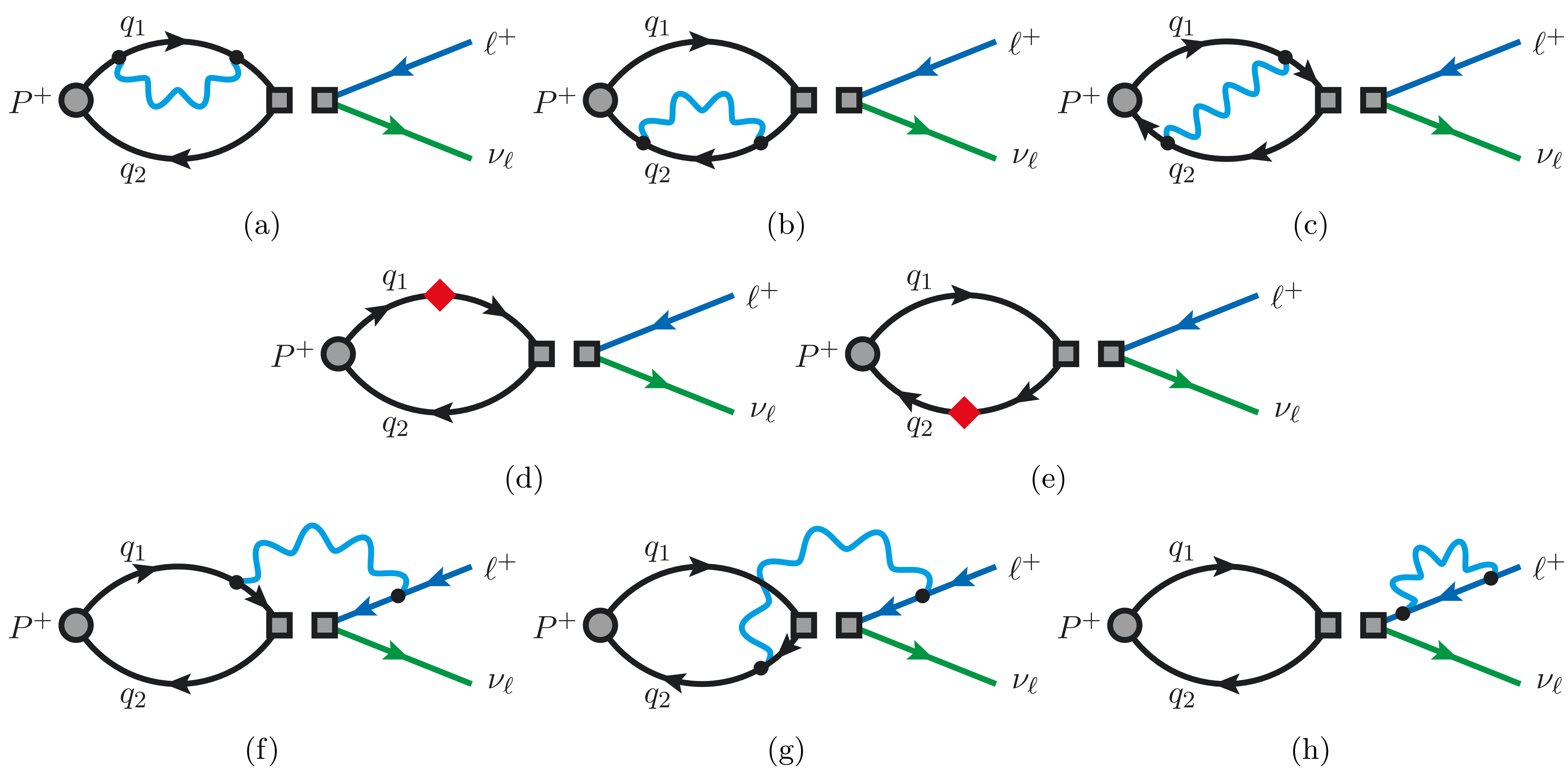
Energies and matrix elements extracted from the large-time behaviour of Euclidean correlation functions

Euclidean time version of LSZ formula

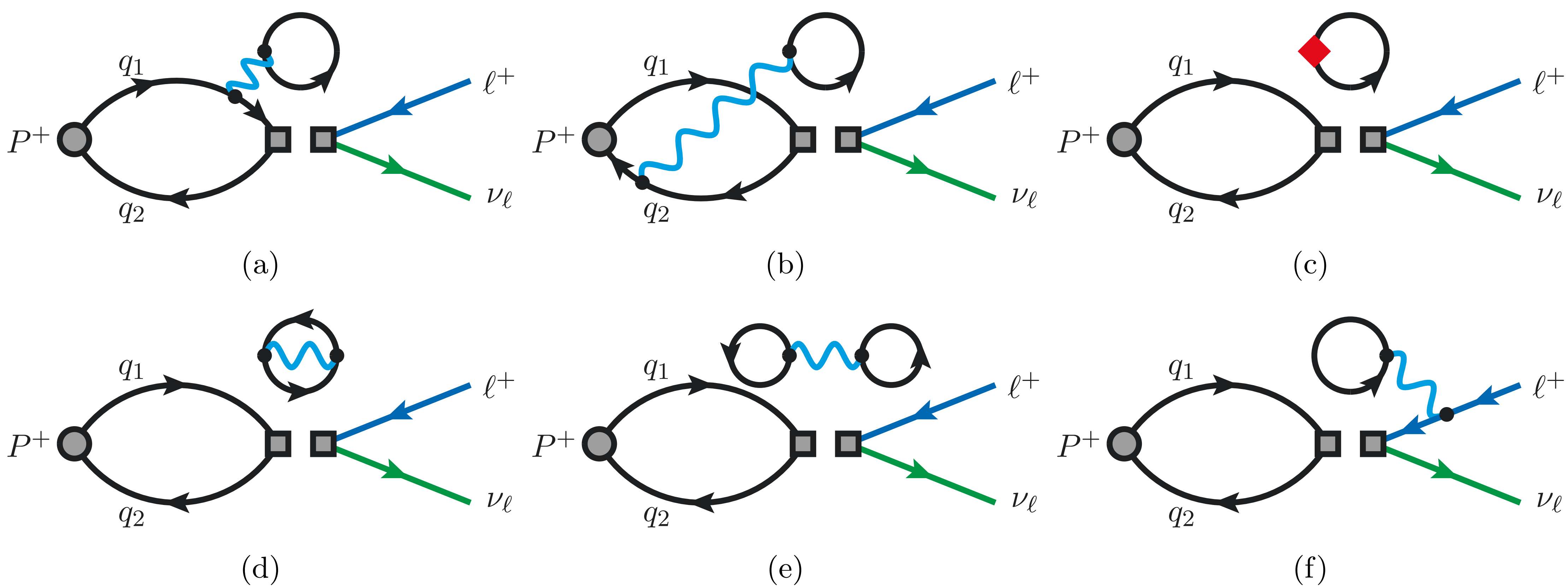


$$C^{(0)}_{\mathrm{P}\ell}(t,t_\ell) = \frac{Z_P\, \mathrm{e}^{-m_P t}\, \mathrm{e}^{-\omega_\ell t_\ell}\, \mathrm{e}^{-\omega_\nu t_\ell}}{8 m_P\, \omega_\ell\, \omega_\nu} \mathscr{A}^{(0)}_P \mathscr{L} + \cdots$$

Quark-connected isospin corrections



Quark-disconnected isospin corrections

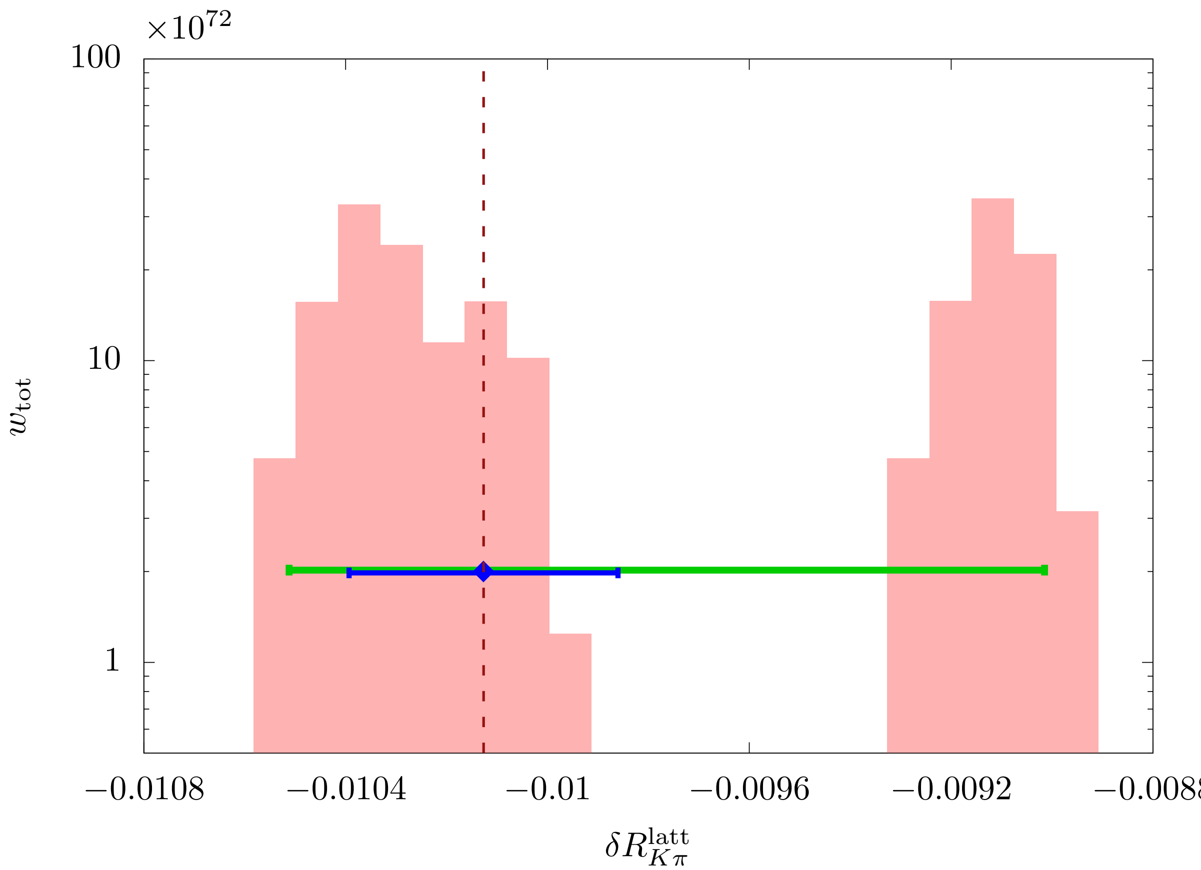


Significant numerical challenge

No computed here (partially quenched calculation)

Data analysis

- $\delta R_{K\pi}$ is predicted from fitting 25 correlators
- Contains fac. and non-fact. corrections, and scale setting
- Genetic selection of 78125 best AIC fits
- Final error budget from AIC-weighted histogram



$$\delta R_{K\pi} = \delta R_K - \delta R_\pi$$

(IB corrections to K and π leptonic decay rate ratio)

Final result

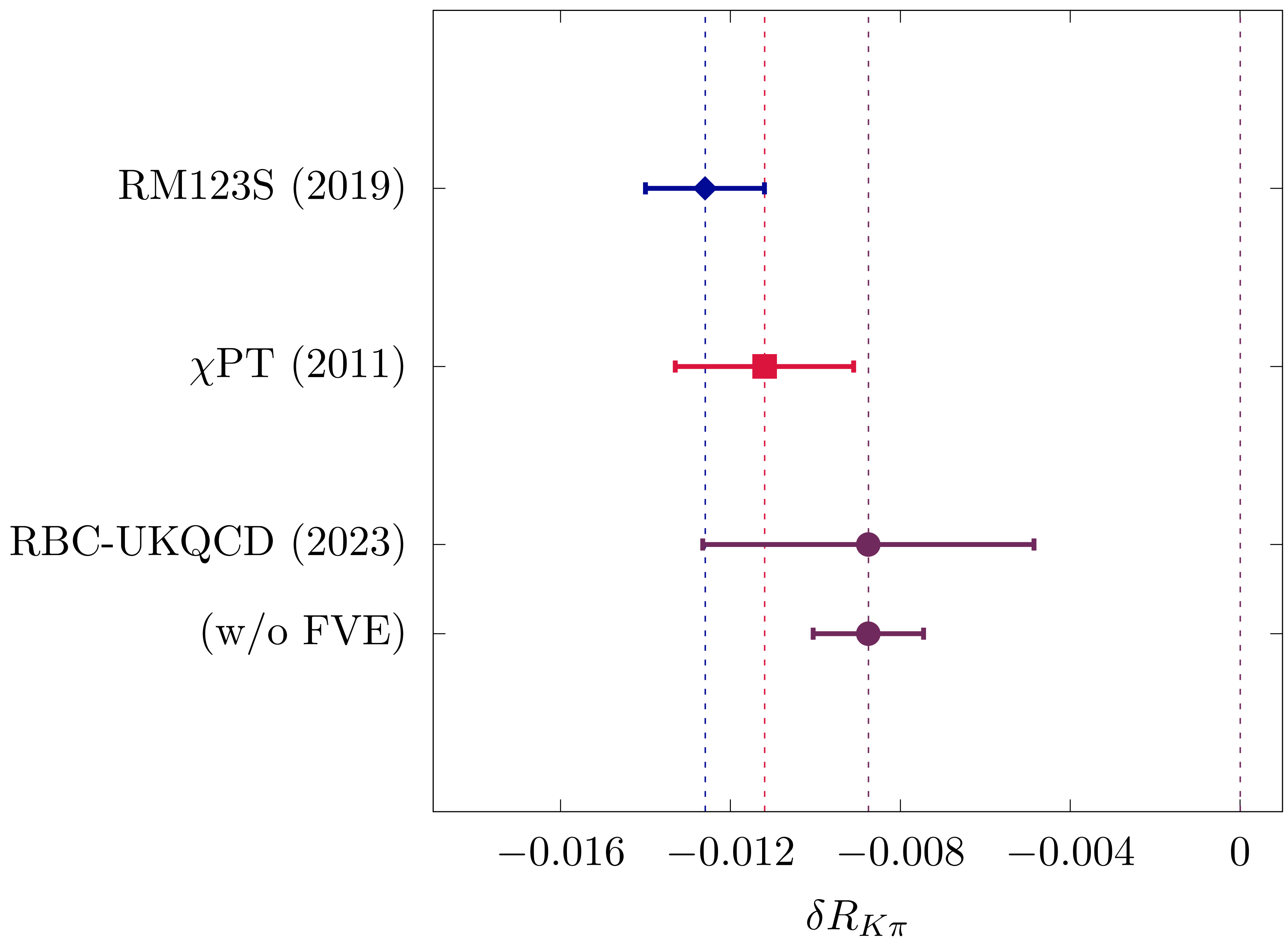
$$\delta R_{K\pi} = -\,0.0086(3)_{\text{stat.}}\left({}^{+11}_{-4}\right)_{\text{fit}}(5)_{\text{disc.}}(5)_{\text{quench.}}(39)_{\text{vol.}}$$

- Error dominated by **finite-volume uncertainties** *(more about that shortly)*

$$|V_{\text{us}}|/|V_{\text{ud}}| = 0.23154(28)_{\text{exp.}}(15)_{\delta R_{K\pi}}(45)_{\delta R_{K\pi},\text{vol.}}(65)_{f_K/f_\pi}$$

- First need better control on volume and f_K/f_π
  Then experimental error dominates

Comparison to other determinations



Finite-volume effects in QED

PHYSICAL REVIEW D **105,** 074509 (2022)

Relativistic, model-independent determination of electromagnetic finite-size effects beyond the pointlike approximation

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(Received 18 November 2021; accepted 8 February 2022; published 27 April 2022)

We present a relativistic and model-independent method to derive structure-dependent electromagnetic finite-size effects. This is a systematic procedure, particularly well-suited for automation, which works at arbitrarily high orders in the large-volume expansion. Structure-dependent coefficients appear as zero-momentum derivatives of physical form factors which can be obtained through experimental measurements or auxiliary lattice calculations. As an application we derive the electromagnetic finite-size effects on the pseudoscalar meson mass and leptonic decay amplitude, through orders $\mathcal{O}(1/L^3)$ and $\mathcal{O}(1/L^2)$, respectively. The structure dependence appears at this order through the meson charge radius and the real radiative leptonic amplitude, which are known experimentally.

DOI: 10.1103/PhysRevD.105.074509

Photon zero-modes

- Photon Green function equation (Feynman gauge)

$$-\Delta G_{\mu\nu}(x) = \delta_{\mu\nu}\delta(x)$$

- *Infinite volume:*
  Laplacian spectrum non-zero a.e., **potentially invertible**

- *Periodic finite-volume:*
  Isolated zero-mode, **non-invertible**

Photon zero-modes

- Finite volume QED loop integrals undefined

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{f(\mathbf{k})}{\mathbf{k}^2} \longmapsto \frac{1}{L^3}\sum_{\mathbf{k}}\frac{f(\mathbf{k})}{\mathbf{k}^2}, \quad \text{with} \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}$$

possibly divergent | isolated $f(0)/0$ term

IR divergences

- $\mathrm{QED_L}$: **remove 3D zero-modes** from photon field

Hayakawa & Uno, PTP 120 413-441 (2008)
BMWc Science 347 1452-1455 (2015)

Non-localities

- $\mathrm{QED_L}$ **non-local in space** (but local in time)
- Potential issues with EFTs and renormalisation
- Alternatives known, $\mathrm{QED_L}$ **most popular choice** so far

Massive photons

Endres, et al. PRL 117(7) 072002 (2016)

C* boundary conditions

Lucini, et al. JHEP02 76 (2016)

Infinite-volume reconstruction

Feng & Jin PRD 100(9), 094509 (2019)

Christ et al. PRD 108(1), 014501 (2023)

Zero-mode regularisation

- In $\mathrm{QED_L}$

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{f(\mathbf{k})}{\mathbf{k}^2} \longmapsto \frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}}\frac{f(\mathbf{k})}{\mathbf{k}^2}$$

- **Finite-volume effects**

$$\Delta'_{\mathbf{k}}\frac{f(\mathbf{k})}{\mathbf{k}^2} = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)\frac{f(\mathbf{k})}{\mathbf{k}^2}$$

- **Soft-photon singularities:** power law in $1/L$ asymptotics

Finite-volume expansion

- Expansion in inverse powers of L, with coefficients

$$c_j(\mathbf{v}) = \Delta'_{\mathbf{n}} \left[\frac{1}{|\mathbf{n}|^j (1 - \mathbf{v} \cdot \hat{\mathbf{n}})} \right]$$

$\Delta'_{\mathbf{n}} = \left(\sum_{\mathbf{n} \neq \mathbf{0}} - \int d^3\mathbf{n} \right)$

$\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$

$\mathbf{v}$: velocity

- For example, **scalar** $\mathrm{QED_L}$ **self-energy FV effects**

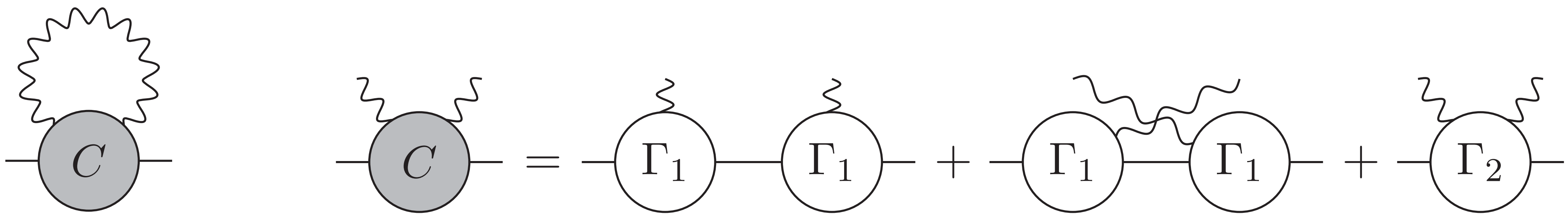
$$\Delta_{\mathrm{FV}} \omega(\mathbf{p})^2 = mq^2 \left[\frac{1}{\gamma(|\mathbf{v}|)} \frac{c_2(\mathbf{v})}{4\pi^2 mL} + \frac{c_1}{2\pi(mL)^2} + \cdots \right]$$

$$\mathbf{v} = \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}$$

Davoudi, AP, et al. PRD99(3), 034510 (2019)

Pseudo-scalar mass corrections in $\mathrm{QED_L}$

$\Sigma(p^2) =$

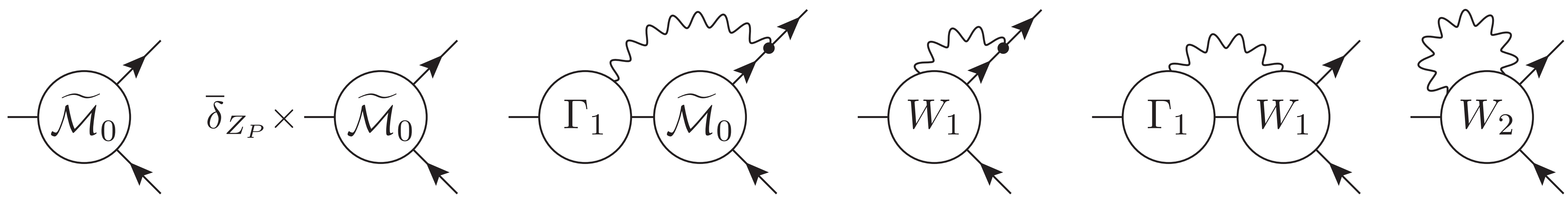


$$\Delta m_P^2(L) = e^2 m_P^2 \left\{ \frac{c_2}{4\pi^2 m_P L} + \frac{c_1}{2\pi (m_P L)^2} - \frac{c_0 \langle r_P^2 \rangle}{3 m_P L^3} - \frac{c_0 \mathscr{C}}{(m_P L)^3} + \mathcal{O}\left[\frac{1}{(m_P L)^4}\right] \right\}$$

- $1/L$ & $1/L^2$ terms are **universal**
- $1/L^3$ term depends on radius and branch-cut contribution
- $1/L^3$ is **purely non-local**
- Higher orders depend on polarisabilities, etc…

Di Carlo, AP, et al. PRD 105(7), 074509 (2022)

Leptonic decay radiative corrections in $\mathrm{QED_L}$



$$\Gamma_0^{(n)}(L) = \Gamma_0^{\text{tree}}\left[1 + 2\frac{\alpha}{4\pi}Y^{(n)}(L)\right] + \mathcal{O}\left(\frac{1}{L^{n+1}}\right)$$

$$\begin{aligned} Y^{(2)}(L) \;=\; & \frac{3}{4} + 4\,\log\left(\frac{m_\ell}{m_W}\right) + 2\,\log\left(\frac{m_W L}{4\pi}\right) + \frac{c_3 - 2\left(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell)\right)}{2\pi} - \\ & -2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{2\pi}\right) + \log\left(\frac{m_\ell L}{2\pi}\right) - 1\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right] + \\ & + \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\left[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\left[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right] \end{aligned}$$

Di Carlo, AP, et al. PRD 105(7), 074509 (2022)

Leptonic decay radiative corrections in $\mathrm{QED_L}$

$$Y^{(2)}(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) + \frac{c_3 - 2\,(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi} -$$

$$-2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{2\pi}\right) + \log\left(\frac{m_\ell L}{2\pi}\right) - 1\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right] +$$

$$+\frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi\,[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

- log & $1/L$ terms **universal**
- $1/L^2$ depends on real radiation form factor F_A

Di Carlo, AP, et al. PRD 105(7), 074509 (2022)

Leptonic decay radiative corrections in $\mathrm{QED_L}$

- New from Lattice 2023: $1/L^3$ contributions

$$\frac{32\pi^2 m_P}{f_P(1-r_\ell^4)(m_P L)^3}\left\{c_0(\mathbf{v}_\ell)\left[F_V - F_A + 2m_P^2 r_\ell^2 A^{(0,1)}(0,-m_P^2)\right] + c_0\mathscr{C}_\ell\right\}$$

Lattice 2023: *Nils Hermansson-Truedsson*

- $\mathscr{C}_\ell$ contains largely unknown branch-cut contributions
- $A^{(0,1)}(0,-m_P^2)$ unknown form factor derivative
- It's ok, *wait a couple of slides…*

$\mathrm{QED_L}$ IR-improvement and $\mathrm{QED_r}$

- Modified QED action, new FV coefficients

$$c_j(\mathbf{v}) = \Delta'_{\mathbf{n}} \left[\frac{1}{|\,\mathbf{n}\,|^j (1 - \mathbf{v} \cdot \hat{\mathbf{n}})} \right] + \sum_{\mathbf{n} \neq \mathbf{0}} \left[\frac{w_{|\mathbf{n}|^2}}{|\,\mathbf{n}\,|^j (1 - \mathbf{v} \cdot \hat{\mathbf{n}})} \right]$$

Davoudi, AP, et al. PRD99(3), 034510 (2019)

- $w_{|\mathbf{n}|^2}$ can be tuned to cancel arbitrary sets of FV coefficients

- Useful choice: $\mathrm{QED_r}$, defined by

$$w_{|\mathbf{n}|^2} = \frac{\delta_{|\mathbf{n}|^2,1}}{6} \qquad \text{which gives} \qquad c_0 = 0$$

Matteo Di Carlo: Lattice 2023 plenary

Consequences of IR improvement

- $\mathrm{QED_r}$ has no $1/L^3$ corrections to the scalar mass
- $\mathrm{QED_r}$ has no $1/L^3$ corrections to the $\pi\pi$ HVP (assuming zero spatial momentum)
- For weak decays it is more complicated because of the presence of $c_0(\mathbf{v}_\ell)$ at $1/L^3$
- More improvement can be done, but will generally require process and kinematics-dependent weights

Davoudi, AP, et al. PRD99(3), 034510 (2019)

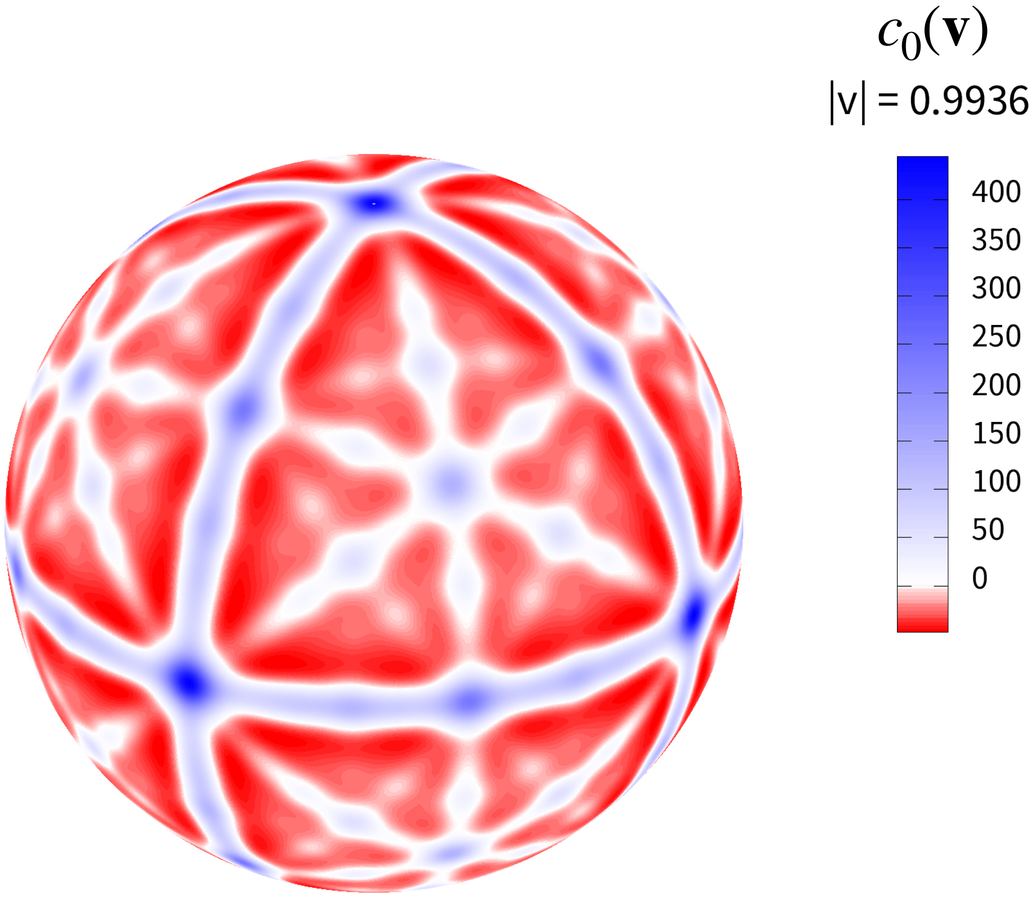
Colinear divergences in finite volume

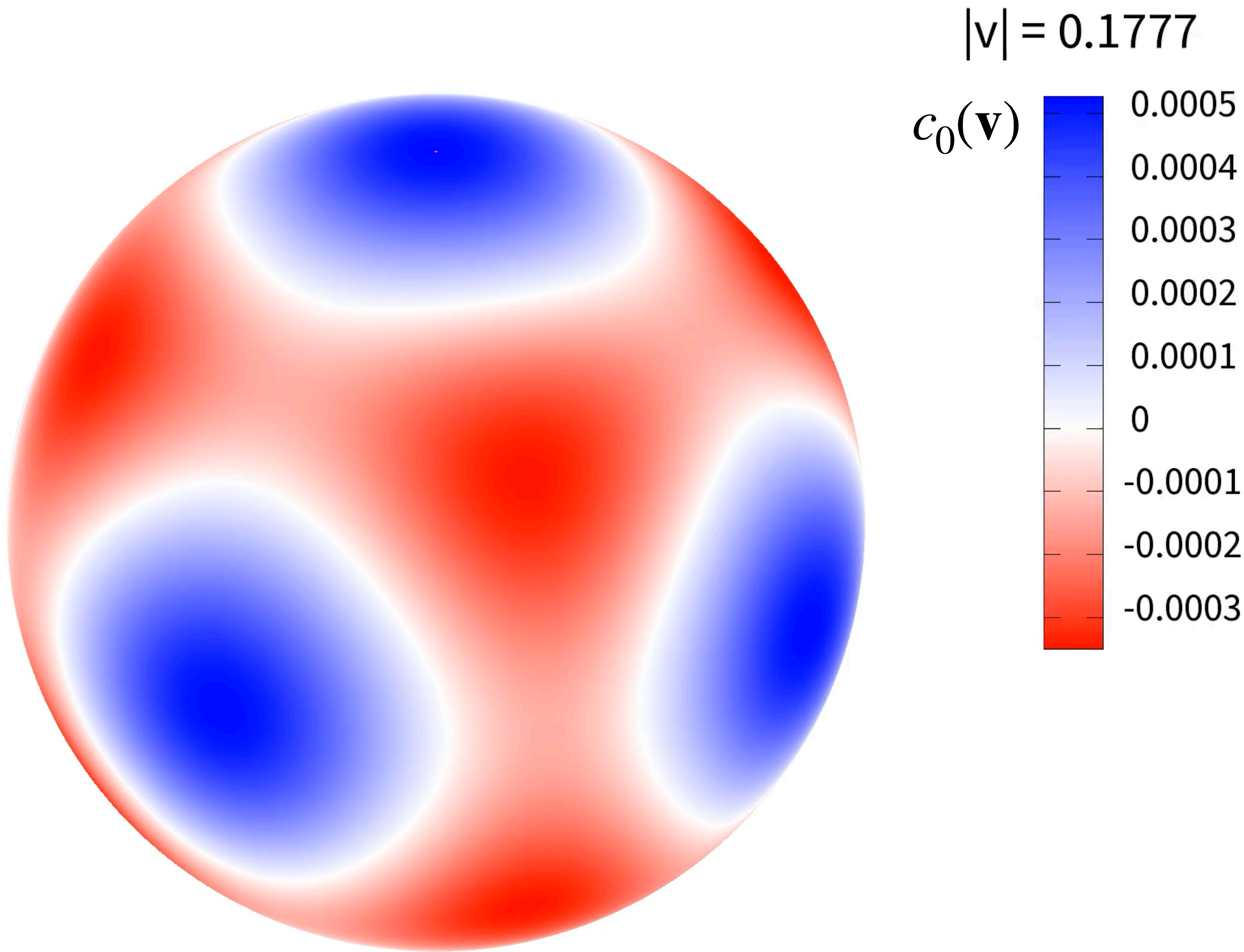
- $c_j(\mathbf{v})$ has a non-trivial angular dependence, and **diverges linearly** with $1 - |\mathbf{v}|$ for $|\mathbf{v}| \to 1$

  AP Lattice 2023

- Relevant for leptonic decays with **ultra-relativistic leptons** in final state (e.g. $D^+ \to \mu^+\nu_\mu$)

- Very different from symmetric, logarithmic behaviour in infinite-volume





|v| = 0.1777
$c_0(\mathbf{v})$
0.0005
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003

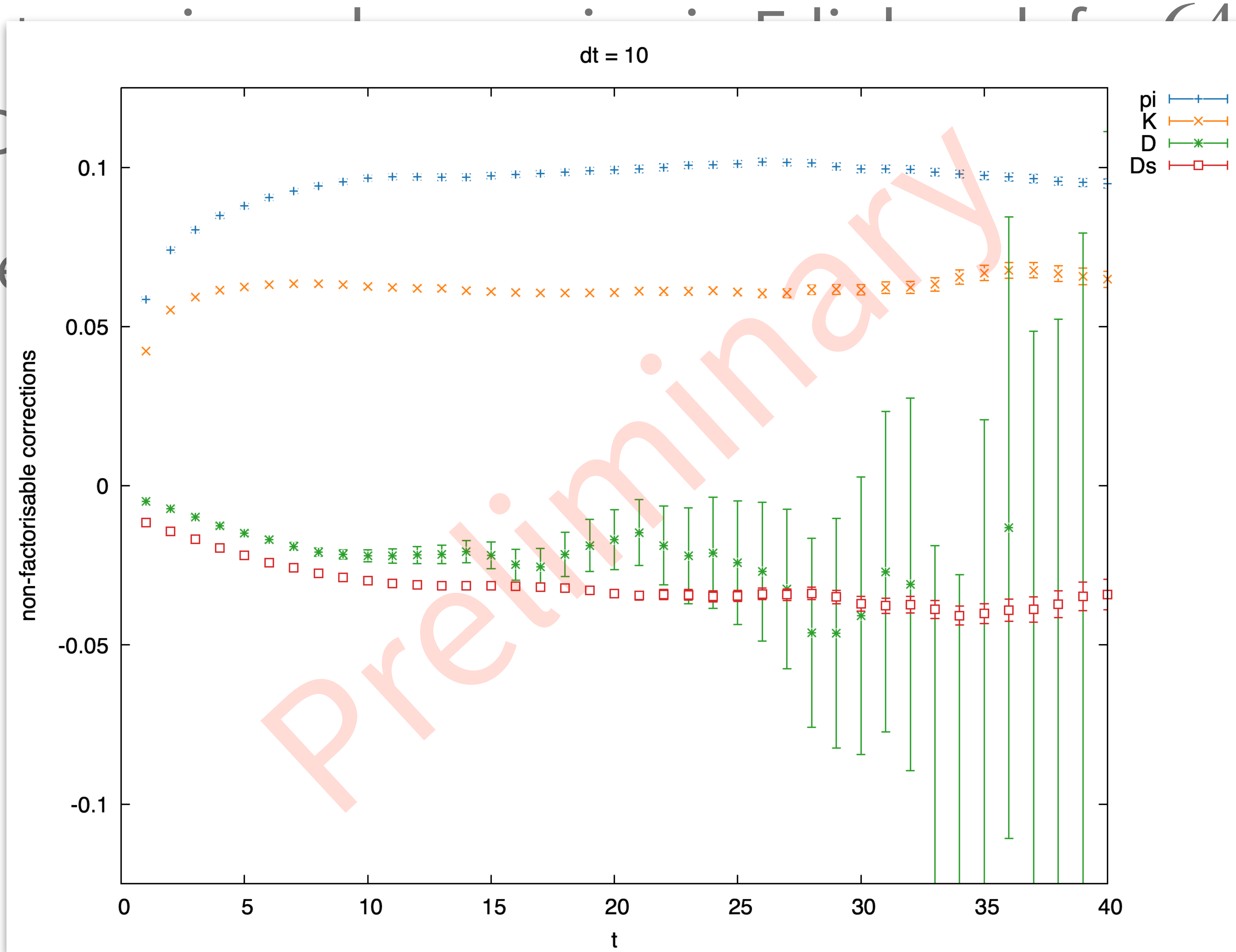
Dealing with $1/L^3$ effects for leptonic decays

- With $\mathrm{QED_r}$, $c_0 = 0$
- Collinear divergences **can be tamed stochastically** averaging momentum direction across measurements (SDA)
- With $\mathrm{QED_r}$, $\langle c_0(\mathbf{v}) \rangle_{\hat{\mathbf{v}}} = 0$
- Alternatively, one can solve $c_0(\mathbf{v}*) = 0$ (**magic angles**)
- **Removes** $1/L^3$ **FV corrections in leptonic decays!**

Outlook

UKQCD current status

- QED$_r$ … 64^3 RBC-UKQCD
- Volume … es
- Discon …



Summary

- Unambiguous and accurate results for radiative corrections to weak meson decays **is crucial for pushing further unitarity tests of the CKM matrix**

- Lattice results **already competitive** for kaons and pions

- Experimental efforts are also required (e.g. NA62/HIKE)

- Lattice should be ready to **move to heavy quarks**

- Recent improvements allow **control of FV effects up to high orders in finite-volume QED**

Hakone 13/04/2024

Thank you! ありがとうございます！

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Edinburgh Consensus on QCD+QED prescriptions

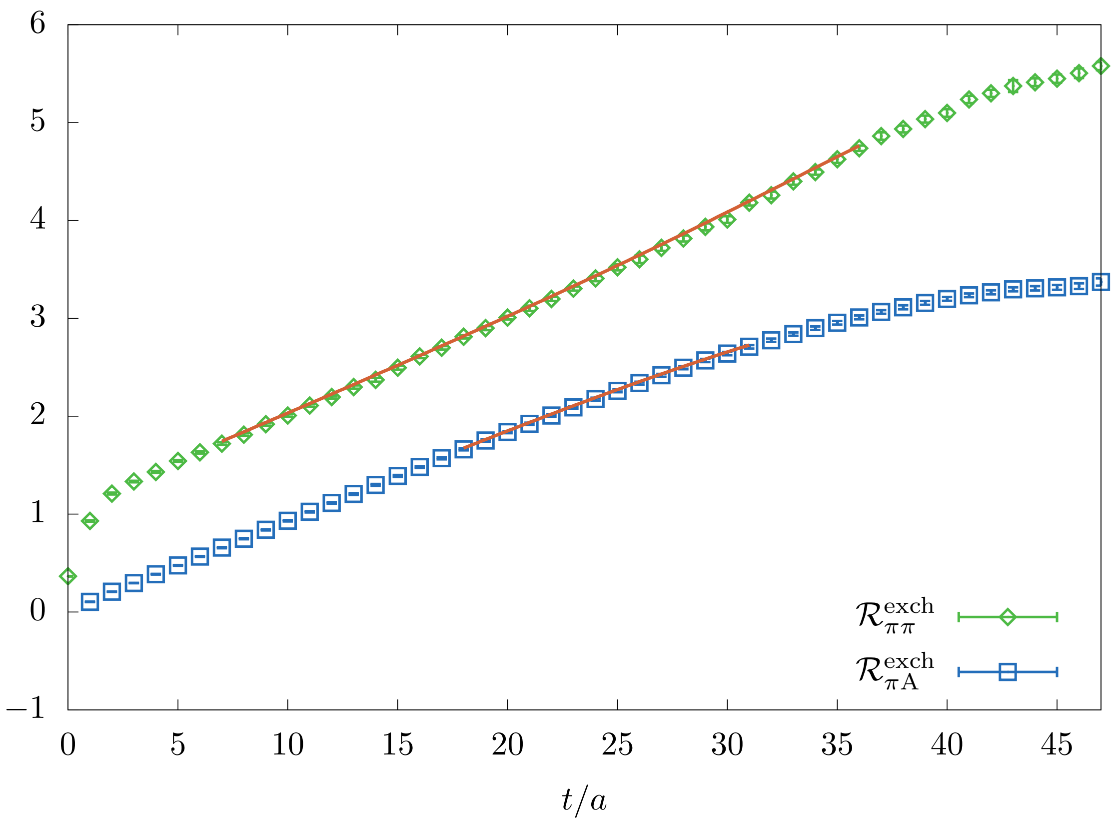
Pure QCD	*Iso-symmetric QCD*
$\hat{M}_{\pi^+} = 135.0 \text{ MeV}$	$\bar{M}_{\pi} = 135.0 \text{ MeV}$
$\hat{M}_{K^+} = 491.6 \text{ MeV}$	$\bar{M}_{K} = 494.6 \text{ MeV}$
$\hat{M}_{K^0} = 497.6 \text{ MeV}$	$\bar{M}_{D_s} = 1967 \text{ MeV}$
$\hat{M}_{D_s} = 1967 \text{ MeV}$	

Scale $\bar{f}_{\pi} = \hat{f}_{\pi} = 130.5 \text{ MeV}$

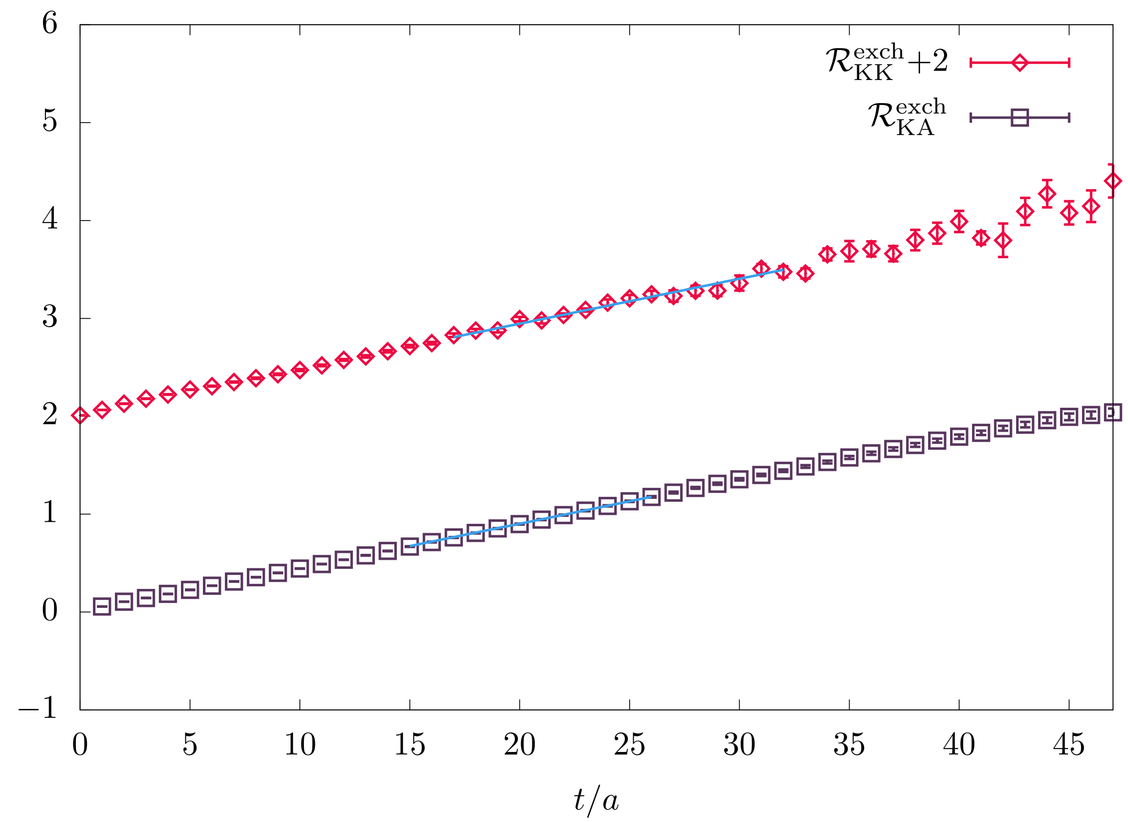
Converging on QCD+QED prescriptions
Edinburgh, 29-31 May 2023

- **Proposed to FLAG and g-2 TI**

Leptonic decays correlation functions examples

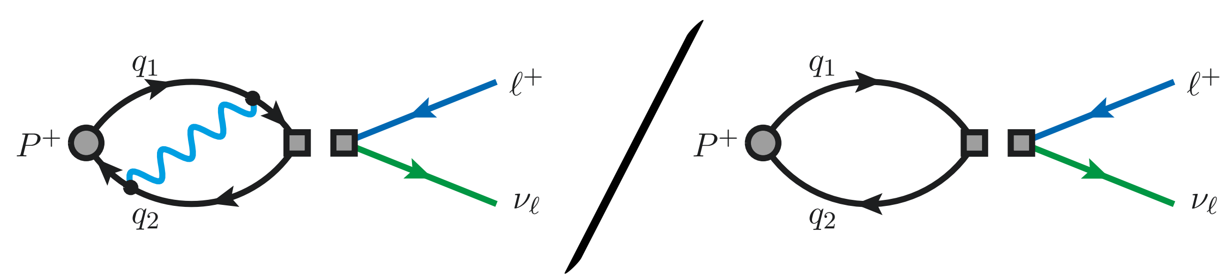


(a) pion



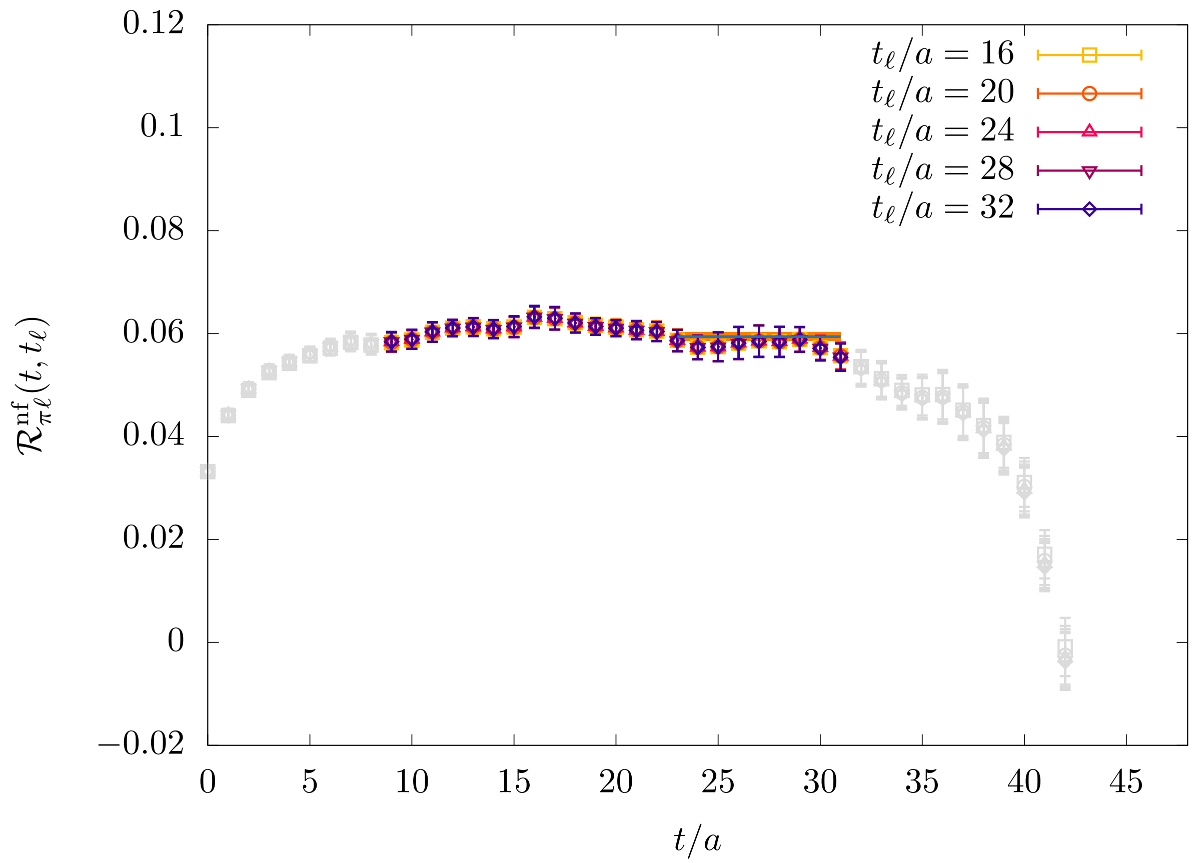
(b) kaon

Ratios

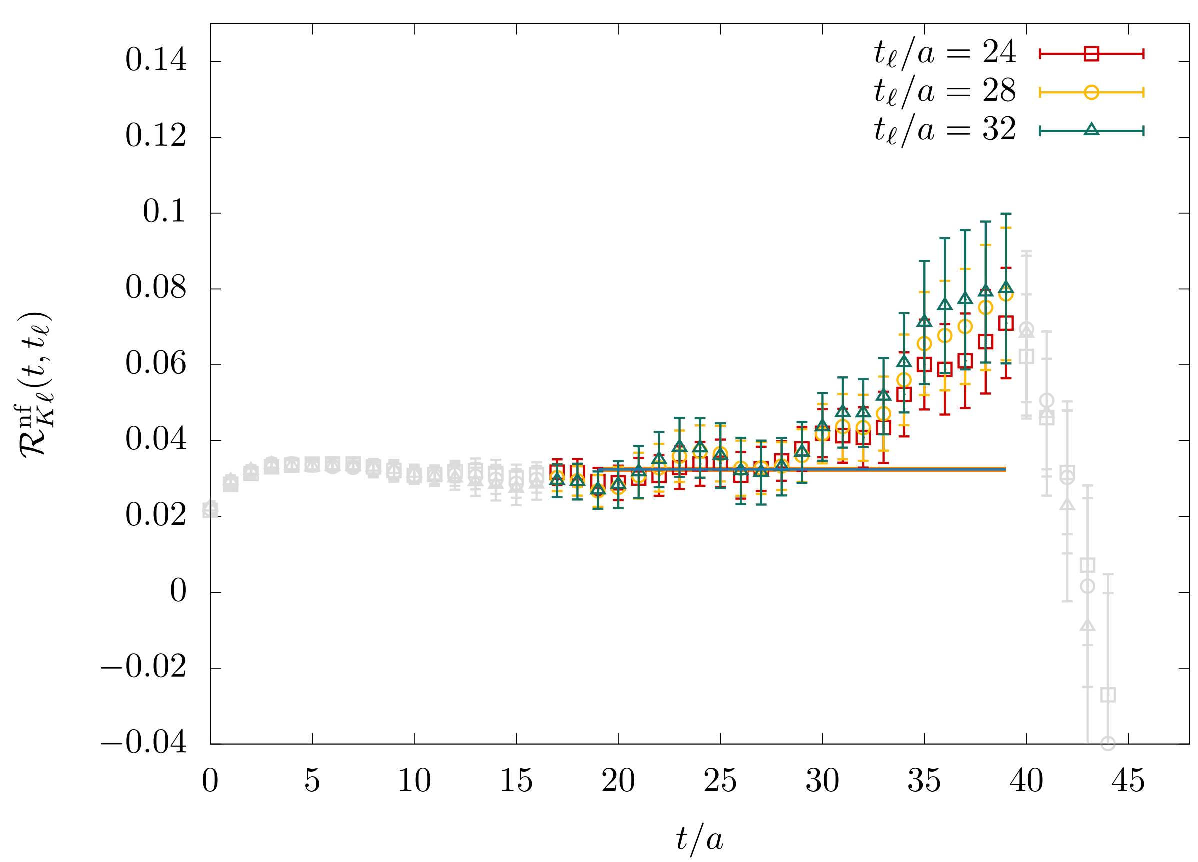


(asymptotic linear in decay with corrections)

Leptonic decays correlation functions examples

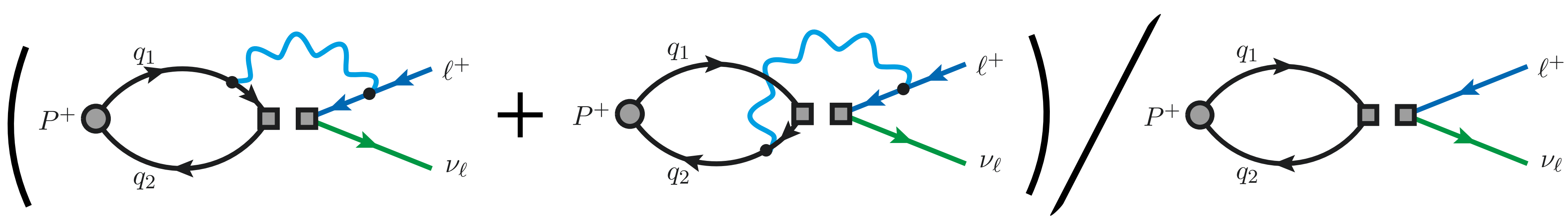


(a) pion



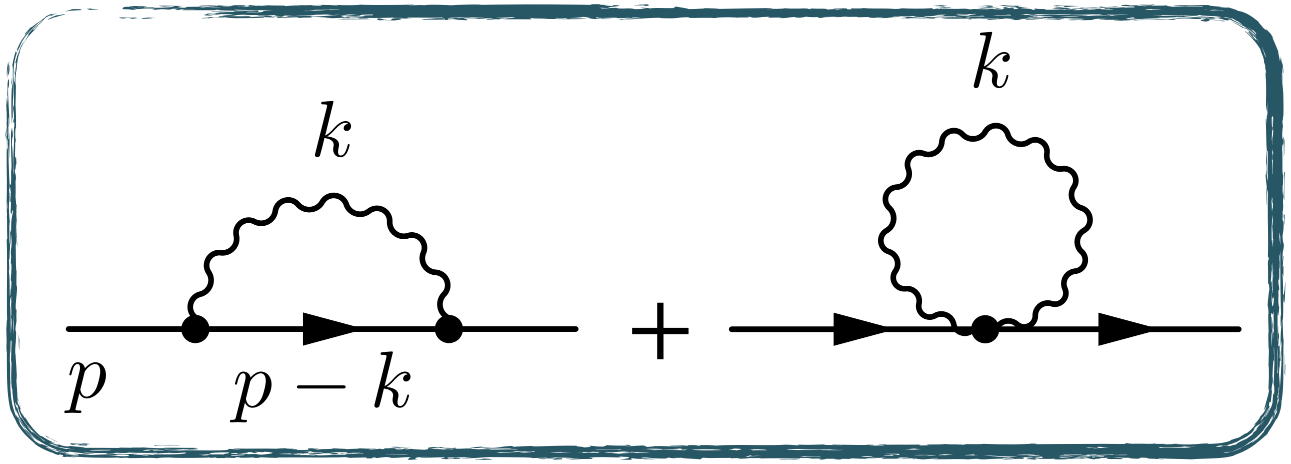
(b) kaon

Ratios



(asymptotically constant in time)

Power-like finite-volume effects: example



$$f(k) = \frac{4}{k^2} - \frac{(2p-k)^2}{k^2[(p-k)^2 + m^2]}$$

$$\int \frac{\mathrm{d}k_0}{2\pi} f(k) = \frac{4m^2\omega_\gamma(\mathbf{k}) + |\mathbf{k}|[-p_0^2 + 3\omega_\gamma(\mathbf{k})^2]}{2\omega(\mathbf{k})|\mathbf{k}|[p_0^2 + \omega_\gamma(\mathbf{k})^2]}$$

$$= \frac{4m^2\omega_\gamma(\mathbf{k}) + |\mathbf{k}|[m^2 + 3\omega_\gamma(\mathbf{k})^2]}{2\omega(\mathbf{k})|\mathbf{k}|[\omega_\gamma(\mathbf{k})^2 - m^2]}$$

$$= \frac{m}{|\mathbf{k}|^2} + \frac{1}{|\mathbf{k}|} + R(\mathbf{k})$$

analytic in $\mathbf{k}$, vanishes at $|\mathbf{k}| = \mathbf{0}$

$$\mathbf{p} = \mathbf{0}$$

$$\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$$

$$\omega_\gamma(\mathbf{k}) = \omega(\mathbf{k}) + |\mathbf{k}|$$

Power-like finite-volume effects: example

- In $\mathrm{QED_L}$, $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ and $\mathbf{k} \neq \mathbf{0}$
- $\Delta'_{\mathbf{k}} = (\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}) = \frac{1}{L^3}\Delta'_{\mathbf{n}}$

$$\Delta_{\mathrm{FV}} m^2(L) = \Delta'_{\mathbf{k}}\left(\frac{m}{|\mathbf{k}|^2} + \frac{1}{|\mathbf{k}|} + R(\mathbf{k})\right)$$

$$= \frac{c_2 m}{4\pi^2 L} + \frac{c_1}{2\pi L^2} + \Delta'_{\mathbf{k}} R(\mathbf{k})$$

- **FV coefficient** $c_j = \Delta'_{\mathbf{n}} |\mathbf{n}|^{-j} = Z_{00}\left(\frac{j}{2}, \mathbf{0}\right)$

Non-localities

- If $f(\mathbf{k})$ is analytic, the sum-integral difference in $\mathbf{k}$ **decays exponentially with** L

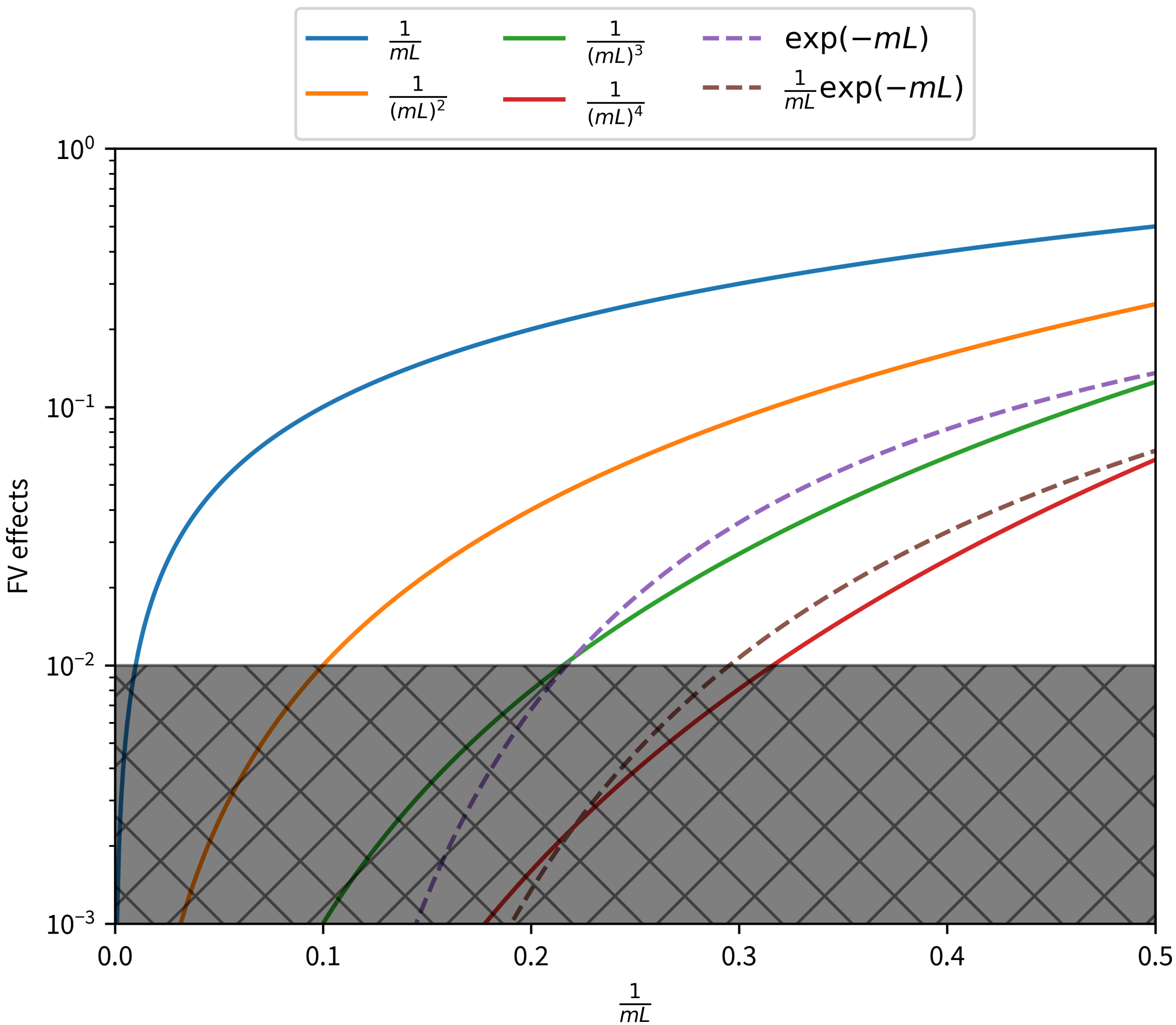
- This is not true in $\mathrm{QED_L}$ because of the missing modes

$$\Delta'_{\mathbf{k}} f(\mathbf{k}) = -\frac{f(\mathbf{0})}{L^3}$$

- Related to FV coefficient $c_0 = \Delta'_{\mathbf{n}}(1) = -1$

- Effects proportional to c_0 are **non-local effects**

Exponential vs power, how much does it matter?



known for masses, g-2 & leptonic decays

exponential similar to powers ≥ 3

unknown NLO IB effects are relevant