Why magnetic monopole becomes dyon in topological insulators

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Based on collaborations with Shoto Aoki, Hidenori Fukaya, Mikito Koshino, Yoshiyuki Matsuki (Osaka U.), arXiv:2304.13954 [cond-mat.mes-hall].

What will happen to a magnetic monopole when it is put inside a topological insulator?

—— We expect that the monopole is observed as a dyon with the electric chage $q_e = -1/2$, because of the Witten effect.

A magnetic monopole: a particle w/ a magnetic charge. It appears in dualities, GUTs, the inflation, etc. (e.g., the Dirac monopole, and the 't Hooft–Polyakov monopole)

A topological insulator: the bulk is the insulator (gapped), but the edge is the gapless. The effective theory of the T-symmetric topological insulator is described by the $\theta = \pi$ vacuum.

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—— We expect that the monopole is observed as a dyon with the electric chage $q_e = -1/2$, because of the Witten effect [Witten ('79)].

The Witten effect

We can write down the θ term as

$$L_\theta = \frac{\theta}{8\pi^2} \int d^3x \, \boldsymbol{E} \cdot \boldsymbol{B}.$$

We put a magnetic monopole with q_m on the $\theta \neq 0$ vacuum,

$$\boldsymbol{E} = -\nabla A^0, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} - q_m \frac{\boldsymbol{r}}{r^3}.$$

Then the θ term is described by

$$L_\theta = -\frac{\theta q_m}{2\pi} \int d^3x \, A^0 \delta^{(3)}(\boldsymbol{r}),$$

This implies that there is a particle with electric charge $q_e = -\theta q_m/(2\pi)$ which is coupled to the A^0 potential.

In the T-symmetry protected topological insulator ($\theta = \pi$), the monopole with $q_m = 1$ obtains the electric charge $q_e = -1/2$.

The effective theory description above is quite simple, but can't answer to the following questions:

(1) what is the origin of the electric charge? (must be electrons)
(2) if the origin is the electrons, why is it bound to monopole?
(3) why is the electric charge fractional?

In this our work [Aoki, Fukaya, Kan, Koshino Matsuki ('23)], we try to give answers to the questions from in terms of a microscopic description.

Contents

2. Naive Dirac equation (review)

Solving a naive Dirac equation

We first review the work by [Yamagishi ('83)].

We put a $U(1)$ gauge flux located at the origin describing the monopole:

$$A_x = \frac{-q_m y}{r(r+z)}, \quad A_y = \frac{q_m x}{r(r+z)}, \quad A_z = 0,$$

of which field strength is

$$F_{ij} = q_m \epsilon_{ijk} \frac{x_k}{r^3} - 4\pi q_m \delta(x)\delta(y)\theta(-z)\epsilon_{ij3},$$

where the second term represents the Dirac string. Due to the Dirac quantization, we assume $q_m = n/2$ with $n \in \mathbb{Z}$.

The orbital angular momentum is modified by the monopole configuration:

$$L_i = -i\epsilon_{ijk}x_j\left(\partial_k - iA_k\right) - n\frac{x_i}{2r},$$

which satisfies

$$[L_i, L_j] = i\epsilon_{ijk}L_k.$$

Their explicit forms in the polar coordinate are given by

$$L_\pm = L_1 \pm iL_2 = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{n}{2}\frac{\cos\theta - 1}{\sin\theta}\right),$$

$$L_3 = -i\frac{\partial}{\partial\phi} - \frac{n}{2}.$$

The Dirac Hamiltonian with a mass m is

$$
\begin{aligned}
H &= \gamma_0 \left(\gamma_i \left(\partial_i - i A_i \right) + m \right), \\
&= \begin{pmatrix} m & \sigma_i \left(\partial_i - i A_i \right) \\ -\sigma_i \left(\partial_i - i A_i \right) & -m \end{pmatrix},
\end{aligned}
$$

where $\gamma_0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma_i = \sigma_1 \otimes \sigma_i$.

The "chiral" operator is

$$
\bar{\gamma} := -i \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \otimes \mathbf{1},
$$

($\neq \gamma_5 = -\gamma_0 \gamma_1 \gamma_2 \gamma_3$). The chiral operator $\bar{\gamma}$ anticommutes with the Hamiltonian H: $\{H, \bar{\gamma}\} = 0$.

The total angular momentum is

$$J_i = L_i \otimes \mathbf{1} + \frac{1}{2}\mathbf{1} \otimes \sigma_i,$$

with $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $[J_i, H] = 0$. Except for the lowest eigenvalue $j = |n/2| - 1/2$, (where the degeneracy is $2j + 1 = |n|$,) there are $2(2j + 1)$ degenerate states.

In addition to J^2 and J_3, there is an operator that commutes with H; $[H, \sigma_3 \otimes D^{S^2}] = 0$, where we define the "spherical" operator

$$D^{S^2} := \sigma_i \left(L_i + \frac{n}{2}\frac{x_i}{r} \right) + 1.$$

Because of

$$\left(D^{S^2}\right)^2 = \nu^2, \qquad \nu := \sqrt{\left(j+\frac{1}{2}\right)^2 - \frac{n^2}{4}},$$

we introduce the eigenstates of D^{S^2} which satisfy for $j > |n/2| - 1/2$,

$$D^{S^2}\chi_{j,j_3,\pm}(\theta,\phi) = \pm\nu\chi_{j,j_3,\pm}(\theta,\phi),$$

and for $j = |n/2| - 1/2$

$$D^{S^2}\chi_{j,j_3,0}(\theta,\phi) = 0,$$

with $\sigma_r\chi_{j,j_3,0} = s\chi_{j,j_3,0}$, where $s = \mathrm{sign}(n)$ and $\sigma_r = \sigma_i x_i/r$.

The solution of the Dirac equation $H\psi = E\psi$ with $j > |n/2| - 1/2$ is

$$\psi_{j,j_3,\pm} = \frac{C_{j,j_3,\pm}}{\sqrt{r}} \begin{pmatrix} (m+E)K_{\nu\mp 1/2}(\sqrt{m^2-E^2}r)\ \chi_{j,j_3,\pm}(\theta,\phi) \\ \sqrt{m^2-E^2}K_{\nu\pm 1/2}(\sqrt{m^2-E^2}r)\ \sigma_r\chi_{j,j_3,\pm}(\theta,\phi) \end{pmatrix}.$$

However, this is NOT the normalizable solution localized at the monopole ($r = 0$).

The only normalizable solution is the state with $j = |n|/2 - 1/2$:

$$\psi_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

with $E = 0$.

The solution is localized at the monopole ($r = 0$). Is this state of the electron with $E = 0$ the cause of the dyon?

$$\psi_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi)$$

1. The state is a chiral eigenstate of $\sigma_1 \otimes \sigma_r$ with the eigenvalue $\mathrm{sign(m)}$. What is the origin?
2. No difference between the positive and negative mass in the solution. The Witten effect predicts the dyon appear only in the topological insulator ($m < 0$). The solution can't explain it w/o imposing "the chiral boundary condition" by hand.
3. Why does the electric charge become $q_e = -1/2$?

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3. Regularized Dirac equation

The Wilson term

In order to answer these questions, we take account of the leading correction from the Pauli–Villars regularization. The partition function is expanded as

$$
\begin{aligned}
Z &= \det\left(\frac{D+m}{D+M_{\mathrm{PV}}}\right), \\
&= \det\Bigg[\frac{1}{M_{\mathrm{PV}}}\Bigg(D+m+\frac{1}{M_{\mathrm{PV}}}D_\mu^\dagger D^\mu \\
&\qquad\qquad + \mathcal{O}(1/M_{\mathrm{PV}}^2, m/M_{\mathrm{PV}}, F_{\mu\nu}/M_{\mathrm{PV}})\Bigg)\Bigg].
\end{aligned}
$$

"The Wilson term" $D_\mu^\dagger D^\mu/M_{\mathrm{PV}}$ appears as the leading correction.

Then the “regularized” Dirac Hamiltonian is given by

$$H_{\rm reg} = \gamma_0 \left(\gamma^i D_i + m + \frac{D_i^\dagger D^i}{M_{\rm PV}} \right).$$

Note that the sign of m is well-defined once the sign of $M_{\rm PV}(> 0)$ is fixed. The Dirac equation is manifestly different between positive and negative m.

By the anomalous $U(1)_A$ transformation,

$$Z = \det\left(\frac{D+m}{D+M_{\mathrm{PV}}}\right) = \det\left(\frac{D+|m|}{D+M_{\mathrm{PV}}}\right)\exp\left(\frac{i\theta}{8\pi^2}\int F\wedge F\right).$$

- For $m > 0$, $\theta = 0$, which implies the normal phase.
- For $m < 0$, $\theta = \pi$, which implies the topological phase.

We note that the cut-off of the fermion field is

$$1/M_{\mathrm{PV}} \sim a \sim 10^{-10}[m],$$

while the size of the monopole is less than

$$r_1 \sim 10^{-20}[m],$$

assuming the ('t Hooft–Polyakov) monopole energy is higher than 10 TeV.

Thus the effect of the Wilson term is important.

In perturbative theory, the regulator usually appears in loop computations only.

However, in nonperturbative lattice regularization, the Wilson term

$$H_{\mathrm{Wilson}} = \gamma_0 \left[\sum_{i=1}^{3} \left(\gamma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right) + m \right],$$

is needed even at the tree-level.

Since the Laplacian $D_i^\dagger D^i$ is always positive, the mass shift due to the Wilson term is always positive when we take M_{PV} positive.

For $m < 0$ (or inside topological insulators), it is possible to locally flip the sign of the "effective" mass

$$m < 0 \quad \rightarrow \quad m_{\mathrm{eff}} = m + \frac{D_i^\dagger D^i}{M_{\mathrm{PV}}} \sim m + \frac{1}{M_{\mathrm{PV}} r_1^2} > 0,$$

when the magnetic flux is concentrated in the region $r < r_1$.

It's implies that the inside region $r < r_1$ becomes a normal insulator, and the (spherical) domain-wall is dynamically created and the chiral edge-mode appears on it! (It doesn't happened in the normal insulator with $m > 0$.)

Solving the regularized Dirac equation

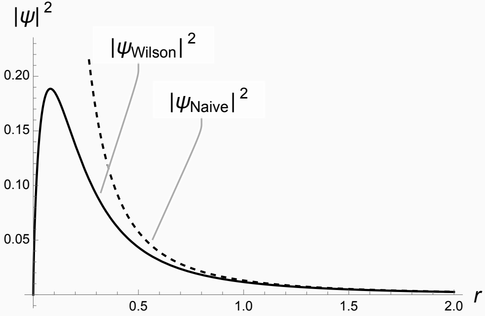
The regularized Hamiltonian is

$$H_{\text{reg}} = \begin{pmatrix} m - D_i D_i / M_{\text{PV}} & \sigma_i \left(\partial_i - i A_i \right) \\ -\sigma_i \left(\partial_i - i A_i \right) & -m + D_i D_i / M_{\text{PV}} \end{pmatrix},$$

The solution of the zero-mode for $r_1 \to 0$ is given by

$$\psi^{\text{mono}}_{j,j_3} = \frac{B e^{-M_{\text{PV}} r/2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\nu = \sqrt{2|n| + 1}/2$, and $\kappa = M_{\text{PV}} \sqrt{1 + 4m/M_{\text{PV}}}/2$.



The plot with $n = 1$, $m = -0.1$, $M_{\mathrm{PV}} = 10$.

- The solution ψ_{Wilson} coincides with ψ_{Naive} for large r.
- A peak at $r = |n|/(2M_{\mathrm{PV}}) \sim 1/M_{\mathrm{PV}}$ is the domain-wall.
- ψ_{Wilson} is zero at $r = 0$.

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4. The Atiyah–Singer index theorem and the half-integral charge

The Dirac operator on the domain-wall

Our solution

$$\psi^{\mathrm{mono}}_{j,j_3} = \frac{Be^{-M_{\mathrm{PV}}r/2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi),$$

is a zero eigenvalue solution of the spherical operator,

$$D^{S^2} = \sigma_i \left(L_i + \frac{n}{2}\frac{x_i}{r} \right) + 1,$$

since $D^{S^2}\chi_{j,j_3,0} = 0$ for $j = |n|/2 - 1/2$.

In fact, D^{S^2} is the Dirac operator on the spherical domain-wall created around the monopole!

With a local Lorentz (or $Spin^c$ to be precise) transformation $R(\theta, \phi) = \exp(i\theta\sigma_y/2)\exp(i\phi(\sigma_z + 1)/2)$, we obtain

$$\begin{aligned} D^{S^2} &= R(\theta,\phi)\left[\sigma^i\left(L_i + \frac{n}{2}\frac{x_i}{r}\right) + 1\right]R(\theta,\phi)^{-1}, \\ &= -\sigma_z\left[\sigma_x\frac{\partial}{\partial\theta} + \sigma_y\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + i\hat{A}_\phi + i\hat{A}^s_\phi\right)\right], \end{aligned}$$

where $\hat{A}_\phi = \frac{n}{2}\frac{\sin\theta}{1+\cos\theta}$ is the vector potential (in units of r_1) generated by the monopole.

The second connection,

$$\hat{A}^s_\phi = \frac{1}{2\sin\theta} - \frac{\cos\theta}{2\sin\theta}\sigma_z,$$

is the induced $Spin^c$ connection on the sphere which is strongly curved with the small radius r_1, i.e., gravity!

Also, D^{S^2} anticommutes with σ_r, which implies that the zero-modes are the chiral zero-modes of not only 3D but also D^{S^2}.

The AS index theorem on the domain-wall

The 2D chirality is

$$\sigma_r \chi_{j,j_3,0}(\theta,\phi) = s\chi_{j,j_3,0}(\theta,\phi), \quad s := \mathrm{sign}(n),$$

and # of the degeneracy is $2j+1 = |n|$. Then the Dirac index is

$$\mathrm{Ind}\, D^{S^2} = n.$$

On the other hand, the topological index is

$$\frac{1}{4\pi}\int_{S^2} d^2x \epsilon^{\mu\nu} F_{\mu\nu} = n.$$

Stability of the zero modes on the domain-wall is topologically protected by the AS index theorem.

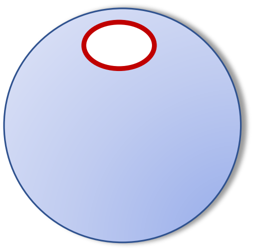
So far, we considered a $\mathbb{R}^3$ space, but in order to discuss topological feature of the fermion zero mode, we also need an IR regularization, such as the one-point compactification, S^3.

Then the topological insulator region with $(m_{\text{eff}} < 0)$ have topology of a disk with a small S^2 boundary at $r = r_1$.

However, due to the cobordism invariance of the AS index,

$$\int_{\partial M} F = \int_M dF = 0,$$

the disk is not possible.



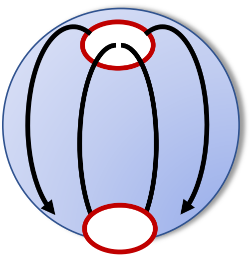
A resolution is: to create another domain-wall at, say, $r = r_0$, outside of the topological insulator.

Another zero mode is localized at the outside domain-wall, and the index is kept trivial.

$$0 = \int_M dF = \int_{\Sigma_{\mathrm{mono}}} F + \int_{\Sigma_{\mathrm{out}}} F,$$

where $\partial M = \Sigma_{\mathrm{mono}} \cup \Sigma_{\mathrm{out}}$.

This implies that the outside of the topological insulator must be a normal insulator (laboratory).



The edge states on the outside domain-wall

The regularized Hamiltonian around the outside domain-wall is

$$H = \gamma_0 \left(\gamma^i D_i + |m|\epsilon(r - r_0) + \frac{D_i^\dagger D^i}{M_{\mathrm{PV}}} \right).$$

The edge-localized state is obtained as

$$\psi_{j,j_3}^{\mathrm{DW}} = \begin{cases} \frac{\exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} \left(B' K_\nu(\kappa_- r) + C' I_\nu(\kappa_- r)\right) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r < r_0), \\ \frac{D' \exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} K_\nu(\kappa_+ r) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r > r_0), \end{cases}$$

where $\kappa_\pm = \frac{M_{\mathrm{PV}}}{2}\sqrt{1 \pm 4|m|/M_{\mathrm{PV}}}$.

The solution

$$\psi^{\mathrm{DW}}_{j,j_3} = \begin{cases} \frac{\exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} \left(B' K_\nu(\kappa_- r) + C' I_\nu(\kappa_- r)\right) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r < r_0), \\ \frac{D' \exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} K_\nu(\kappa_+ r) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r > r_0), \end{cases}$$

is:

- $E = 0$,
- $\sigma_x \otimes \sigma_r = +1$ (which is opposite to $\psi^{\mathrm{mono}}_{j,j_3}$),
- the # of degeneracy is $2j + 1 = |n|$, and
- the 3D chiral state; $\bar{\gamma} = \sigma_x \otimes \mathbf{1} = s$.

At finite r_0, the paired zero-modes near the monopole at $r = r_1$ and the domain-wall at $r = r_0$ are mixed:

$$\psi = \alpha \psi^{\mathrm{mono}}_{j,j_3} + \beta \psi^{\mathrm{DW}}_{j,j_3}.$$

Because of $\{\bar{\gamma}, H\} = 0$,

$$\left(\psi^{\mathrm{mono}}_{j,j_3}\right)^\dagger H \psi^{\mathrm{mono}}_{j,j_3} = \left(\psi^{\mathrm{DW}}_{j,j_3}\right)^\dagger H \psi^{\mathrm{DW}}_{j,j_3} = 0,$$

and

$$\left(\psi^{\mathrm{mono}}_{j,j_3}\right)^\dagger H \psi^{\mathrm{DW}}_{j,j_3} = \left(\psi^{\mathrm{DW}}_{j,j_3}\right)^\dagger H \psi^{\mathrm{mono}}_{j,j_3} =: \Delta \sim \exp(-|m| r_0).$$

Then we can show $E = \pm\Delta$ and $\alpha = \pm\beta$.

$$\psi \sim \frac{1}{\sqrt{2}} \left(\psi_{j,j_3}^{\text{mono}} \pm \psi_{j,j_3}^{\text{DW}}\right)$$

The 50% amplitude is located around the monopole at $r = r_1$.

We conclude that (the expectation value of) the electric charge around the monopole is $q_e = -1/2$!

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5. Numerical analysis

Lattice setup

On a three-dimensional hyper-cubic lattice with size $L = 31$ with open boundary conditions, we put a monopole at $\boldsymbol{x}_m = (L/2, L/2, L/2)$ with a magnetic charge $n/2$. We also put an antimonopole at $\boldsymbol{x}_a = (L/2, L/2, 1/2)$ with the opposite charge $-n/2$.

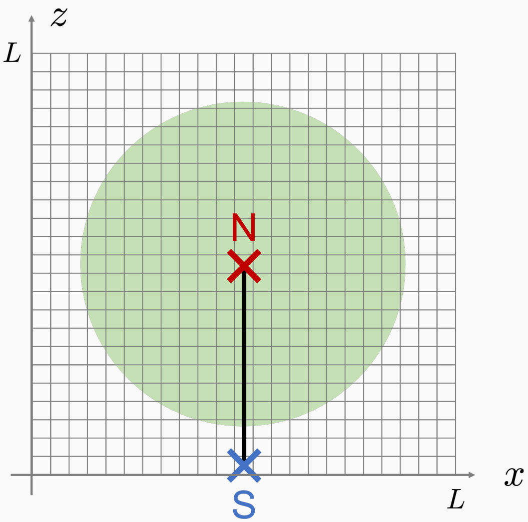
The continuum vector potential at $\boldsymbol{x} = (x, y, z)$ is then given by

$$A_x(\boldsymbol{x}) = q_m \left[\frac{-(y - y_m)}{|\boldsymbol{x} - \boldsymbol{x}_m|(|\boldsymbol{x} - \boldsymbol{x}_m| + (z - z_m))} - \frac{-(y - y_a)}{|\boldsymbol{x} - \boldsymbol{x}_a|(|\boldsymbol{x} - \boldsymbol{x}_a| + (z - z_a))} \right],$$

$$A_y(\boldsymbol{x}) = q_m \left[\frac{x - x_m}{|\boldsymbol{x} - \boldsymbol{x}_m|(|\boldsymbol{x} - \boldsymbol{x}_m| + (z - z_m))} - \frac{x - x_a}{|\boldsymbol{x} - \boldsymbol{x}_a|(|\boldsymbol{x} - \boldsymbol{x}_a| + (z - z_a))} \right],$$

$$A_z(\boldsymbol{x}) = 0,$$

with $q_m = n/2$. Note that the Dirac string extends from $\boldsymbol{x}_a$ to $\boldsymbol{x}_m$.


z
L
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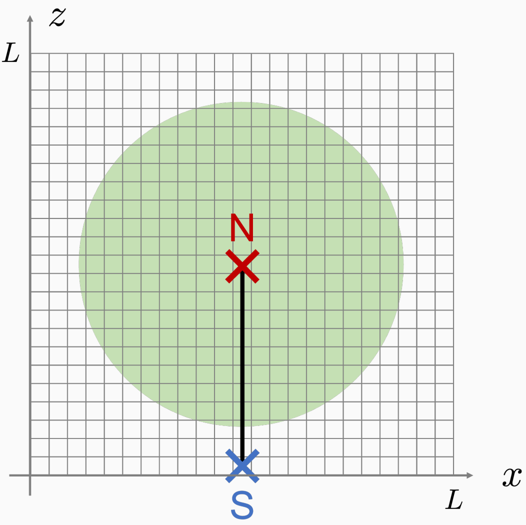
L
x

For the fermion field, we assign a position-dependent mass term to be $m(\boldsymbol{x}) = -m_0$ with $m_0 = 14/L$ for $\sqrt{|\boldsymbol{x} - \boldsymbol{x}_m|} < r_0 = 3L/8$, and $m(\boldsymbol{x}) = +m_0$ otherwise.

Namely, the monopole is located at the center of a spherical topological insulator with radius r_0 surrounded by a normal insulator with the gap m_0, while the anti-monopole sits in the normal insulator region.

We assume that outside of the lattice with open boundary condition corresponds to a "laboratory" with $m(\boldsymbol{x}) = +\infty$.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The Wilson Dirac Hamiltonian is given by

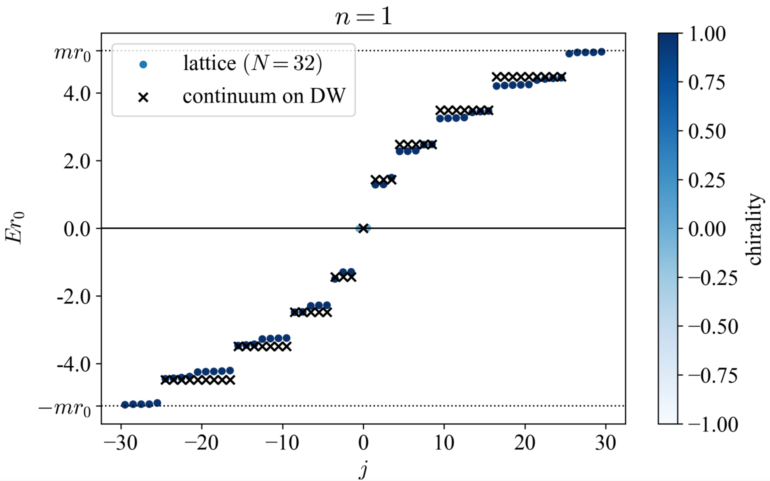
$$H_W = \gamma^0 \left(\sum_{i=1}^{3} \left[\gamma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{1}{2} \nabla_i^f \nabla_i^b \right] + m(\boldsymbol{x}) \right),$$

where $\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x})$ denotes the forward covariant difference and $\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i)$ is the backward difference. Also, $U_i(\boldsymbol{x}) = \exp\left(i \int_0^1 A_i(\boldsymbol{x} + \boldsymbol{e}_i l) dl \right)$ is the link variables.

Note that H_W anti-commutes with $\bar{\gamma} = \gamma_x \otimes 1$ even on a lattice.

Numerical results

The eigenvalue spectrum of H_W w/ $n = 1$ on the $L = 31$ lattice:



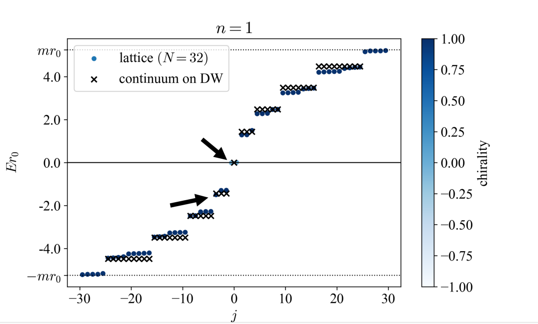
We see that:

- the circle symbols are the numerical results,
- the cross symbols are the continuum results,
- two nearest zero-modes.

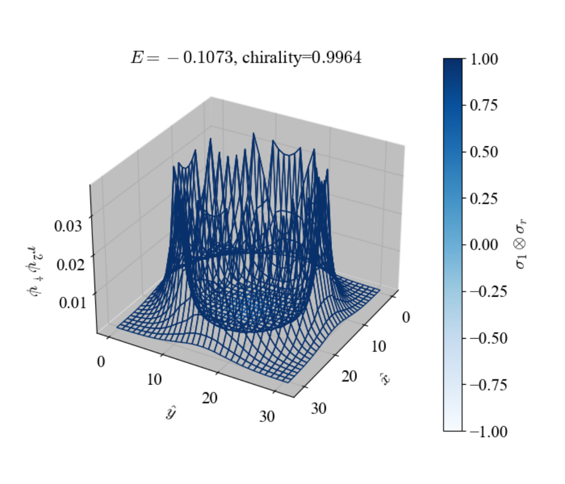
We plot the amplitude,

$$A_k(\boldsymbol{x}) = \phi_k^\dagger(\boldsymbol{x})\phi_k(\boldsymbol{x})r^2,$$

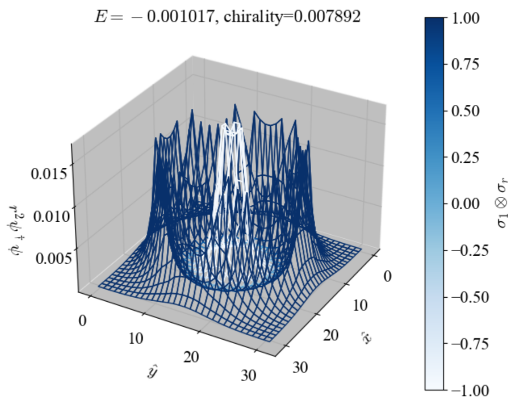
for the negative first and second nearest-zero modes.



The amplitude of the negative second nearest-zero mode for $n = 1$ in $z = (L + 1)/2$ slice:



The amplitude of the negative nearest-zero mode for $n = 1$ in $z = (L+1)/2$ slice:



We see that:

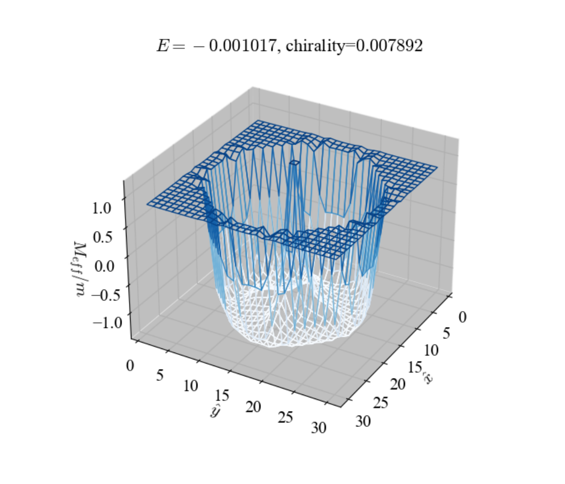
- for the nearest zeromode, the amplitude has two peaks around $r = |\boldsymbol{x} - \boldsymbol{x}_m| = 0$ and $r = r_0$,
- the local chirality near each peak is ~ -1 and $+1$, respectively, although the total chiraity is near zero,
- the 50% of the state is located around the monopole, while the other 50% is located at $r = r_0$: the half electric charge,
- this is only for the nearest zeromode, e.g., for the second nearest zero mode, we have only the edge-localized modes:

To directly confirm creation of the domain-wall near the monopole, we plot distribution of the "effective mass" (normalized by m_0),

$$m_{\text{eff}}(\boldsymbol{x}) = \phi_k(\boldsymbol{x})^\dagger \left[- \sum_{i=1,2,3} \frac{1}{2} \nabla_i^f \nabla_i^b + m(\boldsymbol{x}) \right] \Big/ \phi_k(\boldsymbol{x}) \phi_k(\boldsymbol{x})^\dagger \phi_i(\boldsymbol{x}),$$

on the $z = (L+1)/2$ slice.

The effective mass of the nearest zeromode with $n = 1$ on $z = 16$ slice:



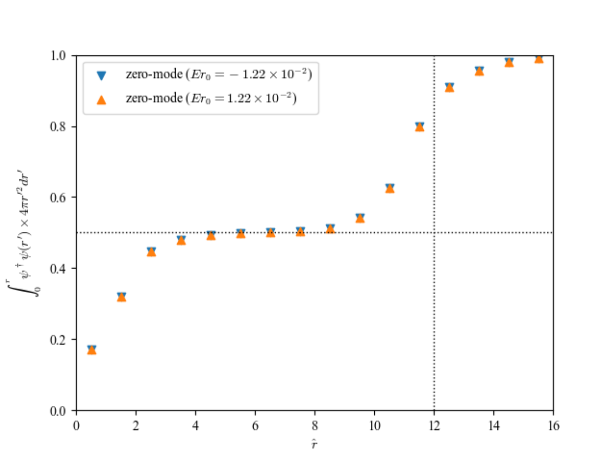
We see that:

- the small island of the normal insulator (or a positive mass region) appears around the monopole: the domain-wall is dynamically created, Let's quantify the electric charge that the monopole gains.

We plot the cumulative distribution of the nearest zero modes:

$$C_k(r) = \int_{|\boldsymbol{x}|<r} d^3x\, \phi_k(\boldsymbol{x})^\dagger \phi_k(\boldsymbol{x}).$$

For $n = 1$:



zero-mode ($Er_0 = -1.22 \times 10^{-2}$)
zero-mode ($Er_0 = 1.22 \times 10^{-2}$)
$\int_0^r \psi^\dagger \psi(r') \times 4\pi r'^2 dr'$
$\hat{r}$
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16

We see that:

- a stable plateau in the middle range $4 < r < 9 = 3r_0/4$ at $C_k(r) \sim 1/2$,
- under the half-filling condition, the monopole gains $|n|/2$ electric charge capturing the half of the occupied $|n|$ zero-mode states of the electron.

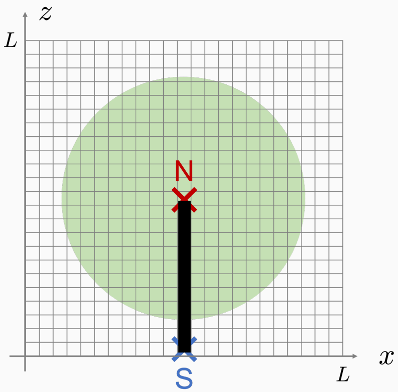
Contents

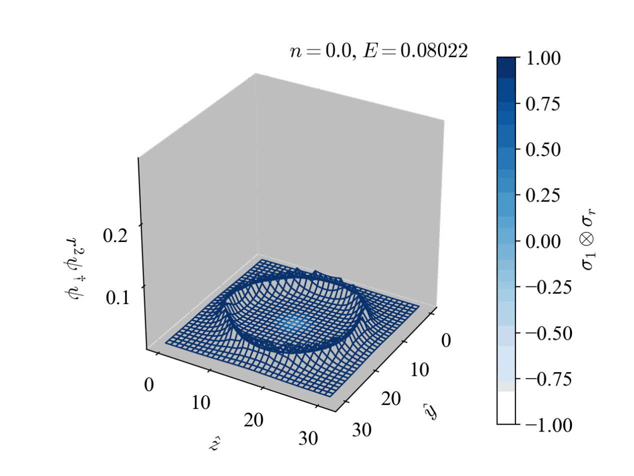
6. Nonintegral magnetic charge

An interesting (thought) experiment

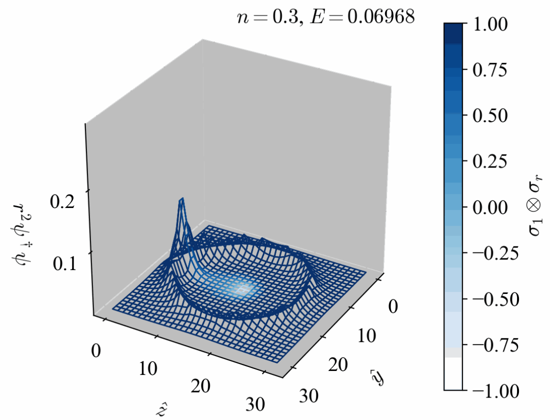
We have a thin solenoid whose radius is comparable to the crystal spacing of a topological insulator. Inserting one end of the solenoid inside the topological insulator while the other end is put outside, we can mimic the monopole-antimonopole system with.

This experiment can be simulated continuously varying the value of n from 0 to 1.

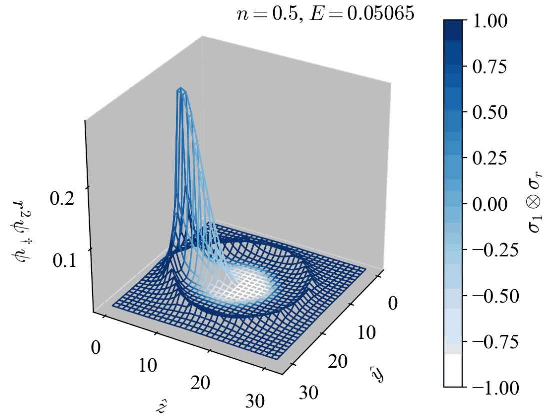




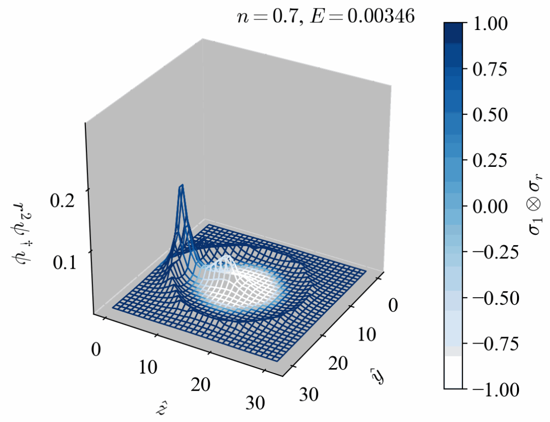
For $n = 0$, the amplitude is uniformly distributed around the sphere of radius $r = r_0$.



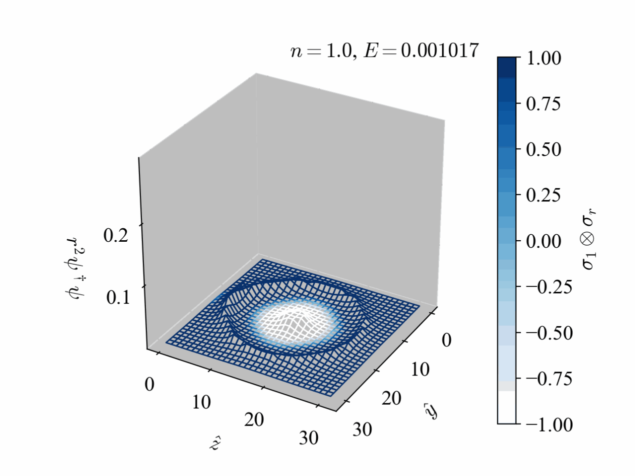
Increasing n, a part of the wave function is gradually swallowed into the topological insulator from bottom of the sphere.



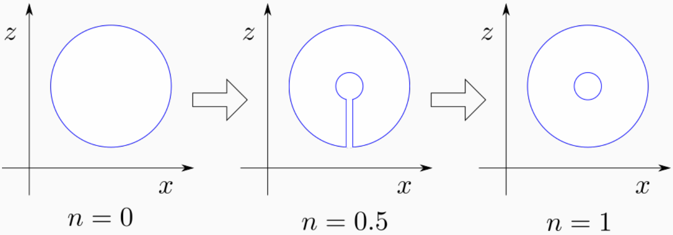
For $n = 0.5$, the circumference of the solenoid becomes the normal insulator, and attracts the electron.



For $n > 0.5$, the amplitude is separated into two, one half is attracted by the monopole, while the other half stays at the original domain-wall.



For $n = 1.0$, the circumference of the solenoid returns to the topological insulator.



Schematic image of topology change of the domain-wall. One spherical domain-wall at $n = 0$ is extended via the Dirac string into the location of the monopole at $n \sim 0.5$, and is separated into two domain-walls at $n = 1$.

Summary

We discussed a microscopic description of the Witten effect with the Wilson term.

How do we distinguish between the normal insulator ($m > 0$) and topological insulator ($m < 0$)?

- It is the topological insulator if the mass is relatively negative compared to the PV mass.

Why are electrons bound to monopole?

- Because of the positive mass correction from the magnetic field of the monopole, the domain-wall is dynamically created (only for the negative mass).

Why do the chiral zero modes appear?

- Because the zero modes localized at the domain-wall are protected by the AS index.

Why is the electric charge fractional?

- Because the 50% of the wavefunction is located around the monopole (the other 50% is located at the surface of the topological insulator).