The pion–nucleon sigma term from lattice QCD

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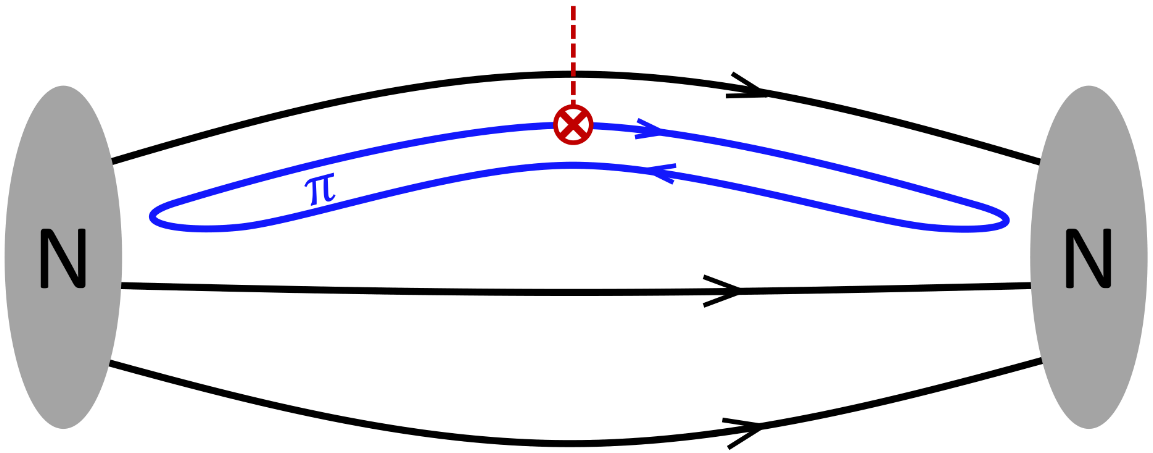

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Reference: PRL 127 (2021) 242002

RIKEN CCS, Kobe, March 27, 2024

$\sigma_{\pi N}$: pion–nucleon sigma term

$$\boxed{\sigma_{\pi N} \equiv m_{ud}\, g_S^{u+d} \equiv m_{ud} \langle N | \bar{u}u + \bar{d}d | N \rangle}$$

- Fundamental parameter of QCD that quantifies the amount of the nucleon mass generated by u and d quarks.
- g_S^2: enters in cross-section of dark matter with nucleons
- Important input in the search of BSM physics

Three methods to calculate $\sigma_{\pi N}$

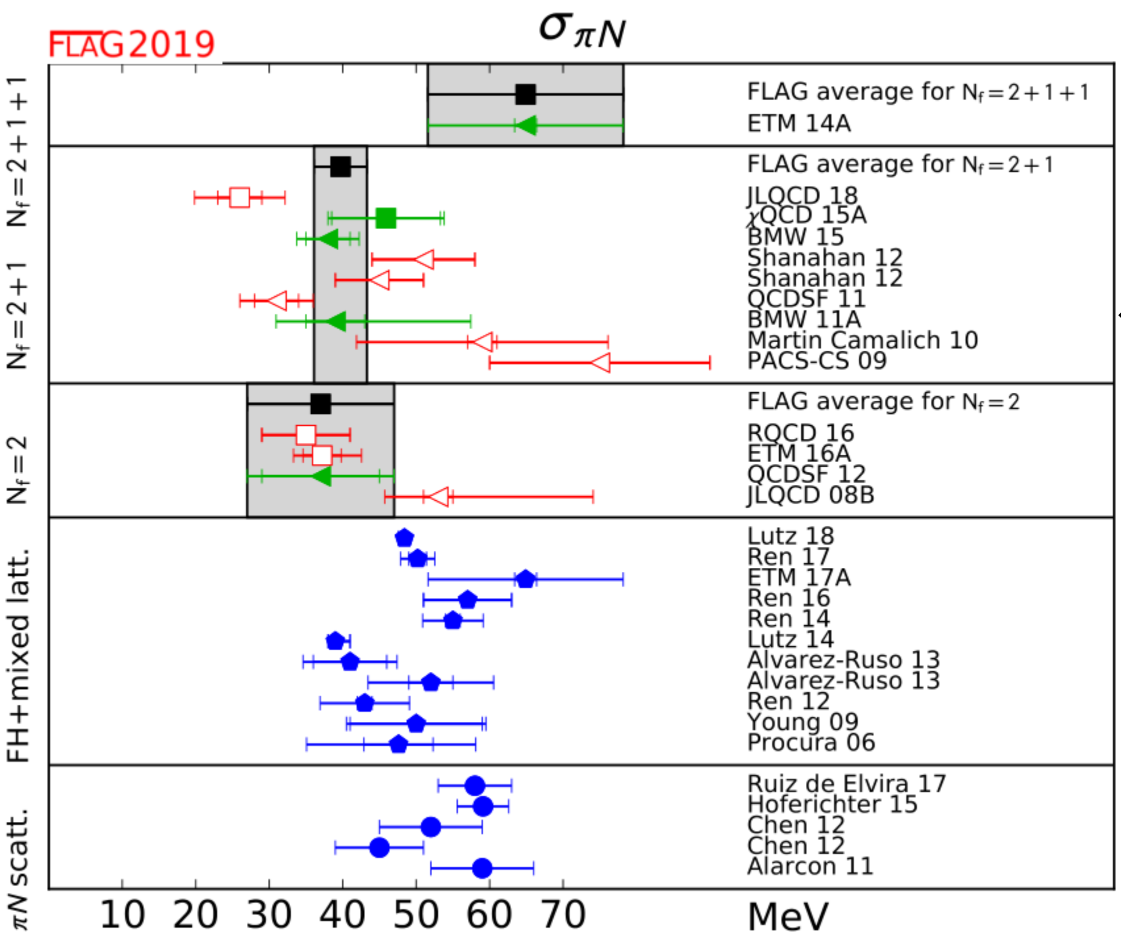
$$\boxed{\sigma_{\pi N} \equiv m_{ud}\, g_S^{u+d} \equiv m_{ud} \langle N | \bar{u}u + \bar{d}d | N \rangle}$$

- Lattice: direct calculation of $\langle N | \bar{u}u + \bar{d}d | N \rangle$
- Lattice: Feynman-Hellmann relation $\left(\frac{g_S^q}{Z_S} = \frac{\partial M_N}{\partial m_q} \right)$
- Phenomenology: connection to πN-scattering amplitude via Cheng-Dashen low-energy theorem

All discussion here assumes isospin symmetry. Effect ~1 MeV

Status of results for $\sigma_{\pi N}$ as of Dec 2020

- $\sigma_{\pi N} \equiv m_{ud} g_S^{u+d} \equiv m_{ud}\langle N|\bar{u}u + \bar{d}d|N\rangle$ in isospin limit



- Lattice
  - With 2+1+1, 2+1, 2-flavors
  - Squares: Direct
  - Triangles: Feynman-Hellmann

- Mixed Lattice: Feynman-Hellmann but taking data from multiple calculations

- Phenomenology: connection to πN-scattering amplitude via Cheng-Dashen low-energy theorem

Tension between lattice and phenomenology

- Lattice results favor ~40 MeV
- Phenomenology favors ~60 MeV
  - $\sigma_{\pi N}$ reduced by 3.1(5) MeV for between M_{π^+} and M_{π^0}

New lattice results post FLAG 2019:

BMW (arXiv:2007.03319) $\sigma_{\pi N}$ = 37.4(5.1) MeV (FH)
ETM (PRD **102**, 054517) $\sigma_{\pi N}$ = 41.6(3.8) MeV (Direct method)
RQCD (PRD **108**, 034512) $\sigma_{\pi N}$ = 42.8(4.7) MeV (Direct method)

Euclidean Field Theory via Wick rotation: $t \to i\tau$

- Converts a QFT into a statistical mechanics system
- Expectation values via path integral $\langle O \rangle = \sum_i O e^{-A}$
- $e^{iE_i t} \to e^{-E_i \tau}$: Spectrum $\{E_i\}$ unchanged under $t \to i\tau$
- Fixed time matrix elements, $\langle j|O|i\rangle$, are the same in Minkowski and Euclidean Time

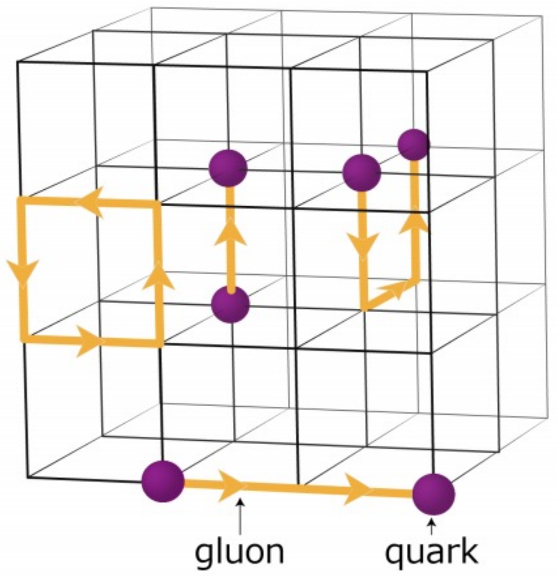
Challenges (complex phase/sign problem)

- Response functions
- Real time dynamics
- Finite chemical potential

Lattice QCD (100^4 4D grid)

Wick rotate the QCD path integral to Euclidean time: $t \rightarrow i\tau$

Discretize QCD on a 4-D hypercubic space-time grid



Links: $U_{x,x+\mu} = e^{-iga\lambda_c A_\mu^c}$ ($\in$ SU(3) matrices)

Sites: $\psi(x)$

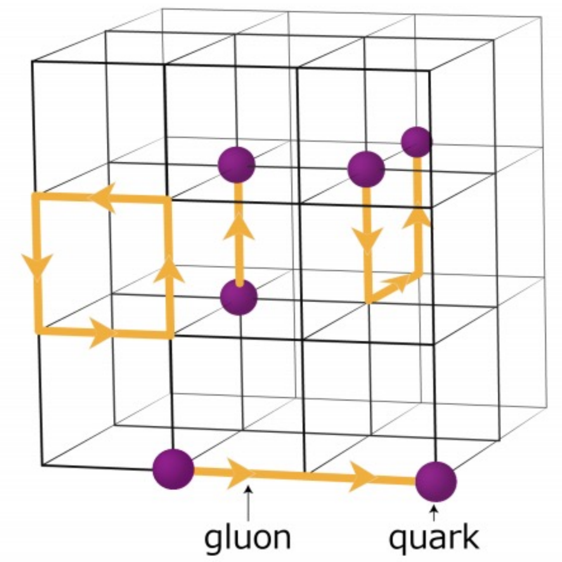
$$S = S_g + S_f = \frac{\beta}{N} Re\, Tr \sum_{x,\mu}(1 - U_p) + S_f$$

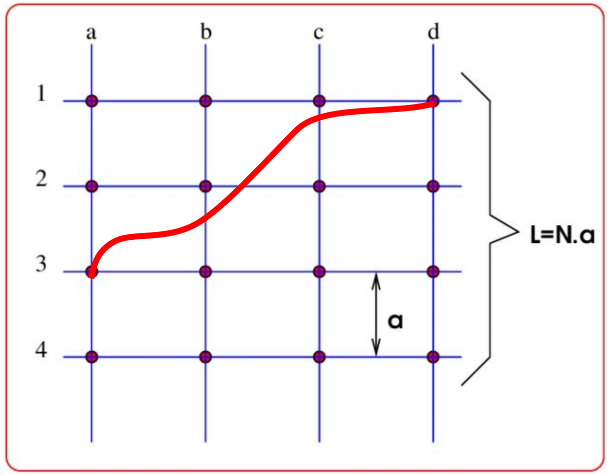
$$S_f = \sum_f \bar{\psi} D_f \psi \equiv \sum_{x,\mu} \bar{\psi}_x \left(U_{x+\mu} + m_f\right) \psi_{x+\mu}$$

- Rules of Grassman integration:
  - $\int d\psi = \int d\bar{\psi} = 0$
  - $\int d\psi \psi = \int d\bar{\psi}\, \bar{\psi} = 1$
- Integrate out the fermions: $S = S_g + \sum_f \text{Ln}\det(D_f[\text{U}])$
- Generate gauge configurations using Boltzmann weight = e^{-S}

(ensemble of configurations is a stochastic representation of the QCD vacuum)

Correlation functions: only links $U_\mu(x)$ and quark propagators P needed





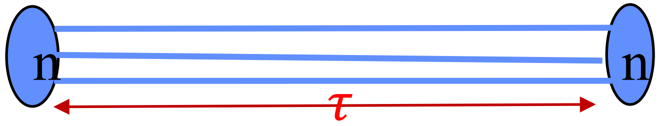
Nucleon 2-point function

$$\Gamma_N^2 = \left\langle \Omega \left| \sum_x \overline{\mathbb{N}}(x,t)\ \mathbb{N}(0,0) \right| \Omega \right\rangle$$

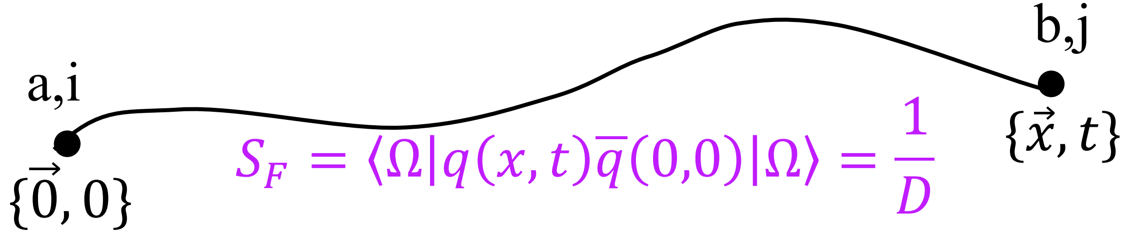
$$= \int d\overline{\psi} d\psi_f\, dU\, e^{-S_f} \{\overline{\mathbb{N}}(x,\tau)\ \mathbb{N}(0,0)\}$$

$$\mathbb{N} = \epsilon^{abc}[q_1^{aT}(x) C\gamma_5 q_2^b(x)] q_1^c(x)$$

- Perform Wick contractions using:
  - → $:\psi_x^{i,a}\, \overline{\psi}_y^{j,b}: = [D^{-1}]_{x;y}^{i,a;j,b} = P$
  - → P = Quark propagator obtained using Krylov solvers for $D\ P = \eta$
- Correlation functions: tie together quark propagators with operators in all possible ways to get quark line diagrams



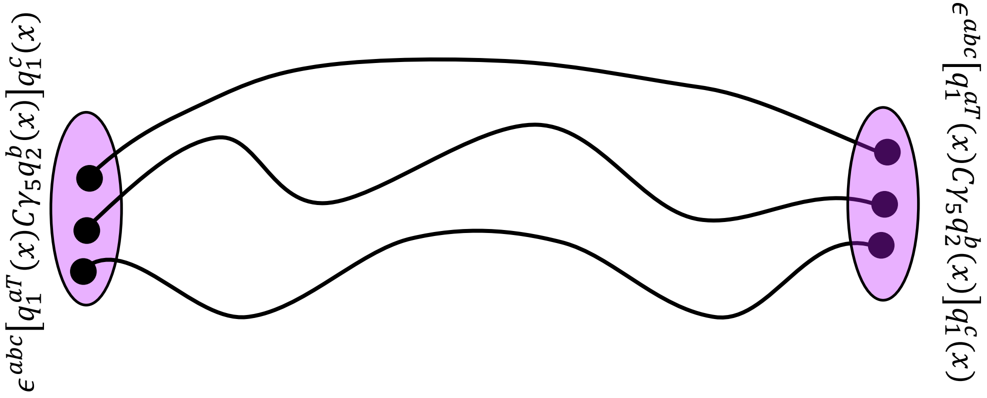
Quark propagator → nucleon 2pt



$$S_F = \langle\Omega|q(x,t)\bar{q}(0,0)|\Omega\rangle = \frac{1}{D}$$

Wick contraction of $\quad \Gamma_N^2 = \left\langle \Omega \left| \sum_x \overline{\mathbb{N}}(x,t)\ \mathbb{N}(0,0) \right| \Omega \right\rangle$

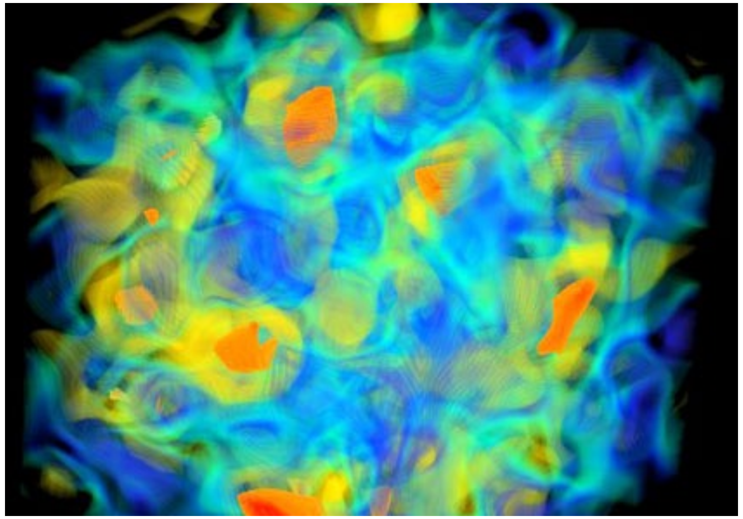
$$= \left\langle \Omega \left| \sum_x \epsilon^{abc}\left[q_1^{aT}(x) C\gamma_5 q_2^b(x)\right] q_1^c(x)\ (x,t)\ \ \epsilon^{def}\,\overline{\left[q_1^{dT}(x) C\gamma_5 q_2^e(x)\right] q_1^f(x)}\ (0,0) \right| \Omega \right\rangle$$



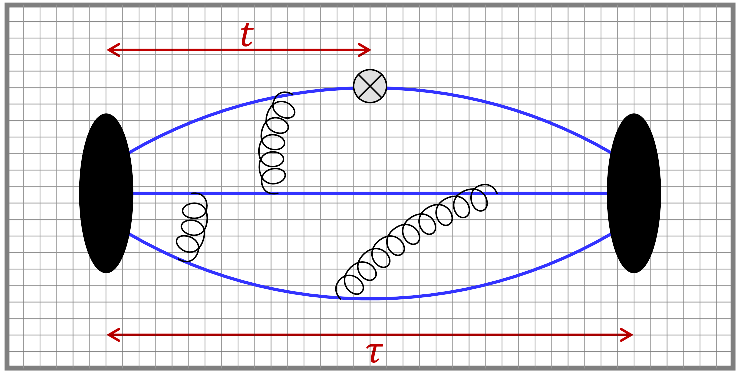
LQCD is QCD (a Quantum Field Theory) discretized on a lattice.

Wick rotation turns QFT into a stochastic computational problem.

Simulations of LQCD provide



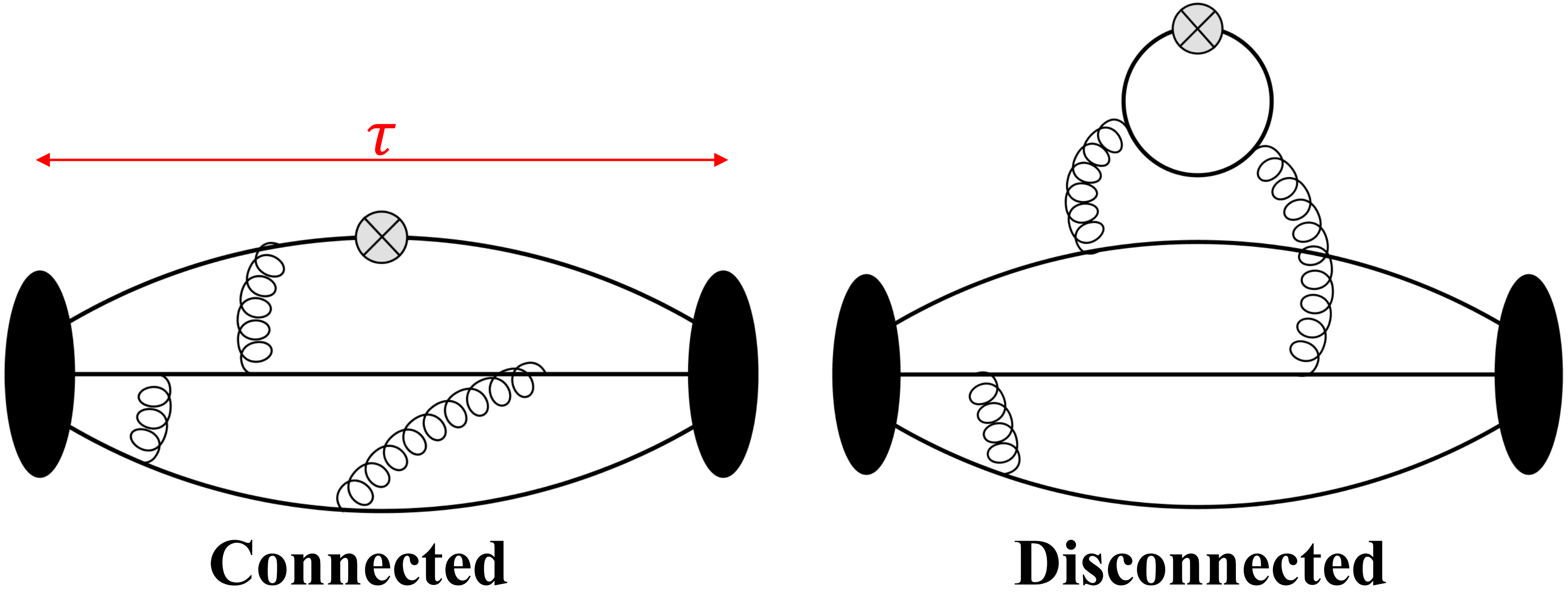
- The quantum vacuum of QCD
  - ➢ensembles of gauge configurations
- Hadrons & interactions are input via external probes
  - ➢N-point correlation functions
- Get quantum wavefunctions of hadronic states
  - ➢Matrix elements: $\langle N(p_f) | \, \mathcal{O}(Q^2) \, | N(p_i) \rangle$



Lattice Methodology is well established

Direct method: Nucleon charges g_A, g_T, and $\boldsymbol{g_S}$ obtained from ME of local quark bilinear operators $\bar{q}_i\,\Gamma\,q_j$ within ground state nucleons: $\langle N|\bar{q}_i\,\Gamma\,q_j|N\rangle$

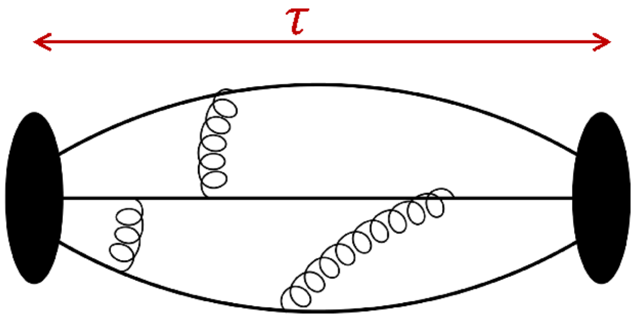
Calculate "connected" and "disconnected" 3-point correlation functions



$$\boldsymbol{g_S^{u+d} = g_S^{u+d,\mathrm{conn}} + 2g_S^{l,disc}}$$

Spectral decomposition of Γ^n

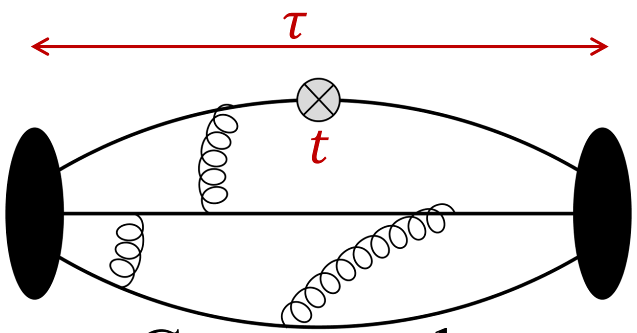
2-point function



$$\langle \Omega \mid \widehat{N_\tau}^\dagger \; \widehat{N}_0 \mid \Omega \rangle$$

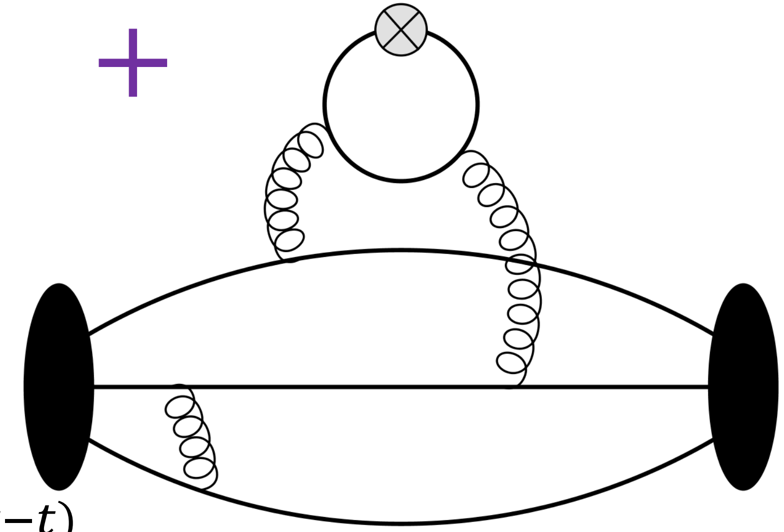
$$\Gamma^{2pt}(\tau) \quad = \sum_i |A_i|^2 e^{-E_i \tau}$$

3-point functions



Connected

+



Disconnected

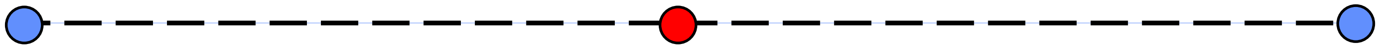
$$\langle \Omega \mid \widehat{N_\tau}^\dagger \, O(t) \, \widehat{N}_0 \mid \Omega \rangle$$

$$\Gamma_O^{3pt}(t,\tau) = \sum_{i,j} A_i^* A_j \langle i | O | j \rangle e^{-E_i t - E_j(\tau - t)}$$

Spectral decomposition of Γ^3

Three-point function for matrix elements of axial current $\mathcal{A}_\mu$

$$\langle\Omega|\mathbb{N}\,\mathcal{A}_\mu(\mathrm{t})\overline{\mathbb{N}}(0)|\Omega\rangle$$



Insert $T = e^{-H\Delta t}\sum_i |n_i\rangle\langle n_i|$ at each Δt with $T\,|n_i\rangle \equiv e^{-H\Delta t}|n_i\rangle = e^{-E_i\Delta t}|n_i\rangle$

$$\langle\Omega|\mathbb{N}(\tau)\cdots e^{-\mathrm{H}\Delta t}\sum_j |n_j\rangle\langle n_j|\,\mathcal{A}_\mu e^{-\mathrm{H}\Delta t}\sum_i |n_i\rangle\langle n_i|\cdots\overline{\mathbb{N}}(0)|\Omega\rangle$$

$$\boxed{\sum_{i,j}\underbrace{\langle\Omega|\mathbb{N}|n_j\rangle}_{A_j^*}\,e^{-\mathrm{E_j}(\tau-t)}\,\underbrace{\langle n_j|A_\mu|n_i\rangle}_{\text{Matrix Elements}}\,e^{-\mathrm{E_i}t}\,\underbrace{\langle n_i|\overline{\mathbb{N}}|\Omega\rangle}_{A_i}}$$

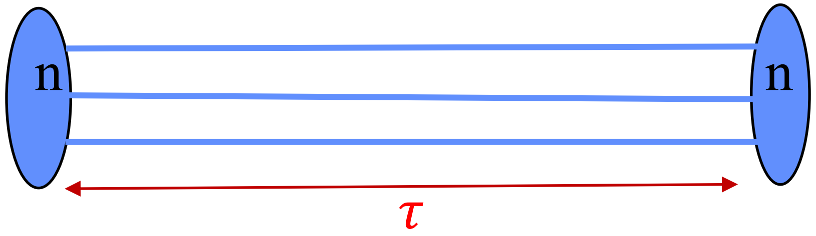
$E_0, E_1, \ldots$ energies of the ground & excited states

$\mathrm{A}_0, \mathrm{A}_1, \ldots$ corresponding amplitudes

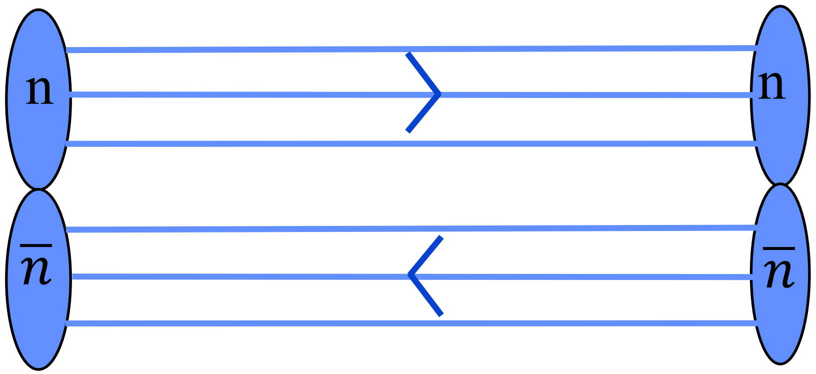
Main issues in lattice calculation

1. Statistical signal
2. Chiral-Continuum-Finite-Volume (CCFV) extrapolation
3. Excited state contributions (ESC)
   a) Towers of multihadron states starting at ~1200 MeV
   b) Which $N\pi$, $N\pi\pi$, ... states contribute to a given ME?
   c) Fits using the spectral decomposition to connected plus disconnected contributions keeping up to 3 states

Signal-to-noise falls as $e^{-(M_N - 1.5M_\pi)\tau}$ in nucleon n-point functions



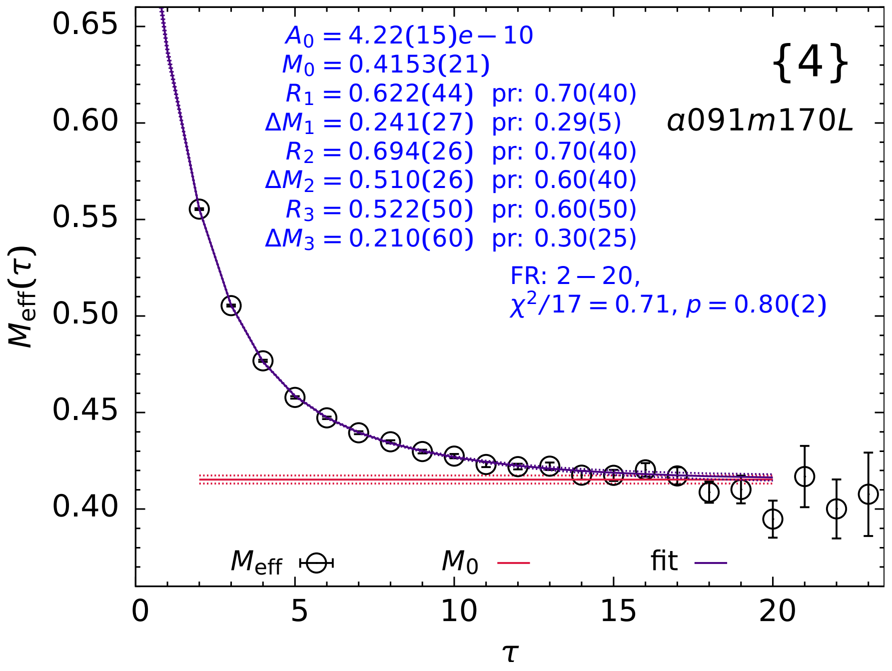
Signal: $\Gamma^2 = e^{-E_N\tau}$



Variance: $e^{-3E_\pi\tau}$

$$\Gamma^{2pt}(\tau) = \sum_i |A_i|^2 e^{-E_i\tau}$$

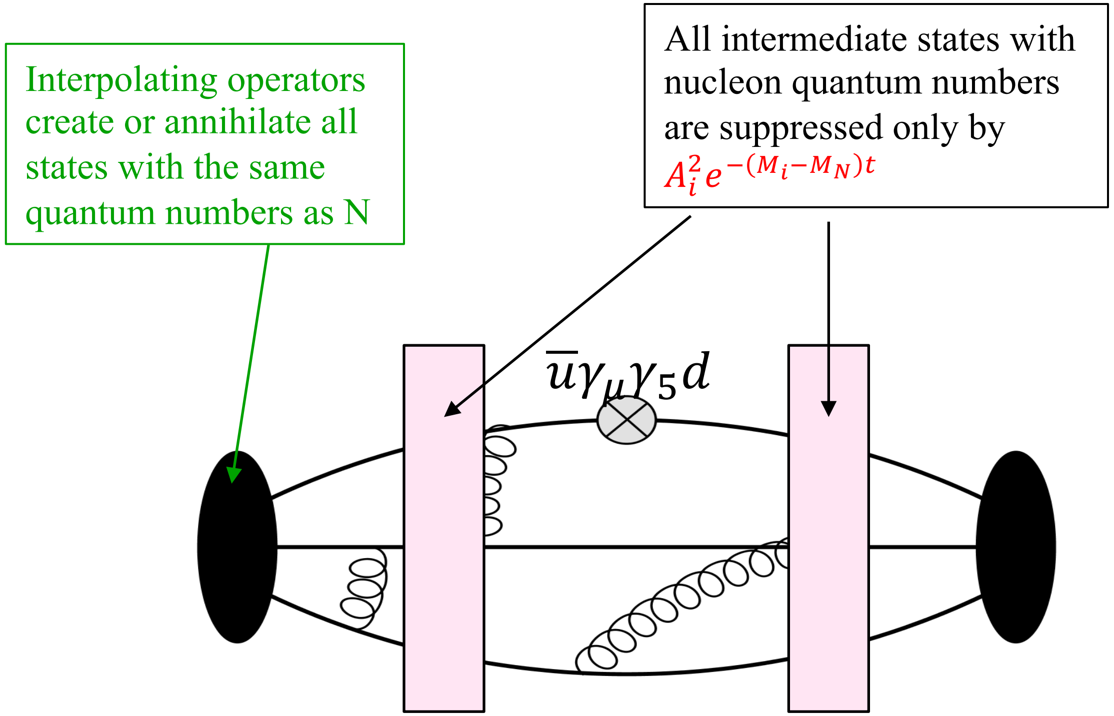
$$M_{eff}(\tau) = \ln\frac{\Gamma^2(\tau)}{\Gamma^2(\tau+1)}$$



➢ To resolve a *small* mass gap $(M_1 - M_0)$ *requires* large *t*

Excited states in correlation functions

Challenge: To get the matrix elements in the ground state of hadrons (nucleons), the contributions of all excited states have to be removed.



Towers of multihadron states

$$N(\vec{p})\pi(-\vec{p})$$
$$N(0)\pi(\vec{p})\pi(-\vec{p})$$
$$N(\vec{p})2\pi(-\vec{p})$$
$$\cdots$$

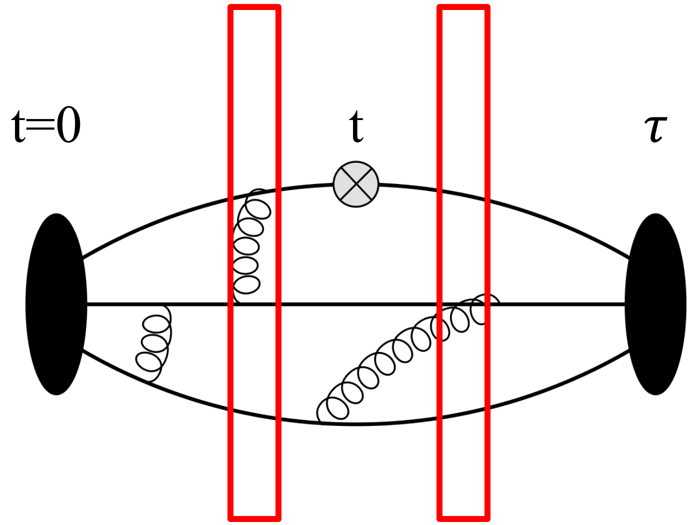
\+ radial excitations

- All states with same quantum numbers as the nucleon are allowed
- Which excited states make significant contributions to a given matrix element?
- What are their energies in a finite box?

Spectral decomposition of 3-point function

$$C^{3pt} = \langle 0|\mathcal{O}|0\rangle |A_0|^2 e^{-M_0\tau} \times \Big[1 + \frac{\langle 1|\mathcal{O}|1\rangle}{\langle 0|\mathcal{O}|0\rangle}\frac{|A_1|^2}{|A_0|^2} e^{-\Delta M_1 \tau} + \frac{\langle 2|\mathcal{O}|2\rangle}{\langle 0|\mathcal{O}|0\rangle}\frac{|A_2|^2}{|A_0|^2} e^{-(\Delta M_2+\Delta M_1)\tau} + \frac{\langle 0|\mathcal{O}|1\rangle}{\langle 0|\mathcal{O}|0\rangle}\frac{|A_1|}{|A_0|} e^{-\Delta M_1 \frac{\tau}{2}} \times 2\cosh\left(\Delta M_1 (t - \frac{\tau}{2})\right) + (\cdots)$$

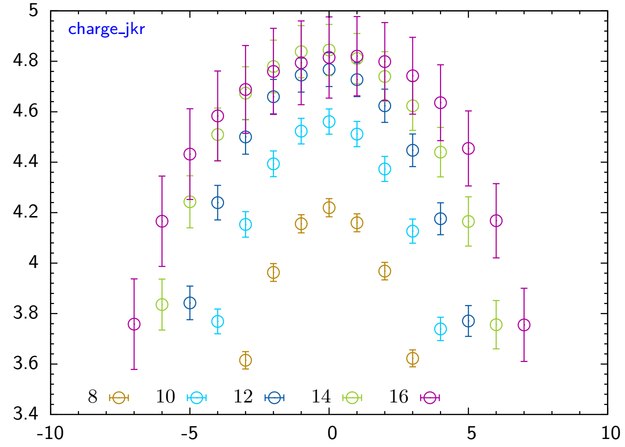
Ground-state matrix element → g_S



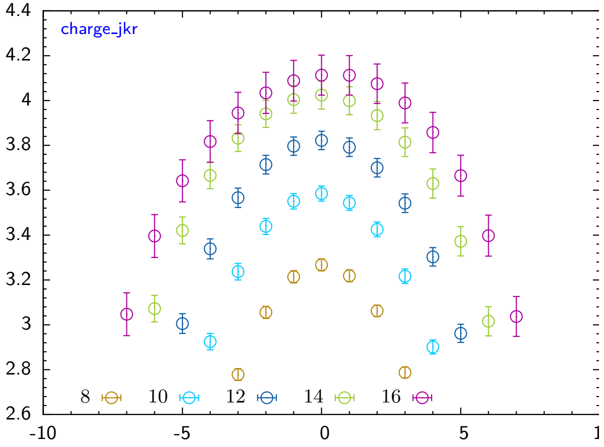
- The transition ME are large in some cases!
- Mass gaps, ΔM_i, of $N\pi, N\pi\pi$, ... are small!
- Need A_0, M_i

Data from physical $M_\pi \approx 135\ MeV$ ensemble @ $a \approx 0.09 fm$, $M_\pi L = 3.9$

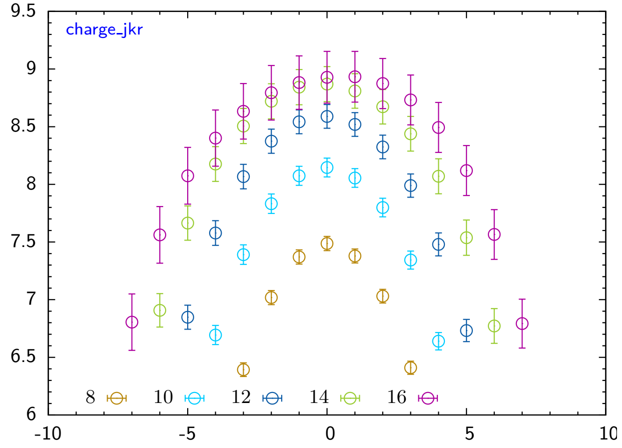
$g_S^{u,conn}$



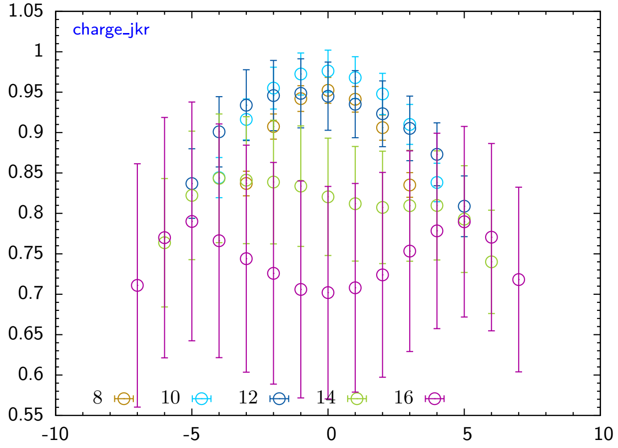
$g_S^{d,conn}$



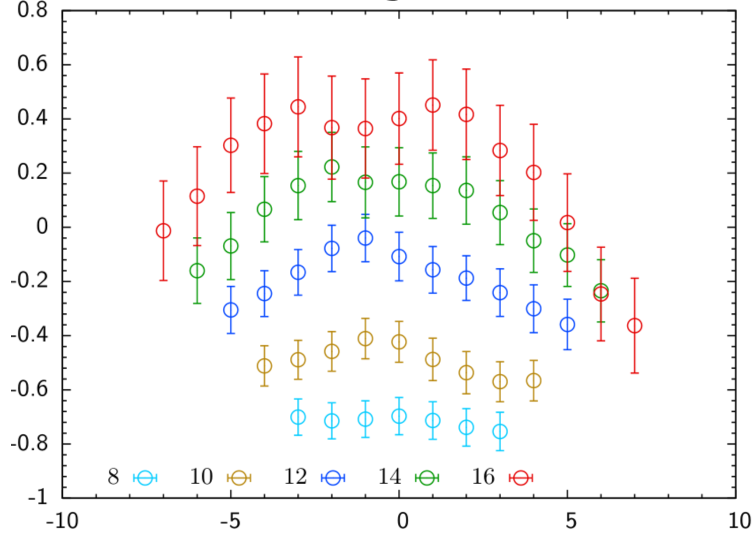
$g_S^{u+d,conn}$



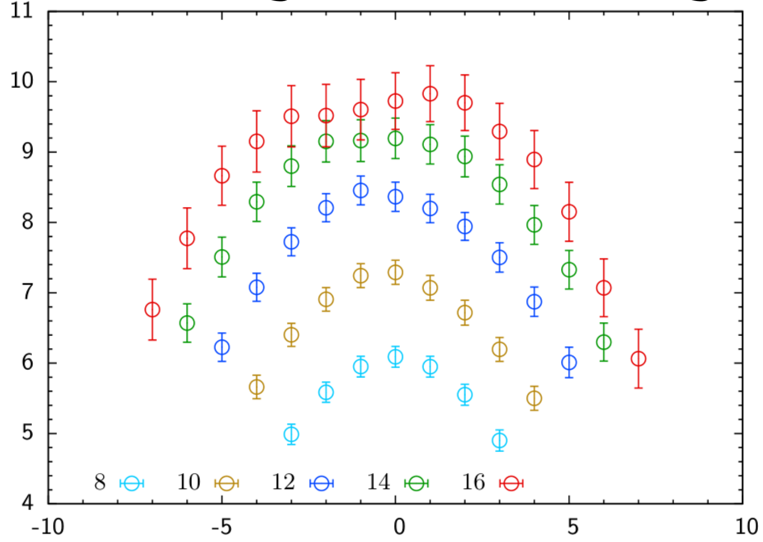
g_S^{u-d}



$g_S^{l,disc}$



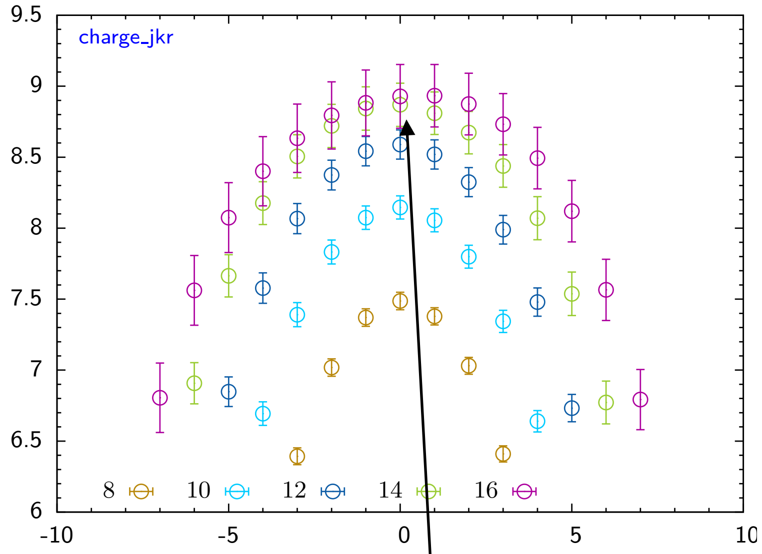
$$g_S^{u+d} = g_S^{u+d,\mathrm{conn}} + 2 g_S^{l,\mathrm{disc}}$$



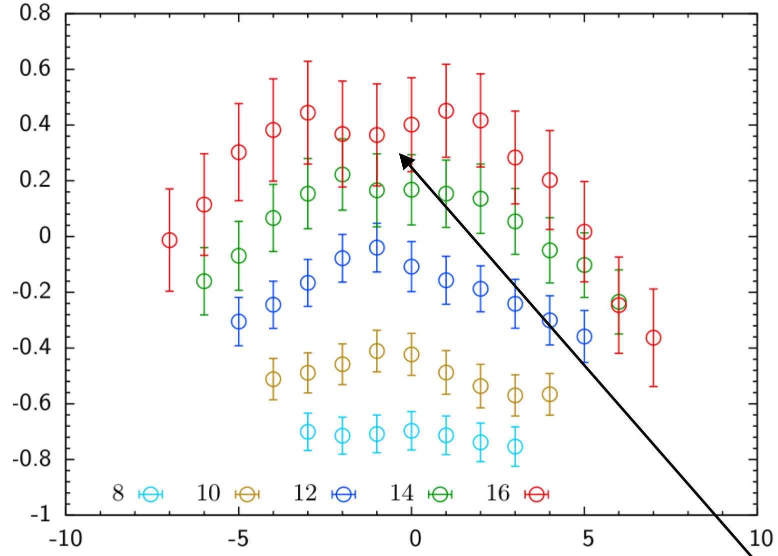
PRL 127 (2021) 242002 . Also See S. Park, *et al.*, 2401.00072

Physical M_π Ensemble: $a \approx 0.09fm,\ M_\pi = 135\ MeV,\ M_\pi L = 3.9$

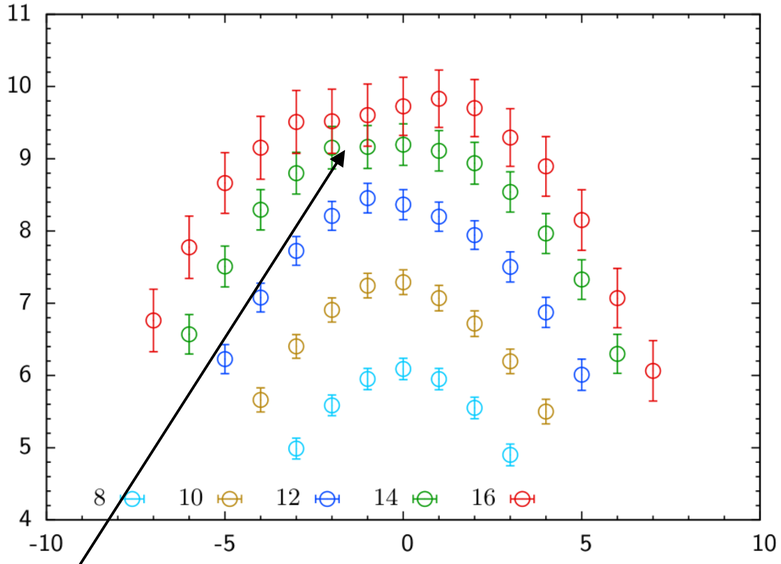
$g_S^{u+d,conn}$



$g_S^{l,disc}$



$g_S^{u+d} = g_S^{u+d,\mathrm{conn}} + 2g_S^{l,\mathrm{disc}}$



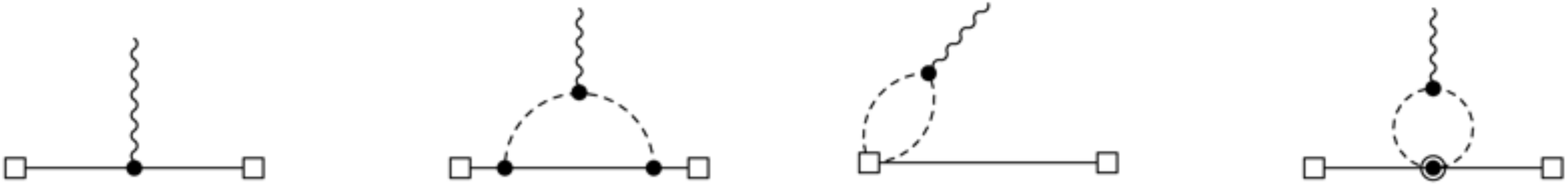
"Appears" to converge as $\tau \to \infty$

"No" convergence as $\tau \to \infty$

Excited-state contribution is large at realizable $\tau \approx 1.5$ fm

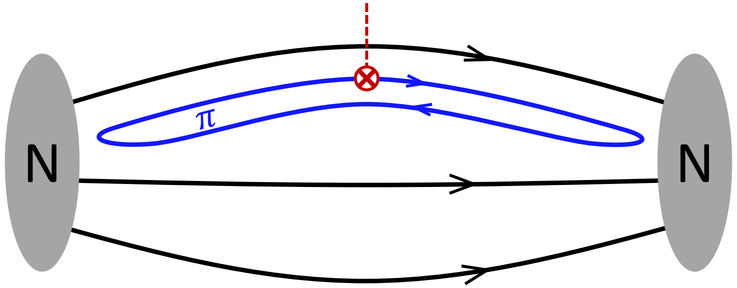
χPT analysis shows $\boldsymbol{N(\vec{k})\pi(-\vec{k})}$ and $\boldsymbol{N(0)\pi(\vec{k})\pi(-\vec{k})}$ states give significant contributions.

Coupling of S to $\boldsymbol{\pi\pi}$ is large

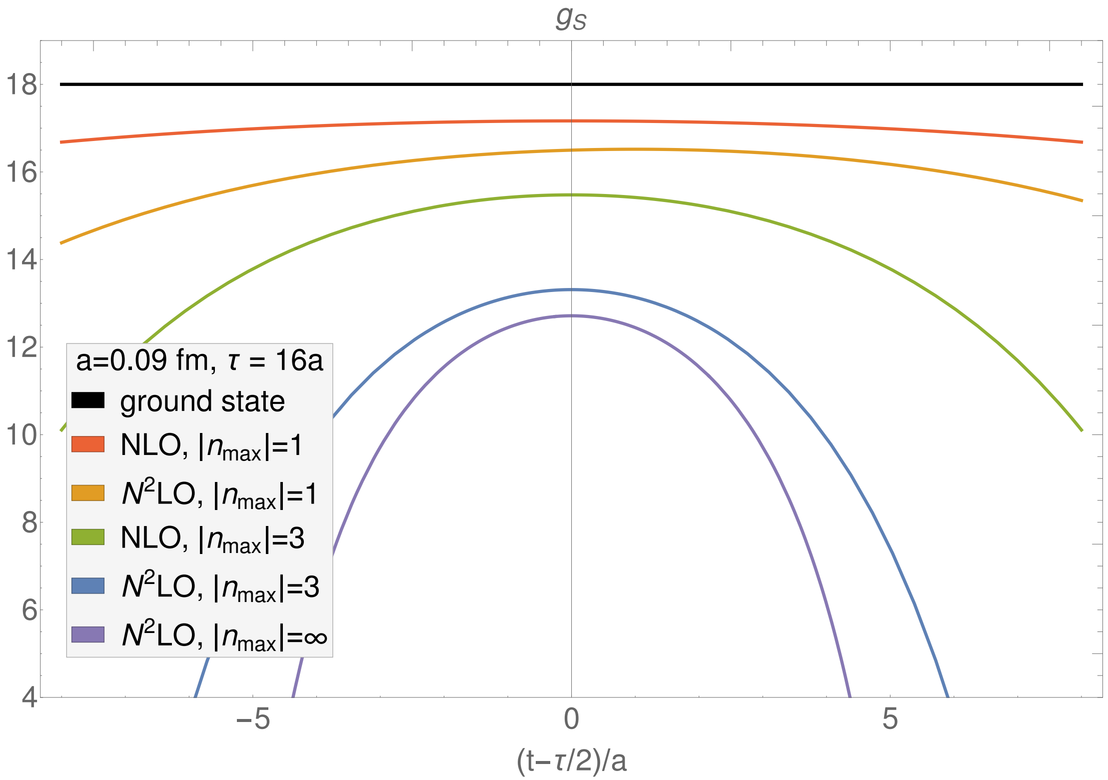


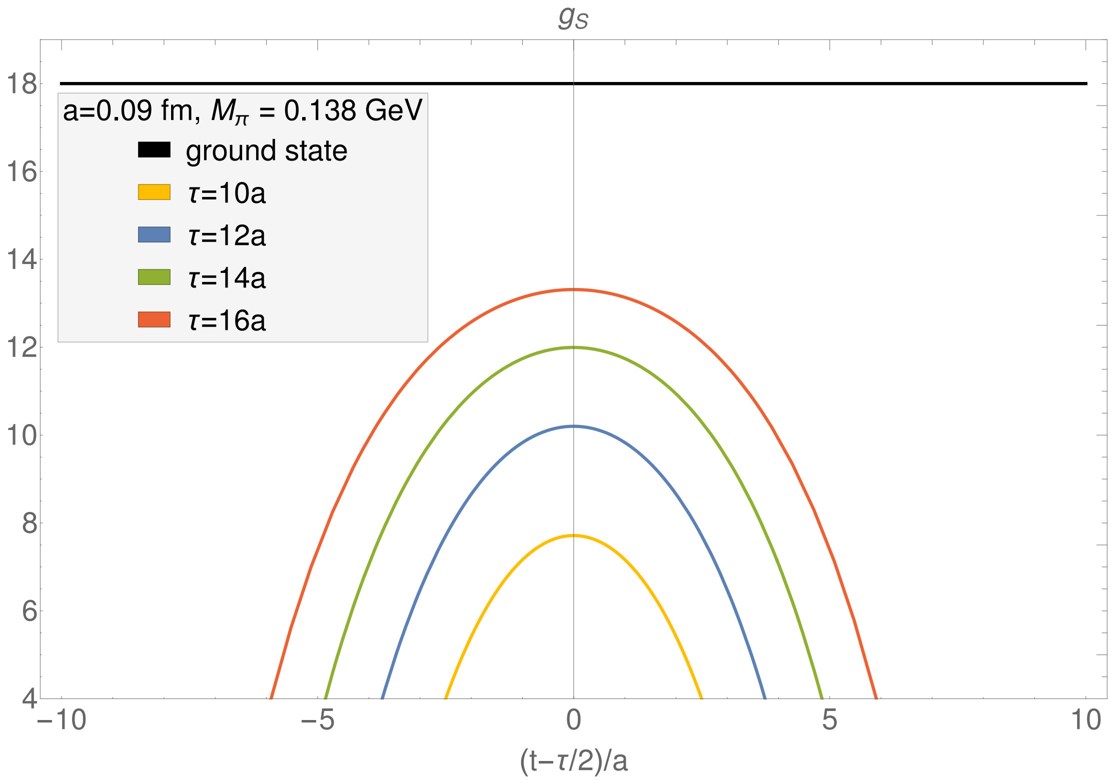
LO NLO NLO N^{2}LO

Why disconnected contribution is large



g_S: ESC from $N\pi$ / $N\pi\pi$ in N^{2}LO χPT





Different truncations (χPT order and $\vec{p}$)

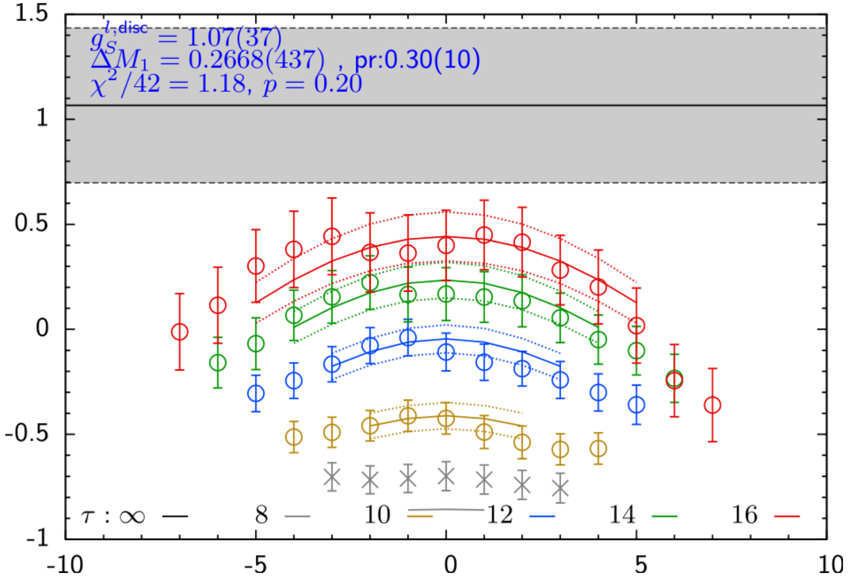
N^{2}LO χPT estimates for $\tau = 10,12,14,16$

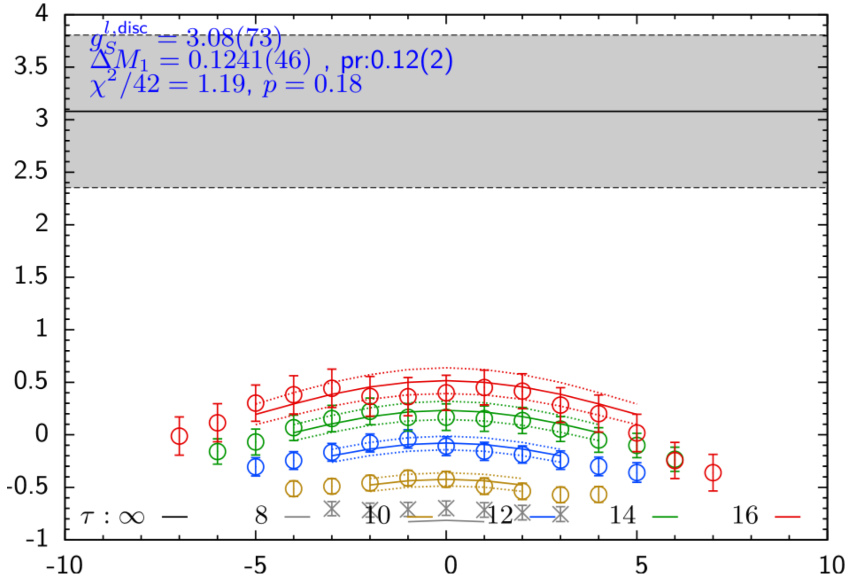
The NLO and N^{2}LO ESC can each reduce $\sigma_{\pi N}$ at a level of 10 MeV

Estimates for the $a \approx 0.09fm$; $M_\pi \approx 135MeV$ ensemble assuming the asymptotic value is 18 without $N\pi/N\pi\pi$ $(M_1 \approx 1.6\ GeV)$

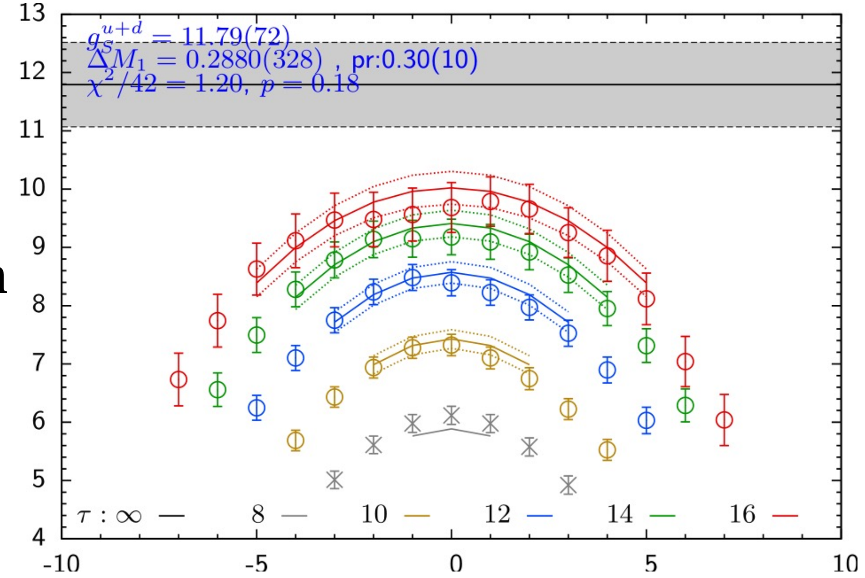
with $N\pi$ / $N\pi\pi$ $(M_1 \approx 1.2\ GeV)$

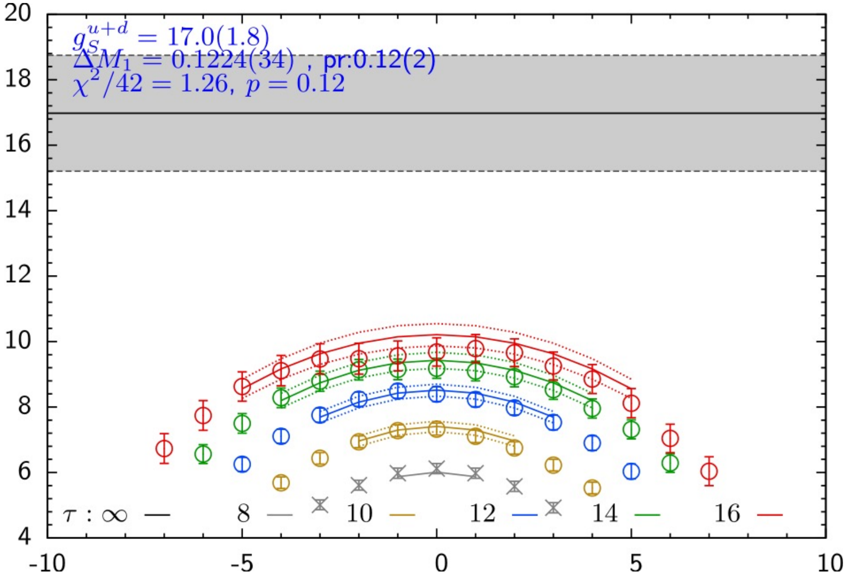
$g_S^{l,\mathrm{disc}}$





$g_S^{u+d} = g_S^{u+d,\mathrm{conn}} + 2g_S^{l,\mathrm{disc}}$





$\sigma_{\pi N} = m_l g_S^{u+d} \sim 40$ MeV

$\sigma_{\pi N} = m_l g_S^{u+d} \sim 60$ MeV

List of Lattice Parameters & Statistics

- All discussion based on Clover-on-HISQ data

Ensemble ID	a (fm)	M_π (MeV)	$(L/a)^3 \times T/a$	$M_\pi L$	$N_{\rm conf}^{\rm 2pt}$	$N_{\rm conf}^{\rm conn}$	$N_{\rm LP}$	$N_{\rm HP}$	$N_{\rm conf}^l$	$N_{\rm src}^l$	$N_{\rm LP}^l/N_{\rm HP}^l$
$a12m310$	0.1207(11)	310(3)	$24^3 \times 64$	4.55	1013	1013	64	8	1013	5000	30
$a12m220$	0.1184(10)	228(2)	$32^3 \times 64$	4.38	959	744	64	4	958	11000	30
$a09m310$	0.0888(08)	313(3)	$32^3 \times 96$	4.51	2263	2263	64	4	–	–	–
$a09m220$	0.0872(07)	226(2)	$48^3 \times 96$	4.79	964	964	128	8	712	8000	30
$a09m130$	0.0871(06)	138(1)	$64^3 \times 96$	3.90	1274	1290	128	4	1270	10000	50
$a06m310$	0.0582(04)	320(2)	$48^3 \times 144$	4.52	977	500	128	4	808	12000	50
$a06m220$	0.0578(04)	235(2)	$64^3 \times 144$	4.41	1010	649	64	4	1001	10000	50

2pt and connected 3pt

disconnected 3pt

Physical pion mass ensemble

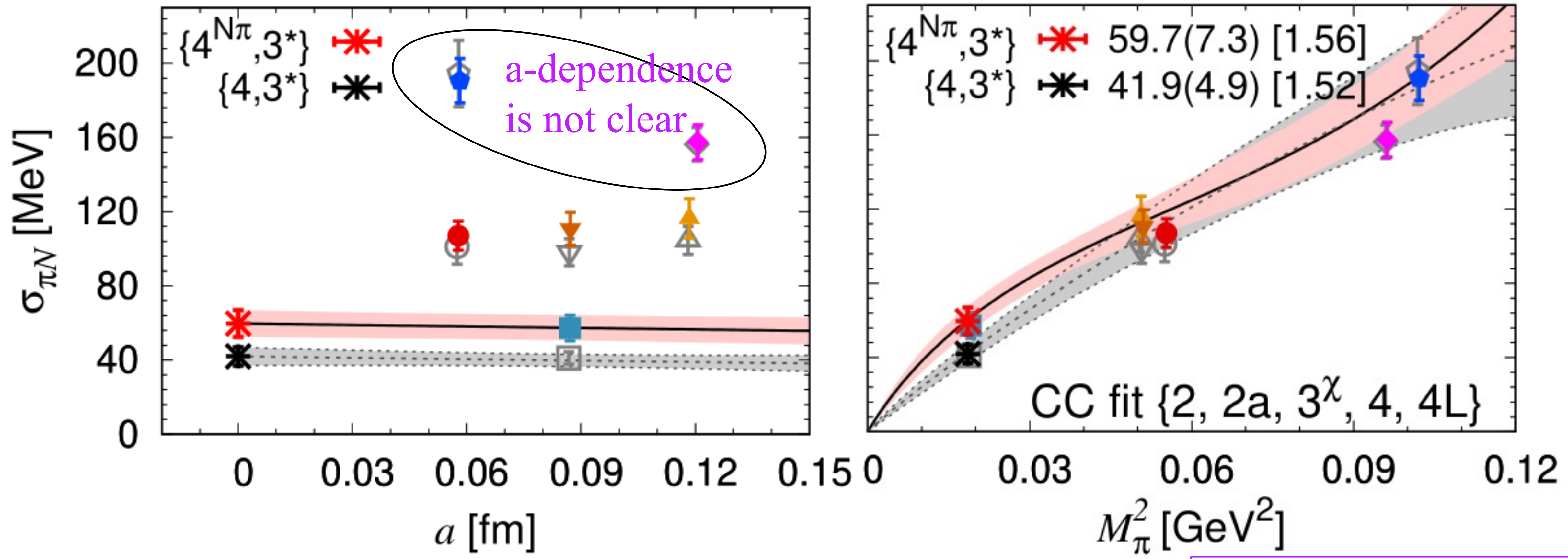
PRL 127 (2021) 242002

Last systematics in lattice calculations

3. Chiral-Continuum-Finite-Volume (CCFV) extrapolation

$$\sigma_{\pi N}(\mathrm{a}, \mathrm{M}_\pi, \mathrm{M}_\pi \mathrm{L}) = \sigma_{\pi N}(0, \mathrm{M}_\pi = 135\mathrm{MeV}, \infty) + \cdots$$

Chiral-Continuum fits to 6 points

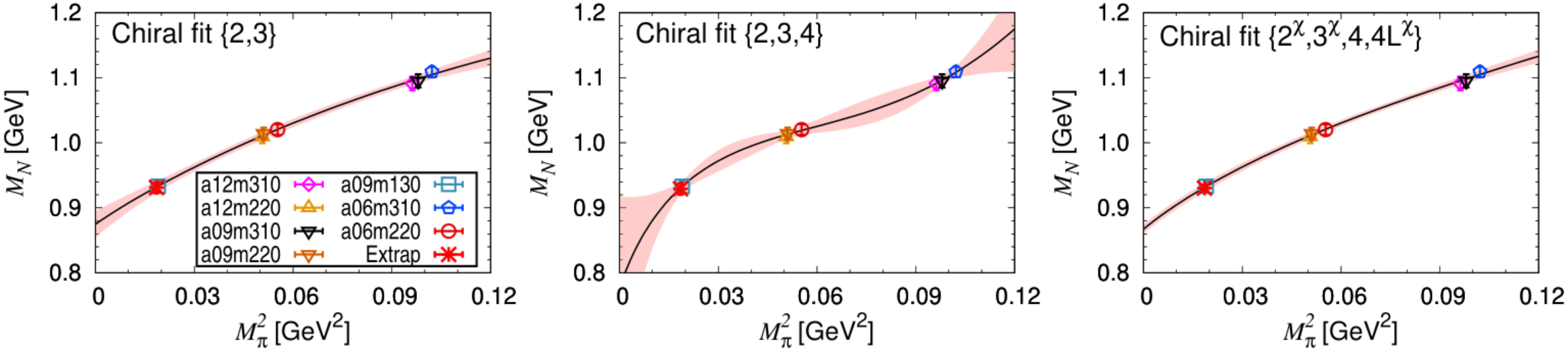


$$\sigma_{\pi N} = d_2 M_\pi^2 + d_2^a M_\pi^2 a + d_3^\chi M_\pi^3 + d_4 M_\pi^4 + d_{4L} M_\pi^4 \log M_\pi^2$$

d_3^χ: fixed to χPT prediction

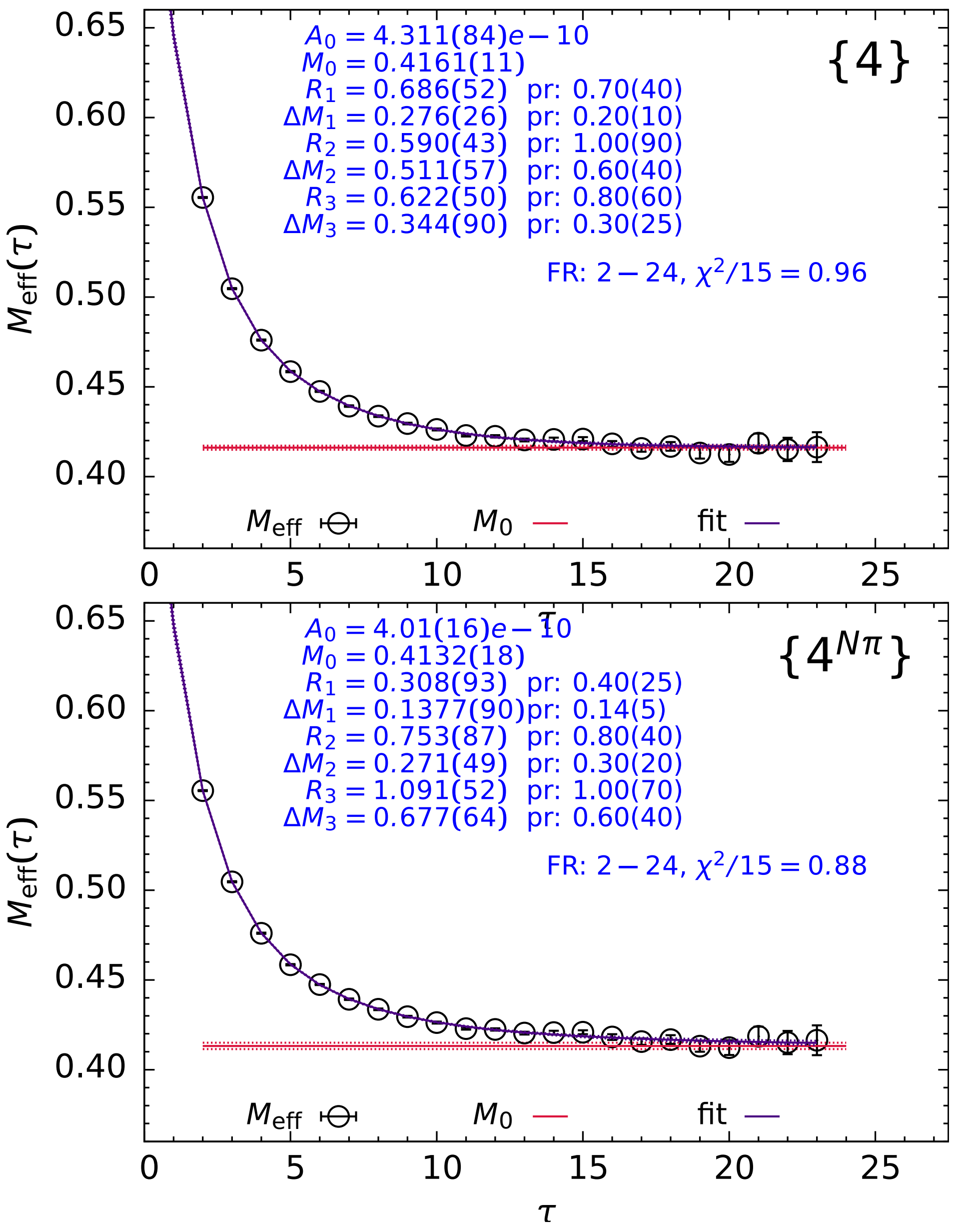
Other colors
$\{4^{N\pi},3*\}$: with $N\pi$
Light gray
$\{4,3*\}$: without $N\pi$

F-H method: Chiral fit to M_N



- χPT ansatz for M_N similar to that for $\sigma_{\pi N}$
- The most constrained fit does yield a good description of the data
- The resulting uncertainty in $\sigma_{\pi N}$ via FH method is too large with our data to compete with the direct method

Need to resolve excited states in F-H method also



$M_\pi = 170$ MeV ensemble
$a \approx 0.091$ fm
3000×320 measurements

- No plateau (ground state domination) even at ≈ 2 fm
- 4-state fits with $\Delta M_1 \approx 320$ and 640 MeV give similar χ^2
- For a state with $\Delta M_1 \approx 300$ MeV, the suppression $e^{-\Delta M_1 \tau}$=0.1 only at 1.5 fm

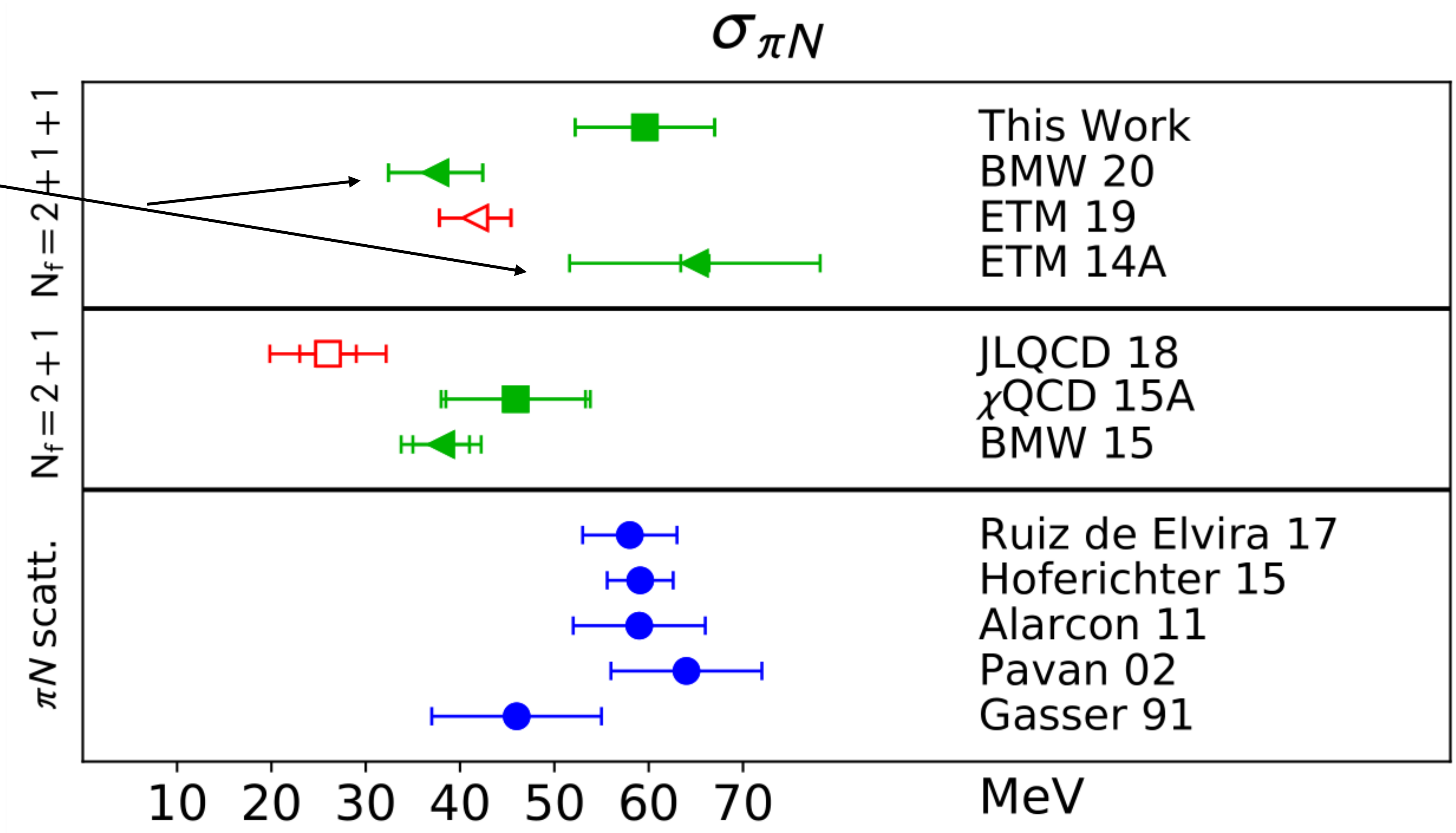
FLAG Summary + **arXiv:2105.12095**

Tension in
F-H method.

Both with $M_\pi \gtrsim$ 200 MeV data

Chiral extrapolation is different!

→ Need data at $M_\pi \approx 135$ MeV



Hoferichter et al, (Phys. Lett. B 843 (2023) 138001. $\sigma_{\pi N}$ reduced by 3.1(5) MeV between π^+ and π^0

Summary

- ESC large in g_S^{u+d} : Data show large τ dependence at $\tau \approx 1.5$ fm
- χPT suggests large contribution of $N\pi$ & $N\pi\pi$ states to g_S^{u+d}
- Contribution increases as $M_\pi \to 135$ MeV and $\Delta\vec{p} \to 0$
- Fits are consistent with coefficients predicted by χPT
- $\sigma_{\pi N}$ changes from ~40 MeV to ~60 MeV on including the $N\pi$ and $N\pi\pi$ excited states
- Need more data to improve the chiral-continuum-finite-volume extrapolation of $\sigma_{\pi N}$
- Large $N\pi$ contribution also seen in axial/pseudoscalar form factors. Including $N\pi$ states in analysis required for FF to satisfy PCAC relation

Extras