# Two-pion scattering & $K \to \pi\pi$ decay calculations with periodic boundaries

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# Introduction

# K → ππ & CP violation

$$\underset{\text{CP odd}}{|K_L\rangle = |K_2\rangle} + \varepsilon\,\underset{\text{CP even}}{|K_1\rangle}$$

direct CPV: $|K_2\rangle \xrightarrow{\varepsilon'} |\pi\pi\rangle$

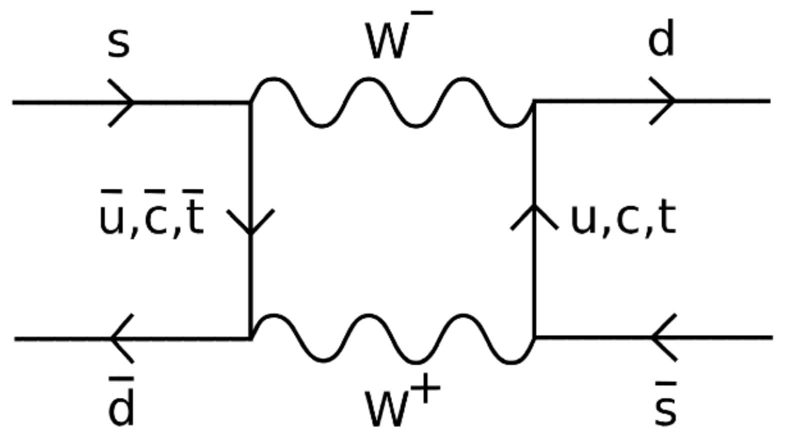
indirect CPV: $|K_1\rangle \xrightarrow{\varepsilon} |\pi\pi\rangle$

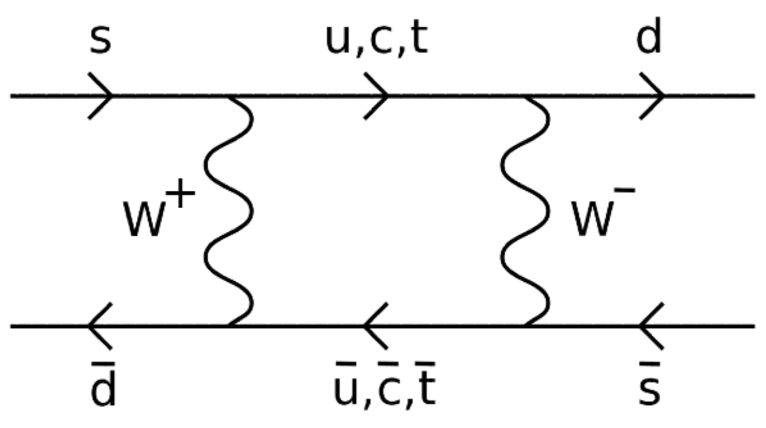
$|\pi\pi\rangle$ CP even

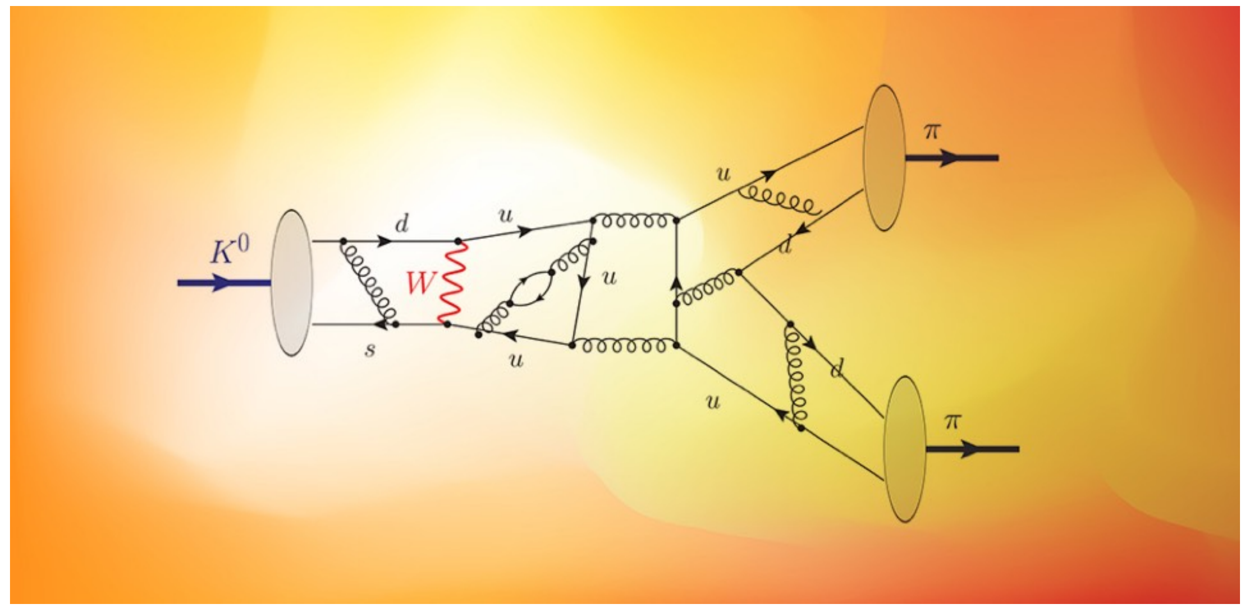
Discovered in 1964

$$\frac{\Gamma(K_L\to\pi^0\pi^0)}{\Gamma(K_S\to\pi^0\pi^0)}\Big/\frac{\Gamma(K_L\to\pi^+\pi^-)}{\Gamma(K_S\to\pi^+\pi^-)} = 1-6\times\mathrm{Re}(\epsilon'/\epsilon)$$

- ε from "odd" mixing b/w $K^0$ & $\overline{K}^0$
- ε' only found in decays Discovered in 1999
  - $\mathrm{Re}(\epsilon'/\epsilon)_{\text{exp}} = 16.6(2.3)\times 10^{-4}$
  - Consistent with SM?



# CPV physics: good source for BSM search

- CPV in SM believed to be insufficient to explain the matter-antimatter imbalance in the present universe
- Testing SM via CPV physics can provide a good source for searching BSM
- Lattice QCD capable of testing hadronic sectors K, D, B, …
- Direct CP violation measure $\varepsilon'/\varepsilon$ in $K \to \pi\pi$ highly demanded

# Anticipated sensitivity of ε′ to BSM

- s → d: most suppressed within SM

$$\mathrm{Re}(\varepsilon'/\varepsilon) \propto \mathrm{Im}(V_{td}V_{ts}^*)$$

$$V_{CKM} \sim \begin{array}{cc} \begin{array}{ccc} \mathbf{d} & \mathbf{s} & \mathbf{b} \end{array} & \\ \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \boxed{\lambda^3} & \boxed{-\lambda^2} & 1 \end{pmatrix} & \begin{array}{c} \mathbf{u} \\ \mathbf{c} \\ \mathbf{t} \end{array} \end{array}$$

$\lambda \approx 0.23$

$$\underbrace{|V_{td}V_{ts}^*| \sim 5 \times 10^{-4}}_{\mathbf{s \to d}} \ll \underbrace{|V_{td}V_{tb}^*| \sim 1 \times 10^{-2}}_{\mathbf{b \to d}},\ \underbrace{|V_{ts}V_{tb}^*| \sim 4 \times 10^{-2}}_{\mathbf{b \to s}}$$

- ε′ highly sensitive to BSM & highly demanded by pheno

# I = 0 & I = 2 decay modes

$$\langle(\pi\pi)_{I=0}| = \sqrt{1/3}\langle\pi^0\pi^0| + \sqrt{2/3}\langle\pi^+\pi^-|, \quad \langle(\pi\pi)^{I_3=0}_{I=2}| = -\sqrt{2/3}\langle\pi^0\pi^0| + \sqrt{1/3}\langle\pi^+\pi^-|$$

- Isospin-definite amplitudes

$$A_I = \langle(\pi\pi)_I|H_W|K\rangle \quad \begin{cases} I = 0 \rightarrow \Delta I = 1/2 \\ I = 2 \rightarrow \Delta I = 3/2 \end{cases}$$

- $A_2$ precisely calculated (PRL108 (2012) 141601, PRD91 (2015) 074502)
  - 2 lattice spacings: 2.36 GeV, 1.73 GeV → continuum limit taken
  - Re $A_2$ = $1.50(4)_{stat}(14)_{sys} \times 10^{-8}$ GeV, Im $A_2$ = $-6.99(20)_{stat}(84)_{sys} \times 10^{-13}$ GeV
    cf: $(\mathrm{Re}\,A_2)_{exp} = 1.479(4) \times 10^{-8}$ GeV
- $A_0$ challenging bc of disconnected diagram and …
- ε': needs both of $A_0$ & $A_2$

# The $\Delta I = 1/2$ rule

## Isospin-specific amplitudes

$$\mathrm{A_I} = \langle (\pi\pi)_\mathrm{I} | \mathrm{H_W} | \mathrm{K} \rangle$$

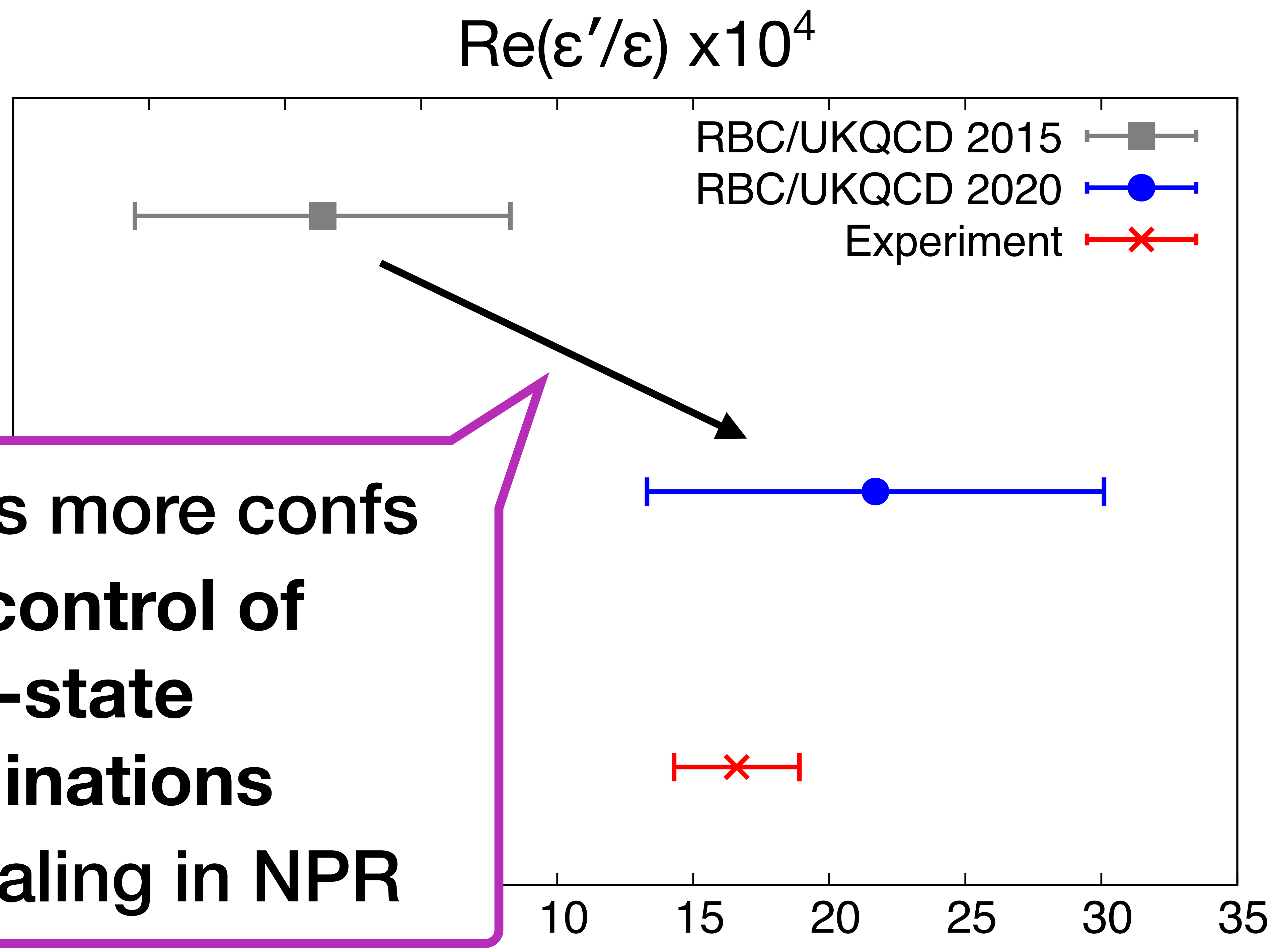
## Experimental fact

$$\frac{\mathrm{Re\,A_0}}{\mathrm{Re\,A_2}} = 22.45(6)$$ : large suppression of $\Delta I = 3/2$ ($A_2$) mode

- Factor 2 can be responsible for Wilson coefs from pQCD [Gaillard & Lee, PRL 33,108 (1974)]
- Remaining factor 10 comes from NP QCD or BSM?

# Previous calculation with G-parity BC

- PRD 102,054509 (2020)




- 3+ times more confs
- Better control of excited-state contaminations
- Step scaling in NPR

- $\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{SM}} = 21.7(2.6)_{\mathrm{stat}}(6.2)_{\mathrm{sys}}(5.0)_{\mathrm{EM/IB}} \times 10^{-4}$

  $\mathrm{Re}(\epsilon'/\epsilon)_{\mathrm{exp}} = 16.6(2.3) \times 10^{-4}$

- ΔI = 1/2 rule

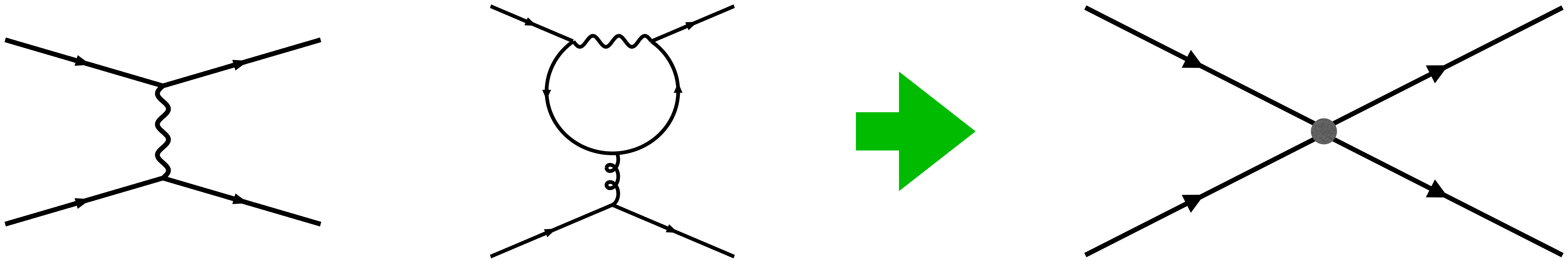
  $(\mathrm{Re}\,A_0/\mathrm{Re}\,A_2)_{\mathrm{SM}} = 19.9(2.3)(4.4)$

  $(\mathrm{Re}\,A_0/\mathrm{Re}\,A_2)_{\mathrm{exp}} = 22.45(6)$

- Independent calculations desired b/c of
  - Phenomenological importance of ε′
  - Relatively large uncertainty compared to exp
  - Complexity of calculation

# Approach to weak decays

- Two typical scales
  - Electroweak scale: $m_W$ = 80 GeV, $m_Z$ = 91 GeV
  - QCD scale: $\Lambda_{QCD}$ ~ 300 MeV
- Low-energy effective interactions @ QCD scale



$$\sim G_F \propto \frac{g^2}{m_W^2}$$

  - $H_W = \Sigma_i \underbrace{c_i(\mu)}_{\text{Wilson coefficients}} \underbrace{O_i(\mu)}_{\text{Effective operators}}$

**Wilson coefficients** **Effective operators**

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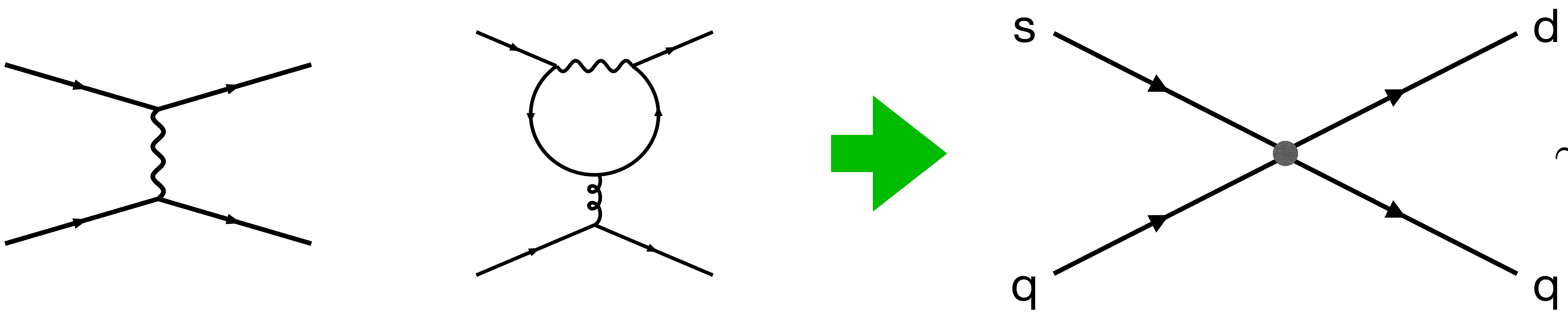


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Wilson coefficients Effective operators

# ΔS = 1 effective operators

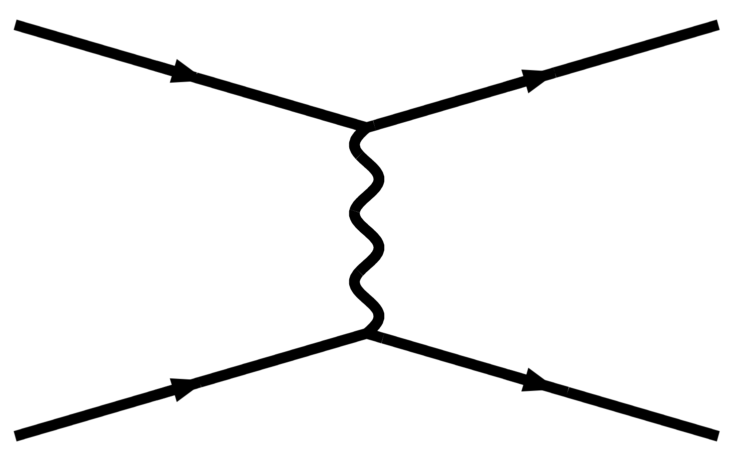
- $(\bar{s}q)_{V-A}(\bar{q}'q'')_{V\pm A} = \bar{s}\gamma_\mu(1-\gamma_5)q' \cdot \bar{q}'\gamma_\mu(1\pm\gamma_5)q''$
- $\alpha, \beta$: color indices

$$
\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\mathrm{V-A}}\,(\bar{u}_\beta d_\alpha)_{\mathrm{V-A}}\,, \\
Q_2 &= (\bar{s}u)_{\mathrm{V-A}}\,(\bar{u}d)_{\mathrm{V-A}}\,, \\
Q_3 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q (\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_4 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q (\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,, \\
Q_5 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q (\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_6 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q (\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_7 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_8 &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_9 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_{10} &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,,
\end{aligned}
$$

**Current-current operators** ($Q_1$, $Q_2$)

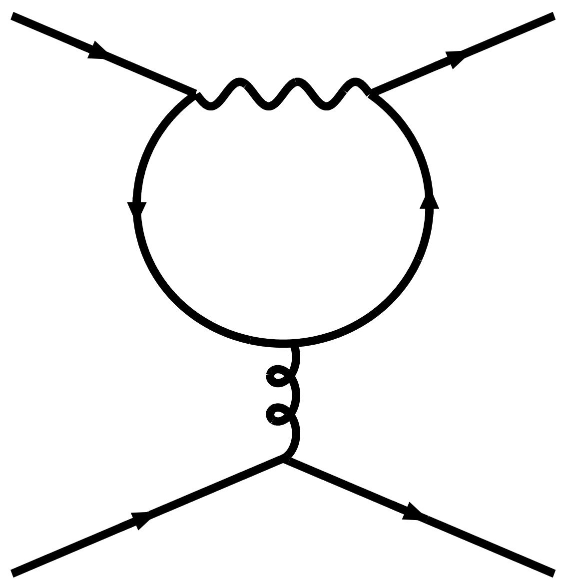
- $Q_1^c = (\bar{s}_\alpha c_\beta)_{\mathrm{V-A}}(\bar{c}_\beta d_\alpha)_{\mathrm{V-A}}$ & $Q_2^c = (\bar{s}c)_{\mathrm{V-A}}(\bar{c}d)_{\mathrm{V-A}}$

  enter when $n_f \geq 4$



**QCD penguin operators** ($Q_3$–$Q_6$)

- sum over q runs for all active quarks



**EW penguin operators** ($Q_7$–$Q_{10}$)

# K → ππ Amplitude and ε′

ππ phase shifts

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right]\right\} \qquad (\omega = \mathrm{Re}\,A_2/\mathrm{Re}\,A_0)$$

Renormalization matrix

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i,j} \underbrace{[z_i(\mu) + \tau y_i(\mu)]}_{\text{Wilson coefs. pQCD}} \underbrace{Z_{ij}(\mu)}_{\text{LQCD (+pQCD)}} \underbrace{\langle (\pi\pi)_I | Q_j^{lat} | K \rangle}_{\text{LQCD}}$$

- Matrix elements $< (\pi\pi)_I \,|\, Q_i^{lat} \,|\, K >$ from 3pt correlation functions

# Systematic errors in 2020

- Systematic errors on Im $A_0$

| | |
|---|---|
| Finite lattice spacing | 12% |
| Wilson coefficients | 12% |
| Lelloch-Lüscher FV correction | 1.5% |
| Residual FV correction | 7% |
| Parametric error | 6% |
| Off-shellness | 5% |
| Renormalization | 4% |
| Missing $G_1$ operator | 3% |
| TOTAL | 21% |

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**Improvement desired** (Finite lattice spacing)

**Improvement desired** (Wilson coefficients)

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- In addition
  - $\varepsilon'$ could be significantly affected by EM/IB effects ($\Delta I = 1/2$ rule → 20%)

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  **Hope to compute near**

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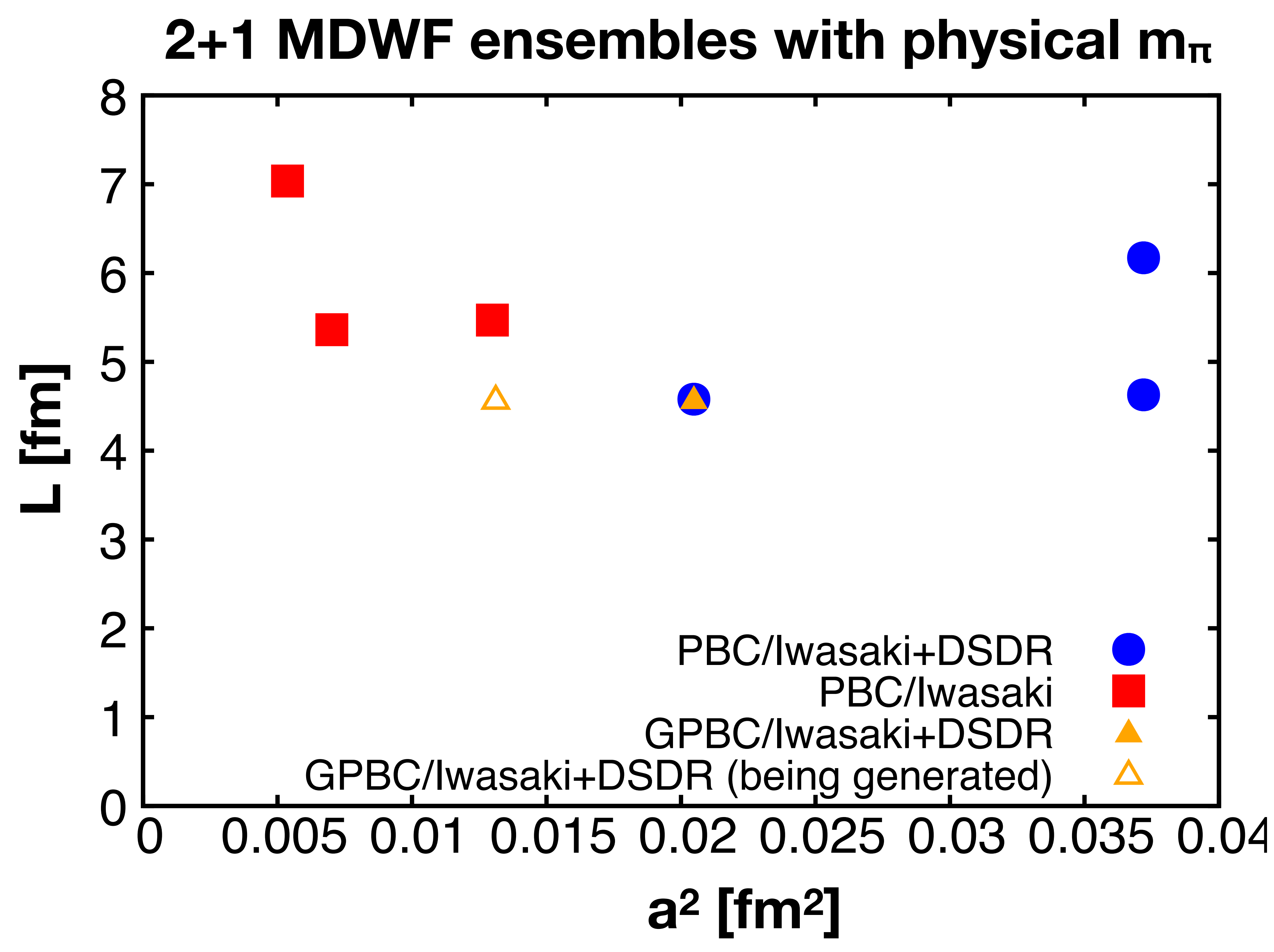
# Finite lattice spacing error

- Can be resolved by taking **continuum limit**
  - Results with different lattice spacings needed
- G-parity BC
  - $32^3 \times 64$, $a^{-1}$ = 1.4 GeV: Done (2020)
  - Ensemble generation speed-up algorithm (Lat23, C. Kelly)
  - $40^3 \times 64$, $a^{-1}$ = 1.7 GeV: Calculation on-going
  - $48^3 \times 64$, $a^{-1}$ = 2.1 GeV: in the future as needed
- <u>Ensembles already generated for PBC</u>



# EM/IB effects

- Usually O(1%) but …

$$\frac{\epsilon'}{\epsilon} = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right] = -\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\left[1 - \frac{1}{\omega}\underline{\frac{\mathrm{Im}\,A_2}{\mathrm{Im}\,A_0}}\right] \qquad (\omega = \mathrm{Re}\,A_2/\mathrm{Re}\,A_0)$$

Even small correction to this can be amplified (1/ω ≈ 22.5 )

Ciligriano et al, JHEP 02, 032 (2020)
NLO ChPT + large $N_c$
(example estimation)

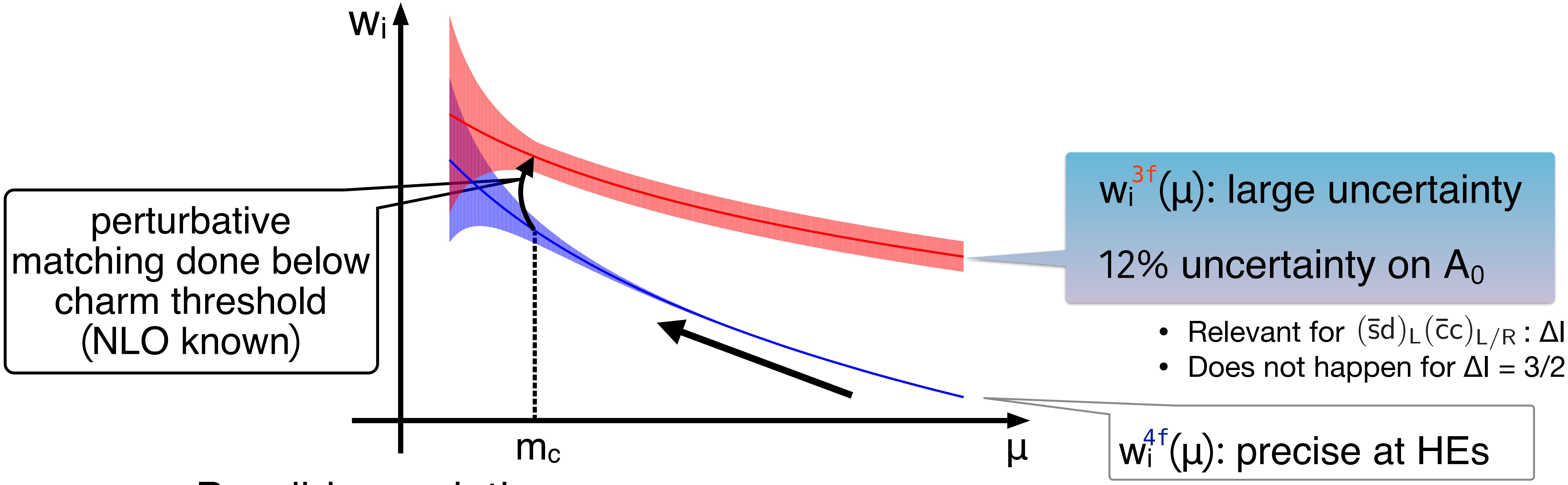
$$\frac{\epsilon'}{\epsilon} = \frac{i\omega_+ e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2^{\mathrm{emp}}}{\mathrm{Re}\,A_2^{(0)}} - \frac{\mathrm{Im}\,A_0^{(0)}}{\mathrm{Re}\,A_0^{(0)}}(1-\hat{\Omega}_{\mathrm{eff}})\right]$$

$$\boxed{\hat{\Omega}_{\mathrm{eff}} = 0.170\left(^{+91}_{-90}\right)}$$

- Developing approaches to introduce QED/IB effects on the lattice
  - ‣ Extension of Lüscher's formalism for treatment of ππ state in a finite box
  - ‣ Coulomb correction to $\pi^+\pi^+$ scattering [Christ et al, PRD106 (2022), 1, 014508]
  - ‣ Computation of transverse radiation contribution still challenging
  - ★ **PBC appear necessary to introduce these effects**

# Wilson coefs

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$




- Relevant for $(\bar{s}d)_L(\bar{c}c)_{L/R}$ : $\Delta I = 1/2$
- Does not happen for $\Delta I = 3/2$

- Possible resolutions
  - NNLO matching only partially done [Cerda-Sevilla et al. Acta Phys.Polon.B 4 (2018) 1087-1096]
  - Nonperturbative matching underway [MT, LATTICE2019]

# Challenge of calculating on-shell $K \to \pi\pi$ matrix elements

# Challenge: realizing on-shell kinematics

- The lightest ππ state with "2 stationary pions," $E_{\pi\pi,0} \approx 280$ MeV → off-shell
  - need $| E_{\pi\pi} = m_K \approx 500 \text{ MeV} \rangle$ state
- Possible approaches

💡 Finite volume → two-pion spectrum not continuous

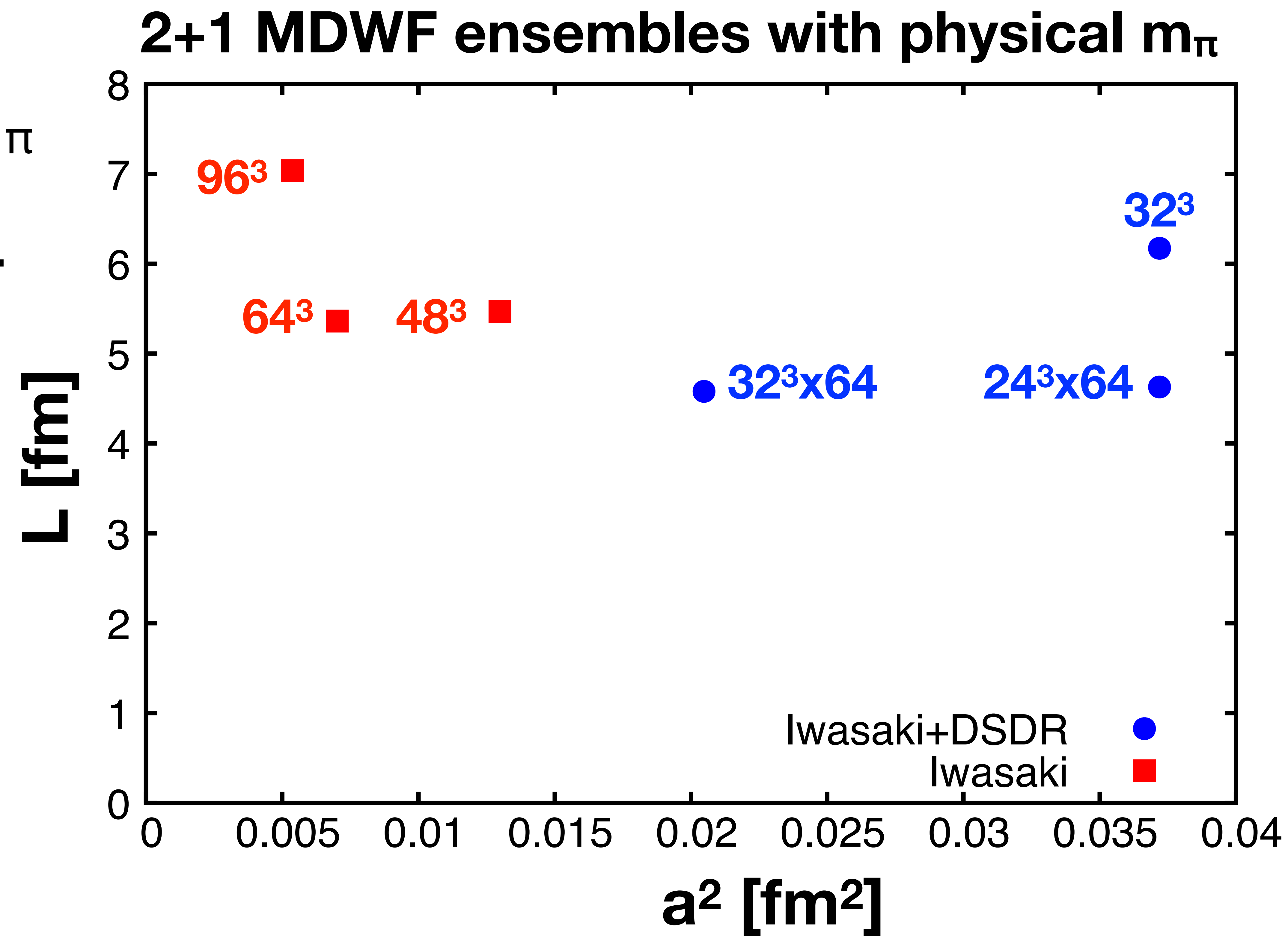
- Moving frame

  e.g. $\sqrt{m_K^2 + p_{tot}^2} = m_\pi + \sqrt{m_\pi^2 + p_{tot}^2}$
- Analyze correlation functions taking multiple states into account (this work)
- Manipulate boundary conditions so that the 280-MeV state vanishes (G-parity BC, previous RBC/UKQCD work)

# PBC calculation

## Advantages

- Already have lattice ensembles with physical $m_\pi$
  - $a^{-1}$ = 1 GeV, $24^3$ x 64, $a^{-1}$ = 1.4 GeV, $32^3$ x 64 & …
  - Continuum limit easier
- Hope to introduce EM/IB effects near future
  - G-parity BC violate charge conservation
  - PBC appear necessary




## Challenge

- Presence of $E_{\pi\pi} = 2m_\pi$ state
  - S/N ratio of $E_{\pi\pi} = m_K$ state should be the same as in G-parity BC: $\sim e^{-(m_K - 2m_\pi)t}$
  - interesting to see feasibility of extracting signal of excited states

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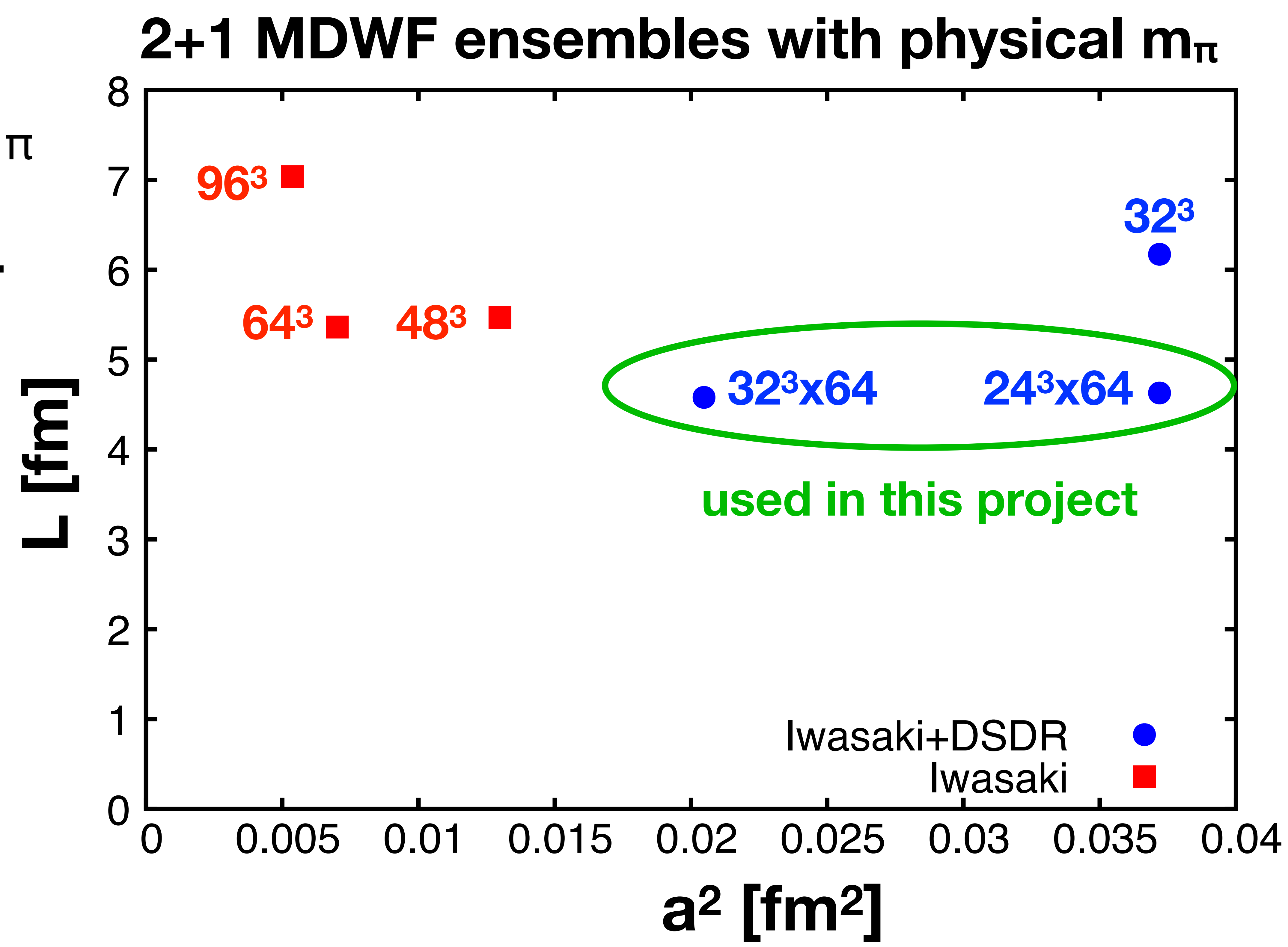


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# Lattice setup

- RBC/UKQCD's 2+1-flavor MDWF ensembles at physical pion & kaon masses
  - $24^3 \times 64$, $a^{-1} = 1.0$ GeV, 258 → 440 confs
  - $32^3 \times 64$, $a^{-1} = 1.4$ GeV, 107 → 470 confs
- Chiral symmetry of DWF → strong constraints on operator mixings
  - with lower-dimensional operators
  - among different representations w.r.t. chiral symmetry (8,1), (8,8) & (27,1)
- All-to-all quark propagators
  - 2,000 low modes for light quarks (no low mode for strange)
  - high-mode part: spin, color and time dilutions => 768 inversions

# Subtopic: $\pi\pi$ scattering

# ππ scattering

- Analysis of $\langle O_{\pi\pi}(t)\, O_{\pi\pi}(0)^\dagger \rangle$
- Needed for calculating K → ππ matrix elements (I = 0, 2)
  - extraction of an excited state needed for PBC
  - we use GEVP
- long-distance contribution to HVP (I = 1, by g-2 group) skipped in this talk
- Access to some quantities
  - phase shifts, scattering length, …
  - very few determinations at physical pion mass so far

# Variational method [Lüscher-Wolf, 1990]

- Solving GEVP (Generalized Eigenvalue Problem)

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0)$$

$C(t)$: N x N correlator matrix $C_{ab}(t) = \langle O_a(t)O_b(0)^\dagger \rangle$

  - $O'_n = \sum_a v^*_{n,a} O_a$ only couples with n-th state (asymptotic region)
  - $\lambda_n(t,t_0) = e^{-E_n(t-t_0)}$

- ππ operators used in this work:
  - $\pi_{p=(0,0,0)}\pi_{p=(0,0,0)}$
  - $\pi_{p=(0,0,1)}\pi_{p=(0,0,-1)}$
  - $\pi_{p=(0,1,1)}\pi_{p=(0,-1,-1)}$
  - $\pi_{p=(1,1,1)}\pi_{p=(-1,-1,-1)}$
  - $\sigma \sim \bar{u}u + \bar{d}d$
  - KK ~ $\bar{K}K + K^+K^-$ : new entry

**I = 2**

**I = 0**

# Before solving GEVP…

- More precise evaluation of 2pt functions on the lattice

$$C_{ab}(t) = \sum_n A_{n,a}A^*_{n,b}e^{-E_n t}$$

– cosh effect $\quad + \sum_n A_{n,a}A^*_{n,b}e^{-E_n(T-t)}$ – taken into account after GEVP

– vacuum effect $\quad + \langle O_a \rangle \langle O_b \rangle$ – needs to be subtracted for I = 0

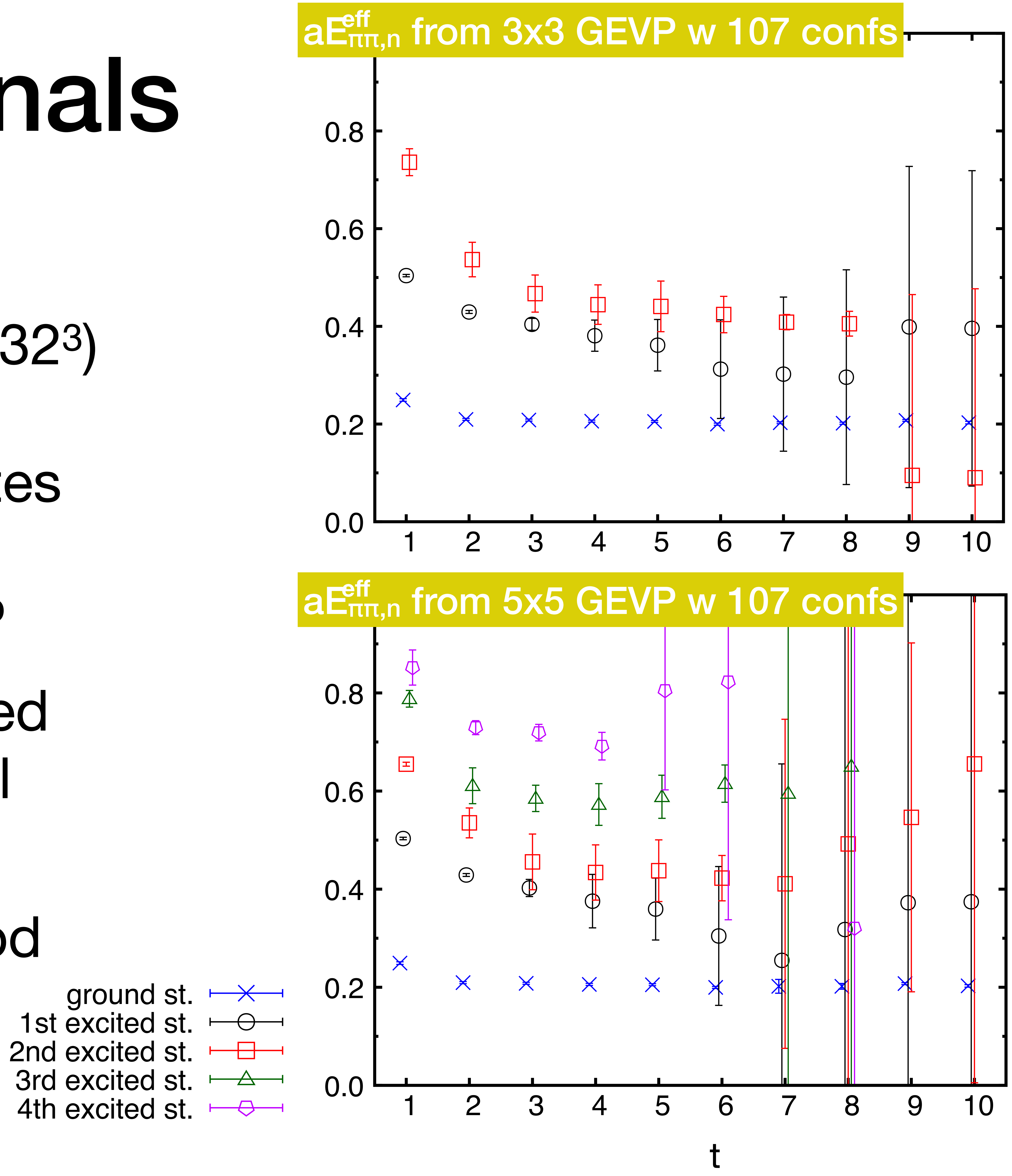
– thermal effect $\quad + \langle \pi | O_a | \pi \rangle \langle \pi | O_b | \pi \rangle e^{-E_\pi T} + \ldots$ – also needs to be subtracted

- Subtraction of vacuum & thermal effects

$$C_{ab}(t) \to C_{ab}(t) - C_{ab}(t+\delta t) = \sum_n A_{n,a}A^*_{n,b}(1 - e^{-E_n \delta t})e^{-E_n t}$$

# Overlap b/w GEVP signals

- Signals from GEVP indistinguishable with insufficient statistics (107confs, $32^3$)
- Plateau not well seen for excited states
- Possible problem of traditional GEVP
  - Solving GEVP including high excited states with big error can even spoil the signal of lower states
  - Small-size GEVP may not give good precision/control of excited states



# Rebased GEVP

- Rebased GEVP
  - Large size GEVP at short time separations
  - Switch to smaller size GEVP at larger time separations but without discarding measured correlation functions
- Example:

$$\begin{pmatrix} O_{(000)} \\ O_{(001)} \\ O_{(011)} \\ O_{(111)} \\ O_{\sigma} \end{pmatrix} \xrightarrow{\text{5x5 GEVP at t = 2}} \left\{ \begin{array}{l} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \\ O_4^{t=2} \end{array} \right.$$

- couples well with the ground st.
- couples well with the 1st excited st.
- …
- incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations

# Rebased GEVP

- Rebased GEVP
  - Large size GEVP at short time separations
  - Switch to smaller size GEVP at larger time separations but without discarding measured correlation functions
- Example:

$$\begin{pmatrix} O_{(000)} \\ O_{(001)} \\ O_{(011)} \\ O_{(111)} \\ O_\sigma \end{pmatrix} \xrightarrow{\textbf{5x5 GEVP at t = 2}} \left\{ \begin{array}{ll} O_0^{t=2} & \text{- couples well with the ground st.} \\ O_1^{t=2} & \text{- couples well with the 1st excited st.} \\ O_2^{t=2} & \text{- …} \\ O_3^{t=2} & \\ \cancel{O_4^{t=2}} & \text{- incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations} \end{array} \right.$$

$$\begin{pmatrix} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \end{pmatrix}$$

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- Example:

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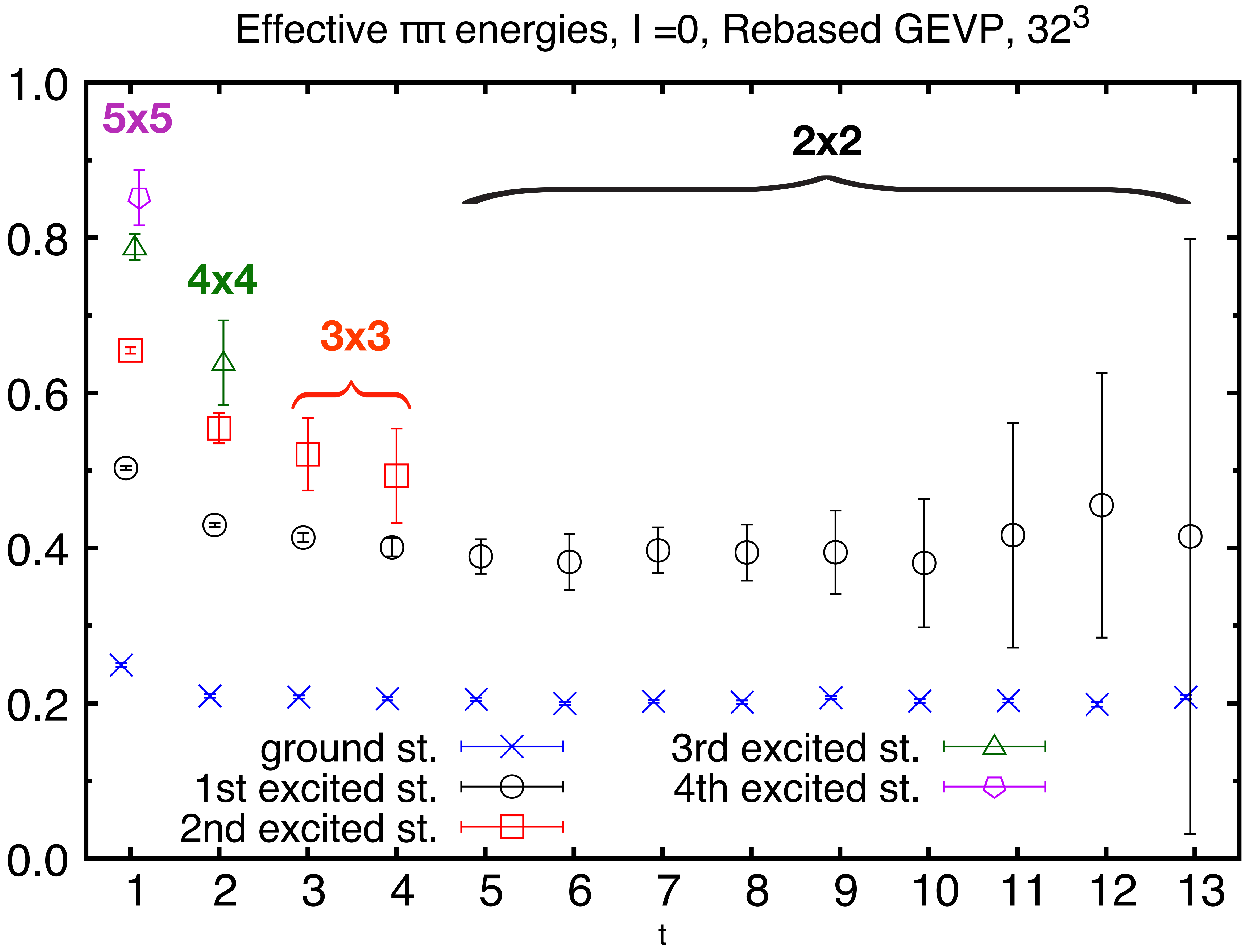
- couples well with the ground st.
- couples well with the 1st excited st.
- …
- incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations

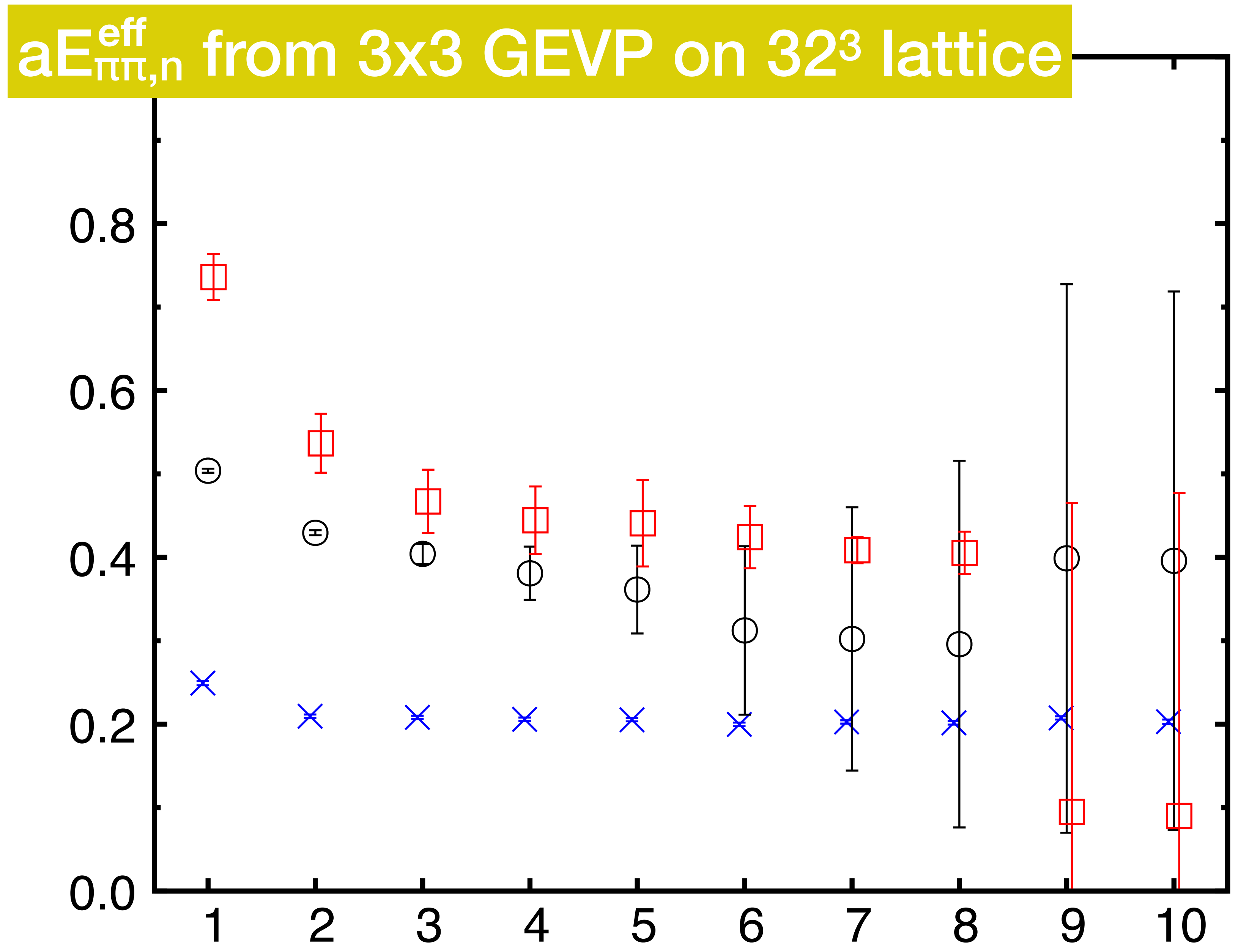
$$\begin{pmatrix} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \end{pmatrix} \xrightarrow{\text{4x4 GEVP at t = 3, ...}}$$

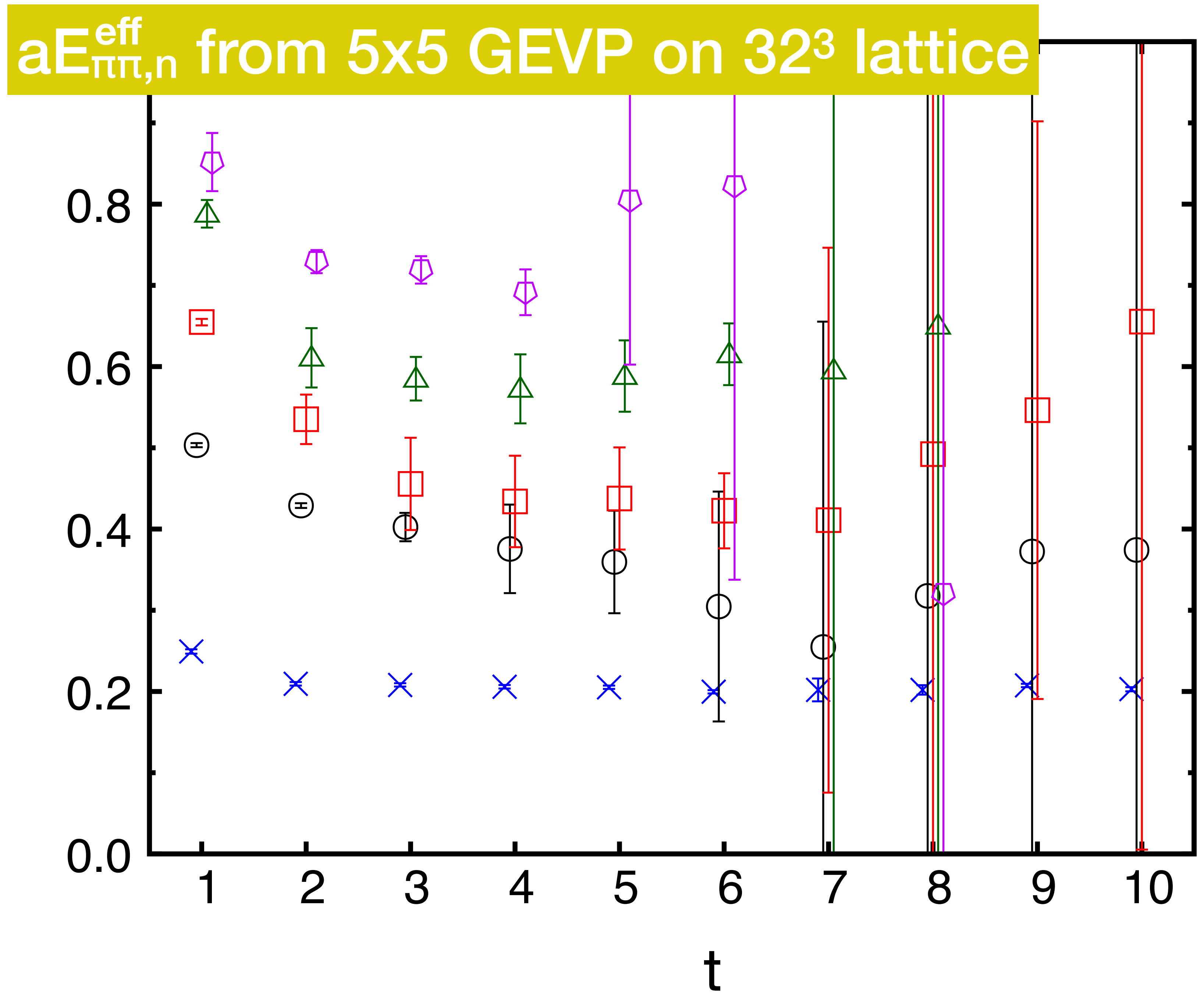
- 3 operators to form next basis
- 1 operator to be excluded from next basis

# Rebased GEVP signals



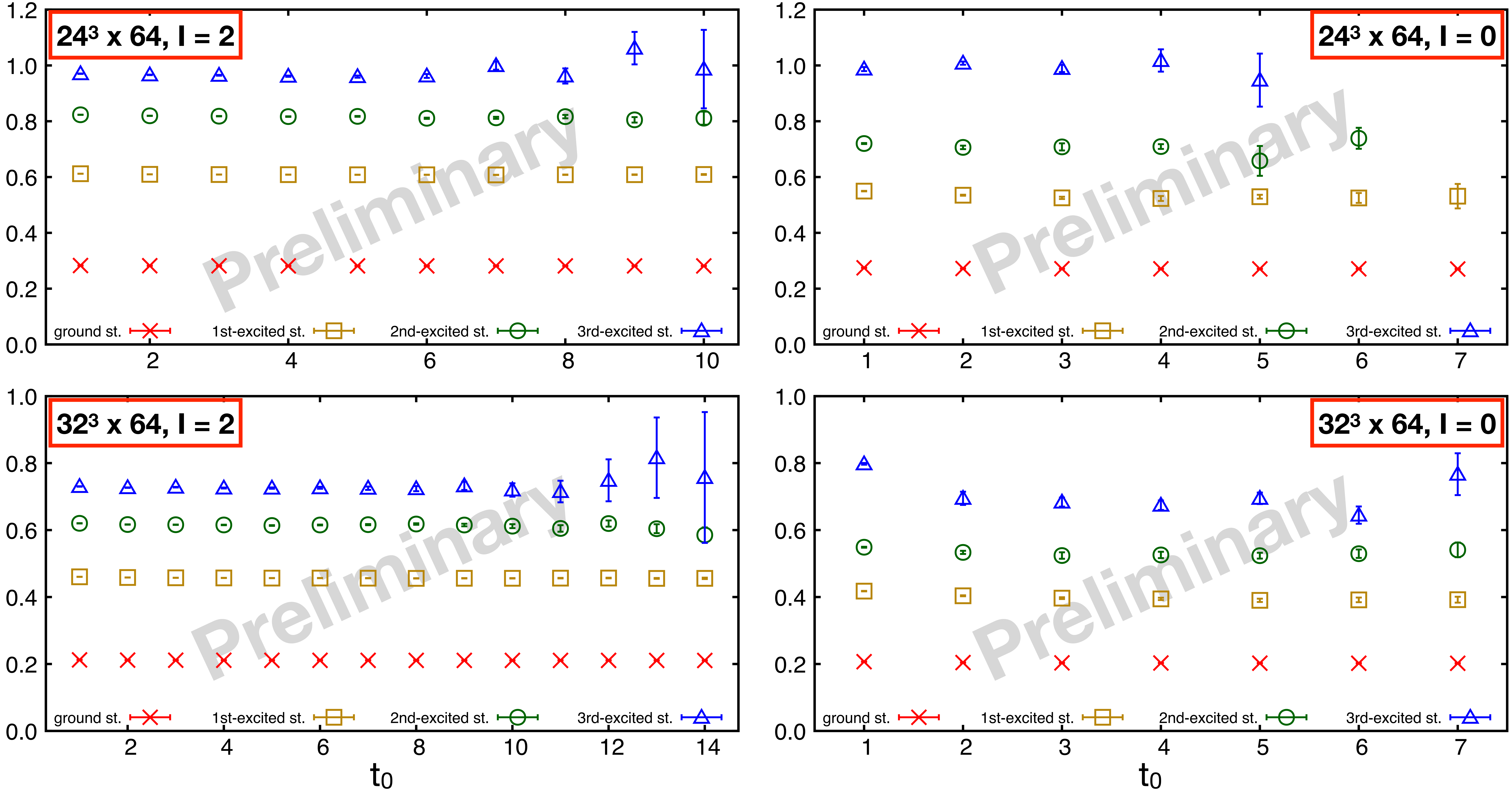

- Plateau seen for the first two states!
- Increasing statistics



# Most recent $aE^{eff}_{\pi\pi}$

All preliminary with new data set



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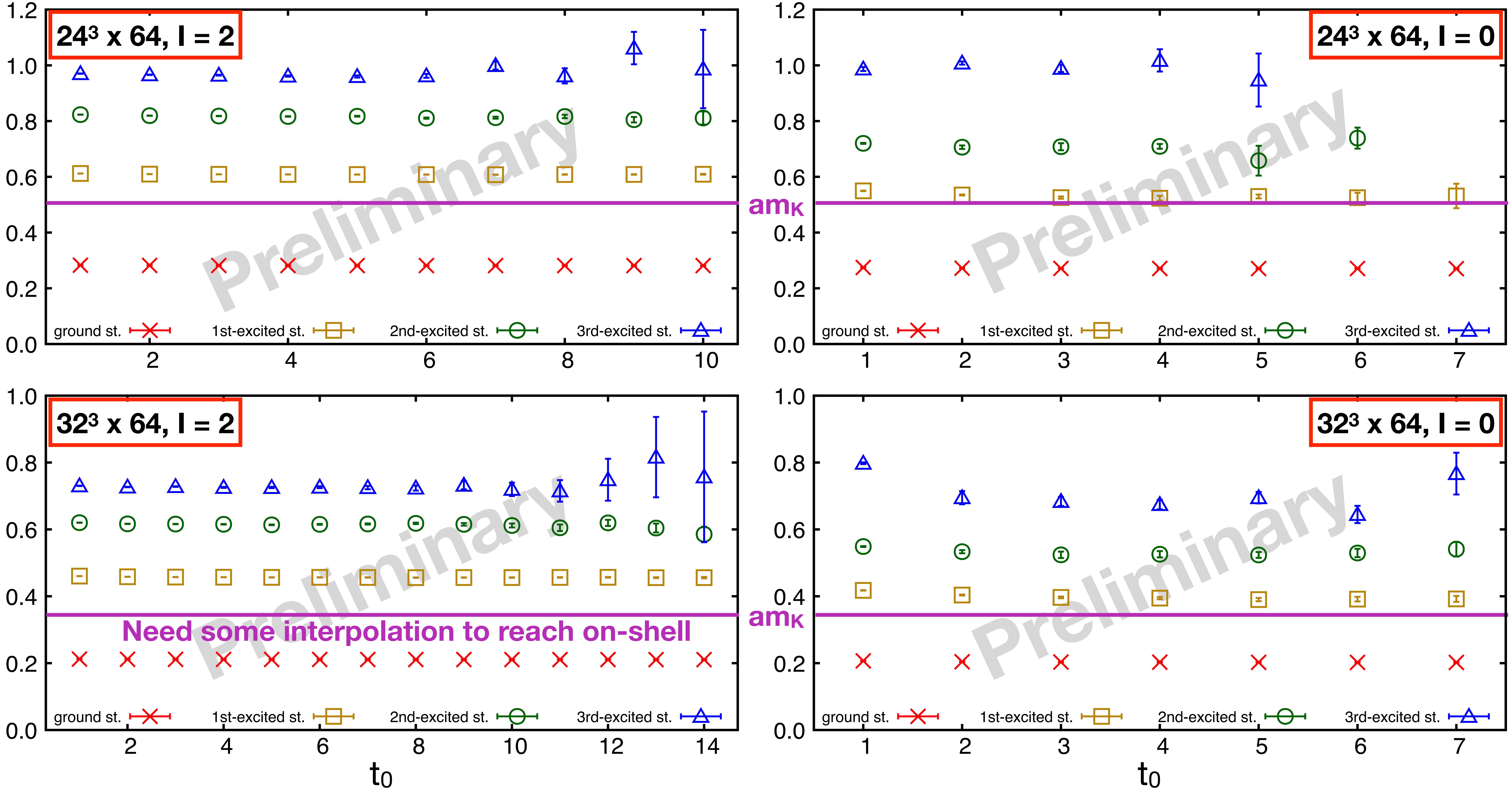


# Cosh effect

- Subtraction of constant artifacts (vacuum & thermal effects)

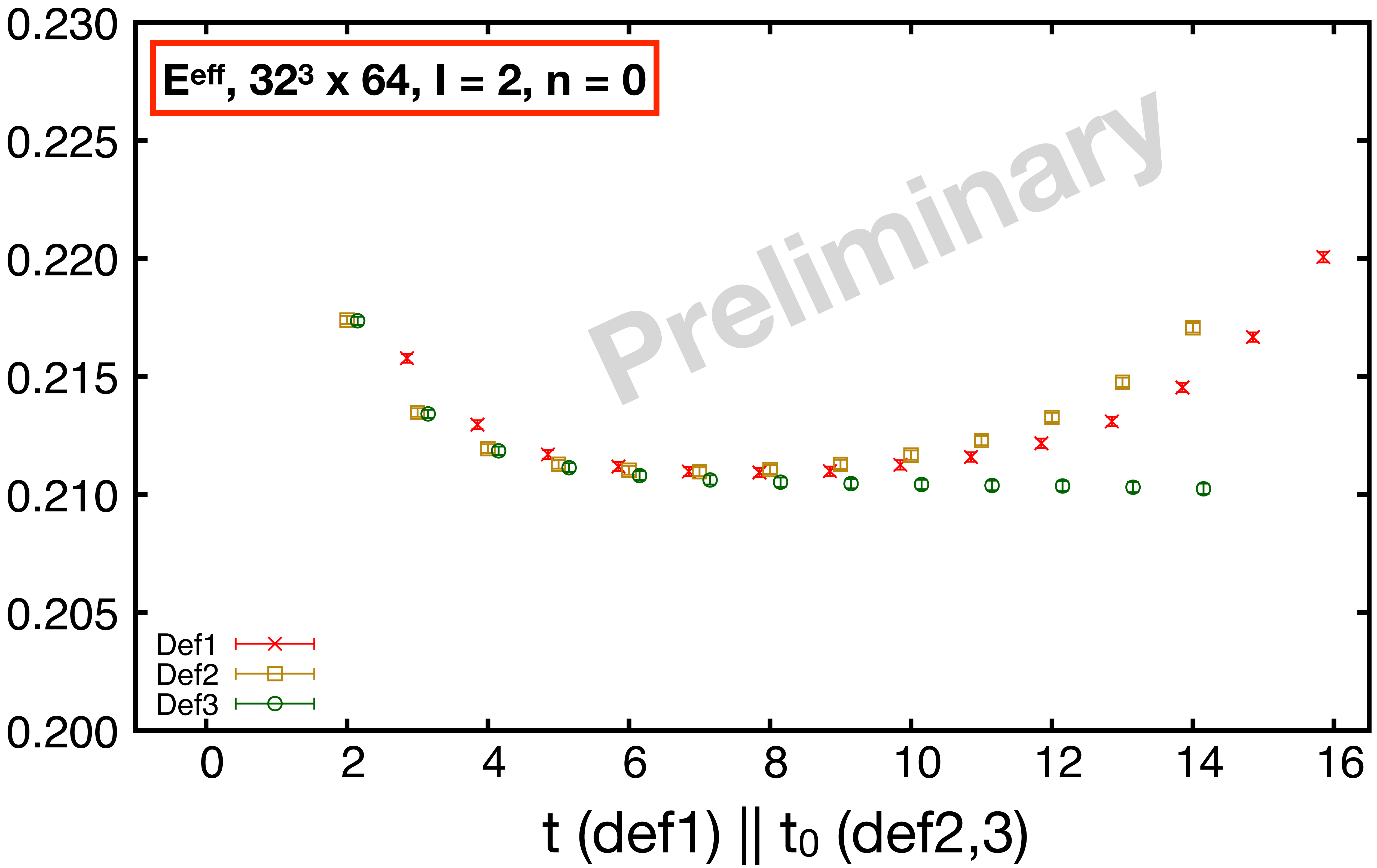
$$C_{ab}(t) \to C_{ab}(t) - C_{ab}(t+\delta_t) = \sum_n A_{n,a}A^*_{n,b}(1-e^{-E_n\delta_t})e^{-E_n t}$$

- GEVP eigenvalue

$$\lambda_n(t,t_0) \to \frac{e^{-E_n t} - e^{-E_n(T'-t)}}{e^{-E_n t_0} - e^{-E_n(T'-t_0)}} \quad \textbf{(*)}$$

$$(T' = T - 2\Delta - \delta_t)$$

- Effective energy definitions
  - def1: $\ln(\lambda_n(t,t_0)/\lambda_n(t+1,t_0))$
  - def2: $-\ln(\lambda_n(t,t_0))/(t-t_0)$
  - def3: solution for **(*)**



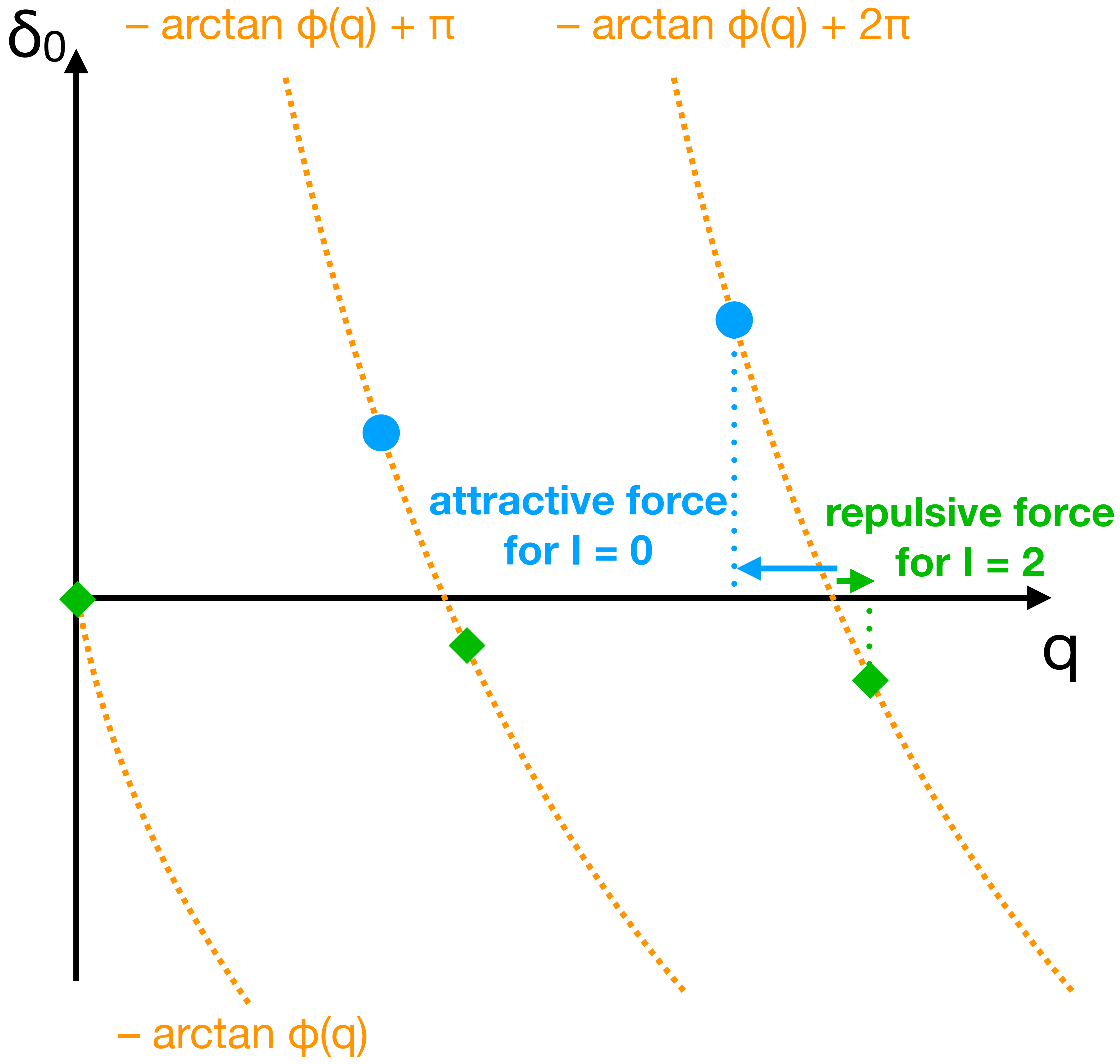
# Phase shifts $\delta_l$

- Lüscher 1991 (valid in $2m_\pi < E_{\pi\pi} < 4m_\pi$)

$$\tan\delta_l = -\frac{\pi^{3/2} q}{Z_{00}(1;q^2)} \equiv -\phi(q)$$

$$q = \frac{L}{2\pi}\sqrt{\frac{E_{\pi\pi}^2}{4} - m_\pi^2}$$

$$Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{\vec{n}}(|\vec{n}|^2 - q^2)^{-s}$$



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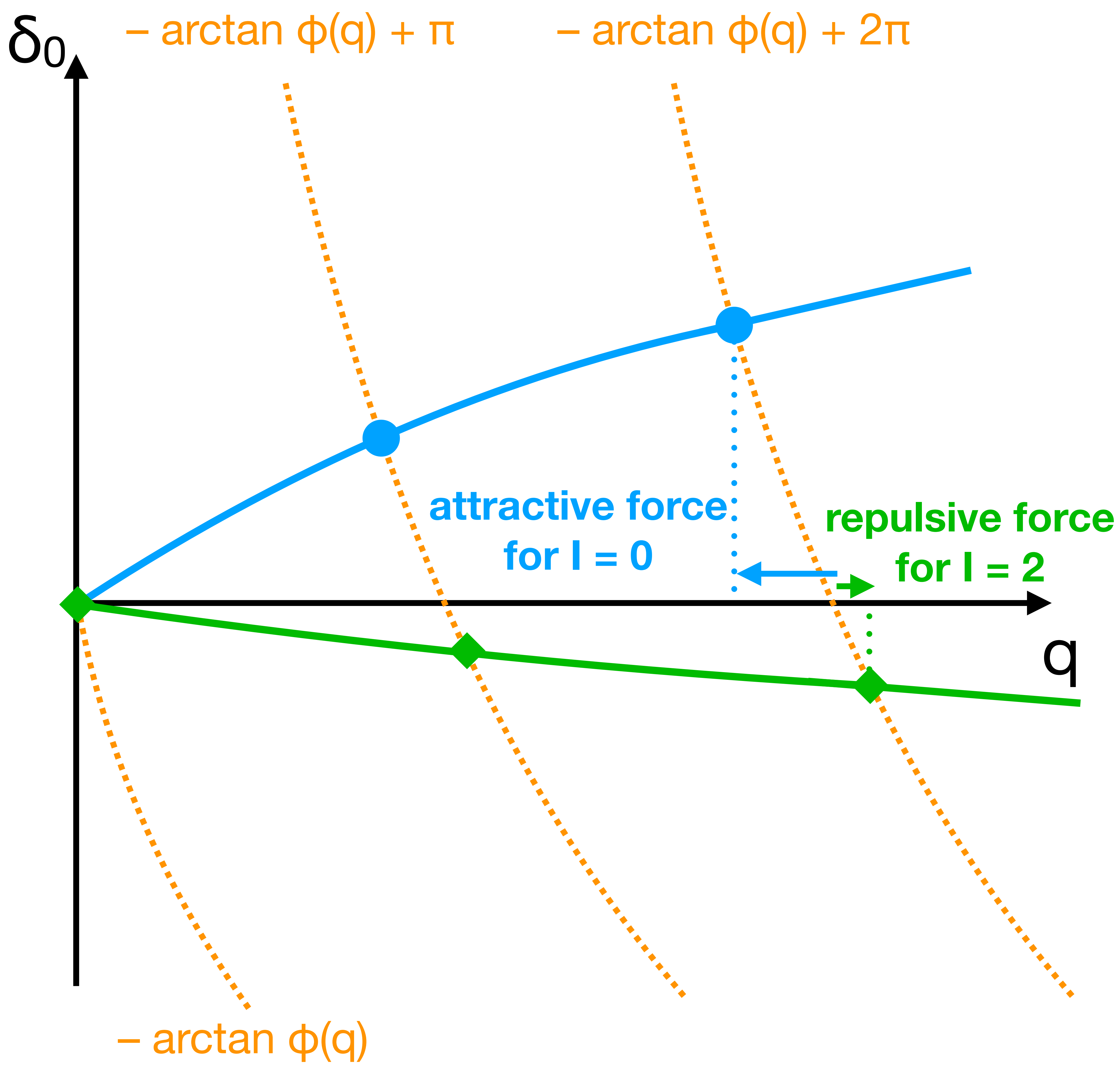

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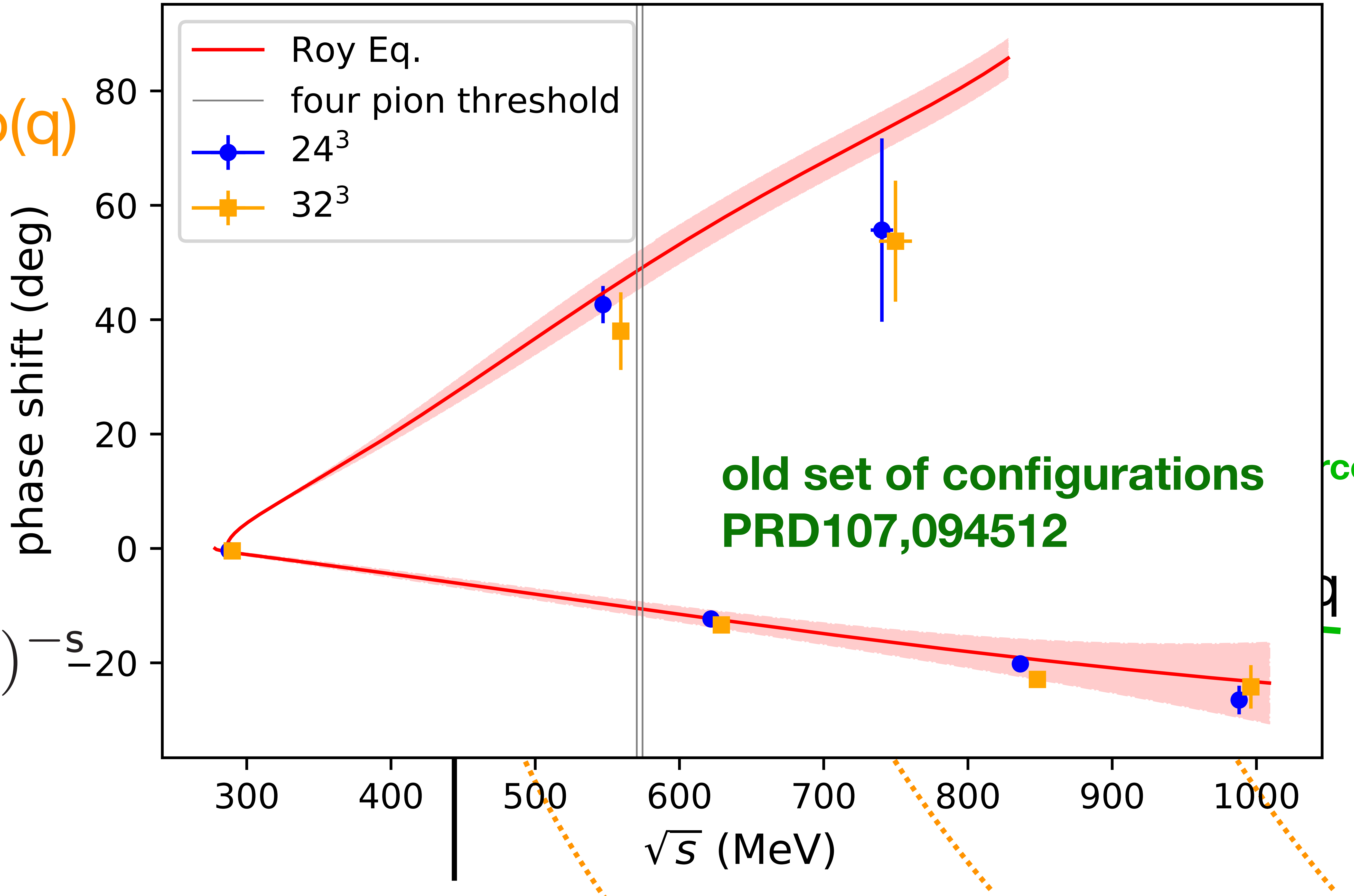



$-\arctan\phi(q)$

# Scattering lengths

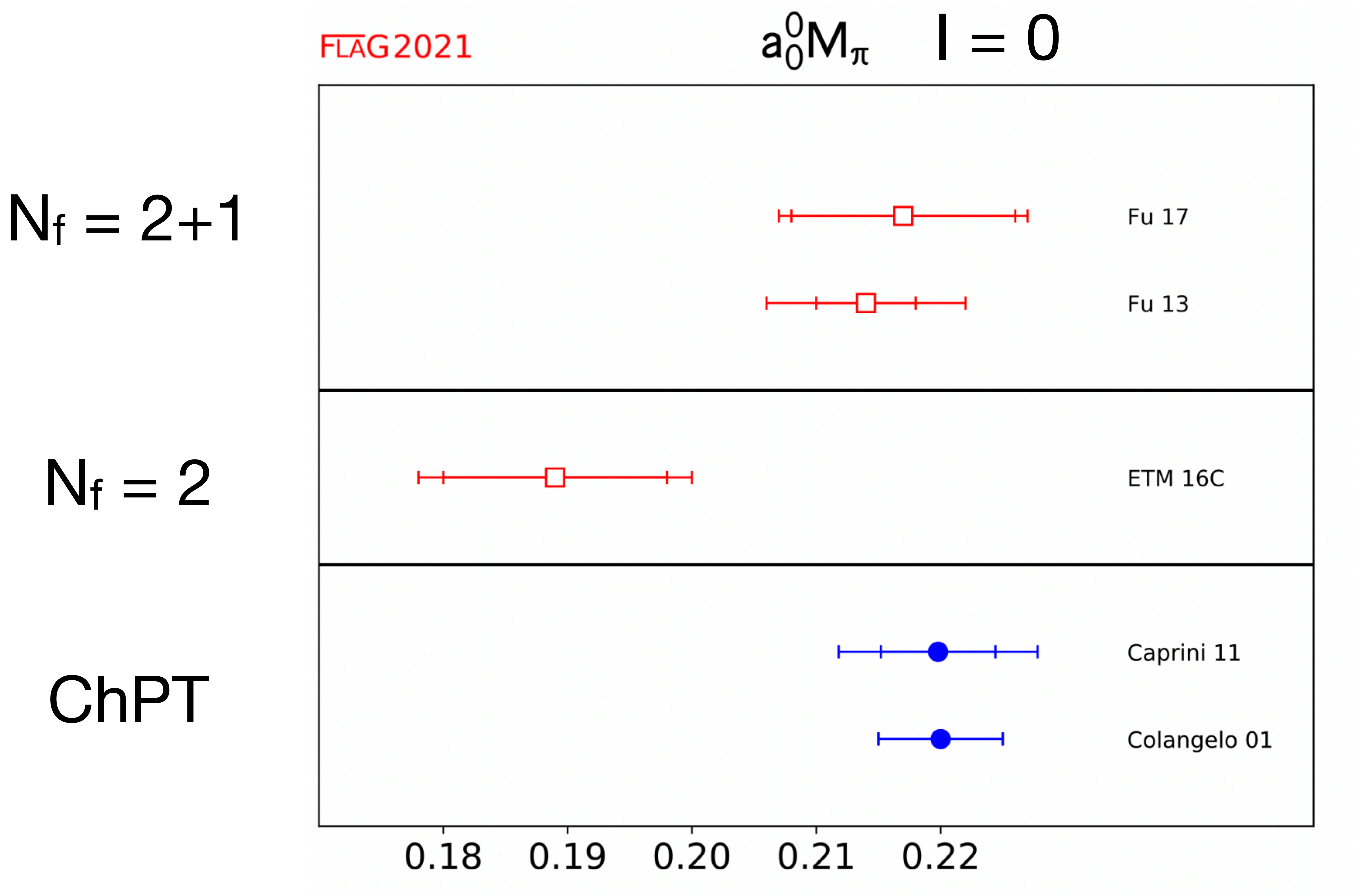
$$k\cot\delta_0^I(k) = \frac{1}{a_0^I} + \frac{1}{2}r_0^I k^2 + O(k^4) \quad \Rightarrow \quad a_0^I \simeq \frac{\tan\delta_0^I(k)}{k}$$

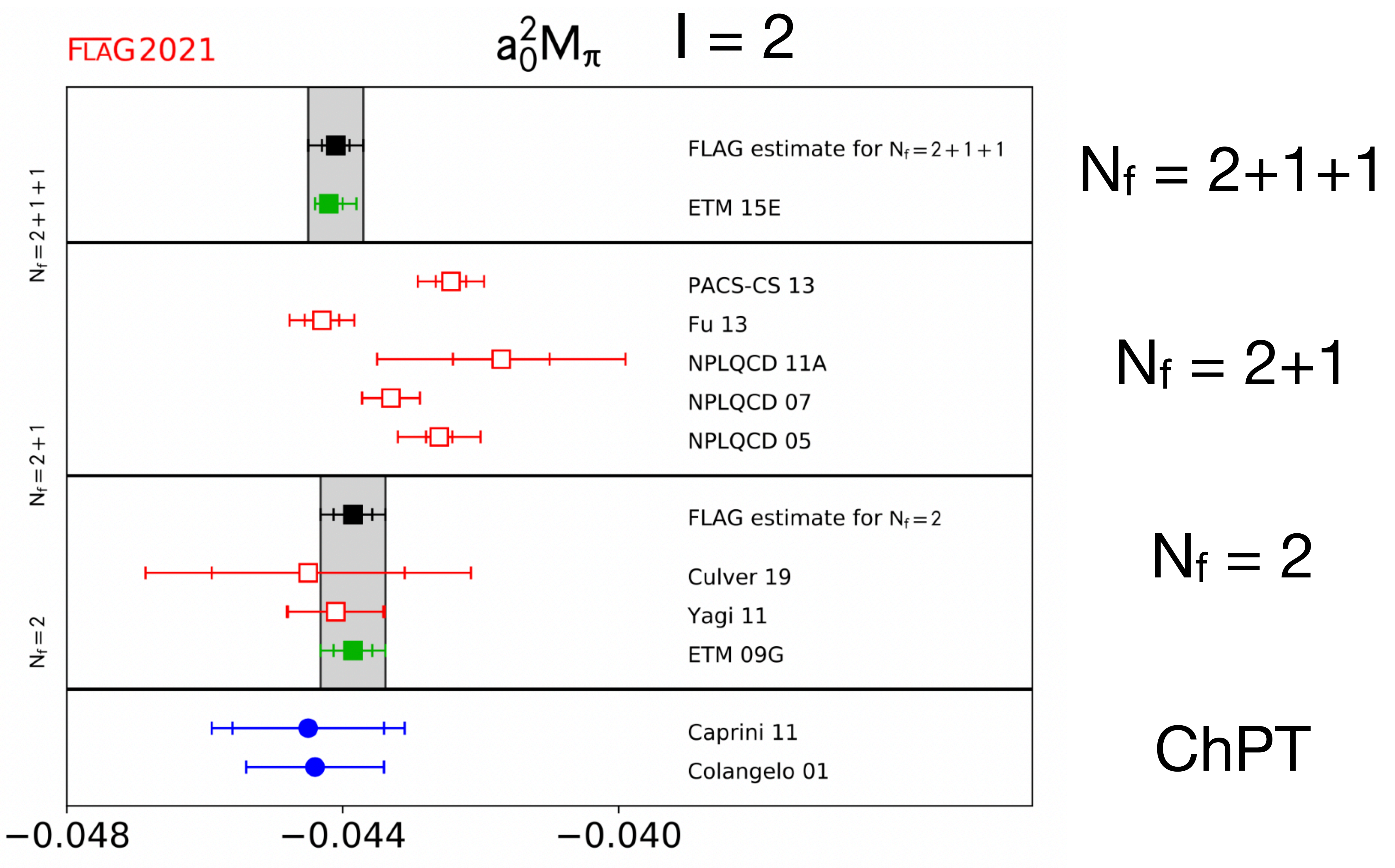
scattering length

for k of the ground state

$$\left(k = \sqrt{\frac{E_{\pi\pi}^2}{4} - m_\pi^2}\right)$$

- FLAG 2021



# Scattering lengths

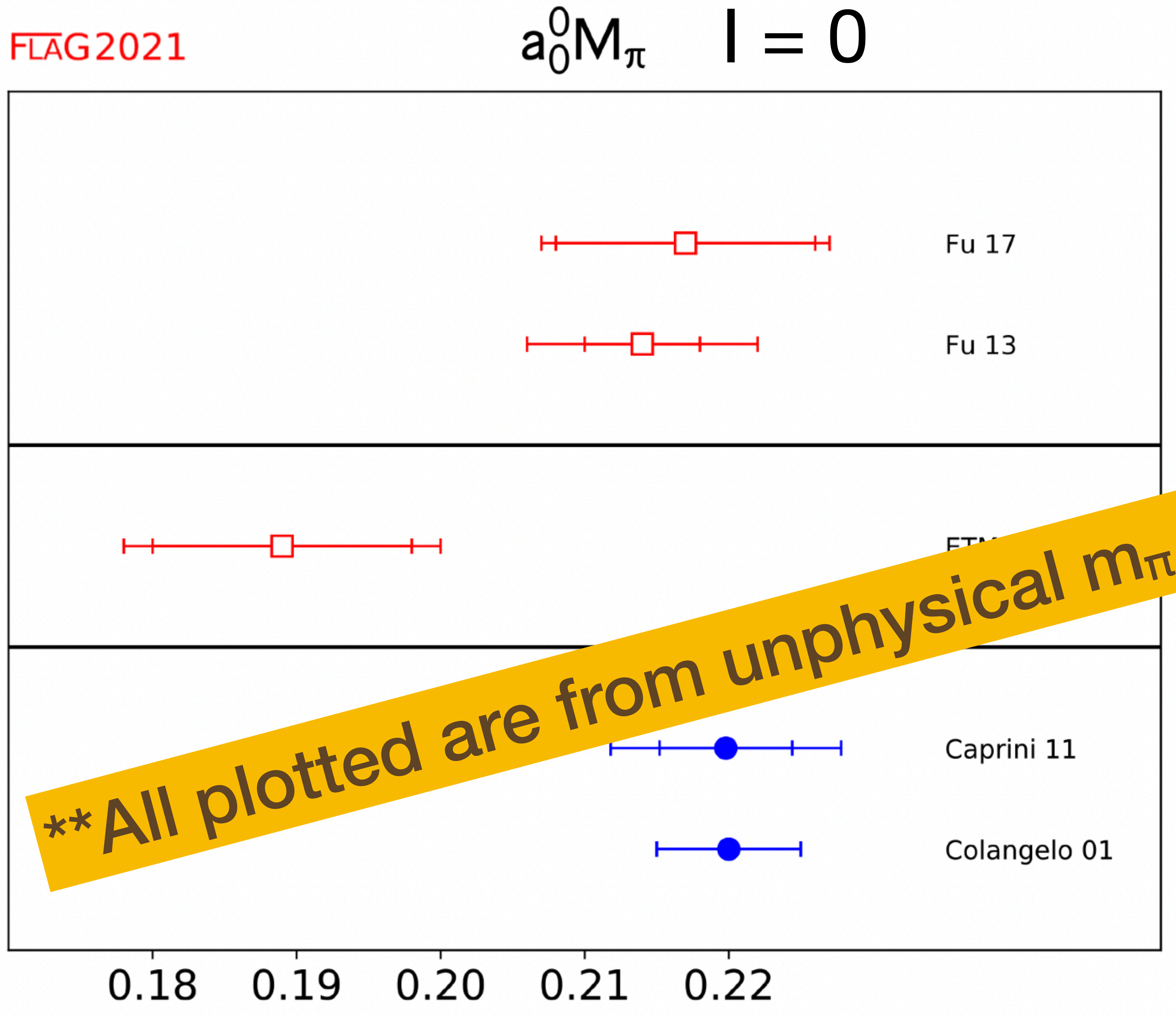
$$k \cot \delta_0^I(k) = \frac{1}{a_0^I} + \frac{1}{2} r_0^I k^2 + O(k^4) \Rightarrow a_0^I \simeq \frac{\tan \delta_0^I(k)}{k} \quad \text{for } k \text{ of the ground state}$$

scattering length

$$\left( k = \sqrt{\frac{E_{\pi\pi}^2}{4} - m_\pi^2} \right)$$

- FLAG 2021

$a_0^0 M_\pi$ I = 0

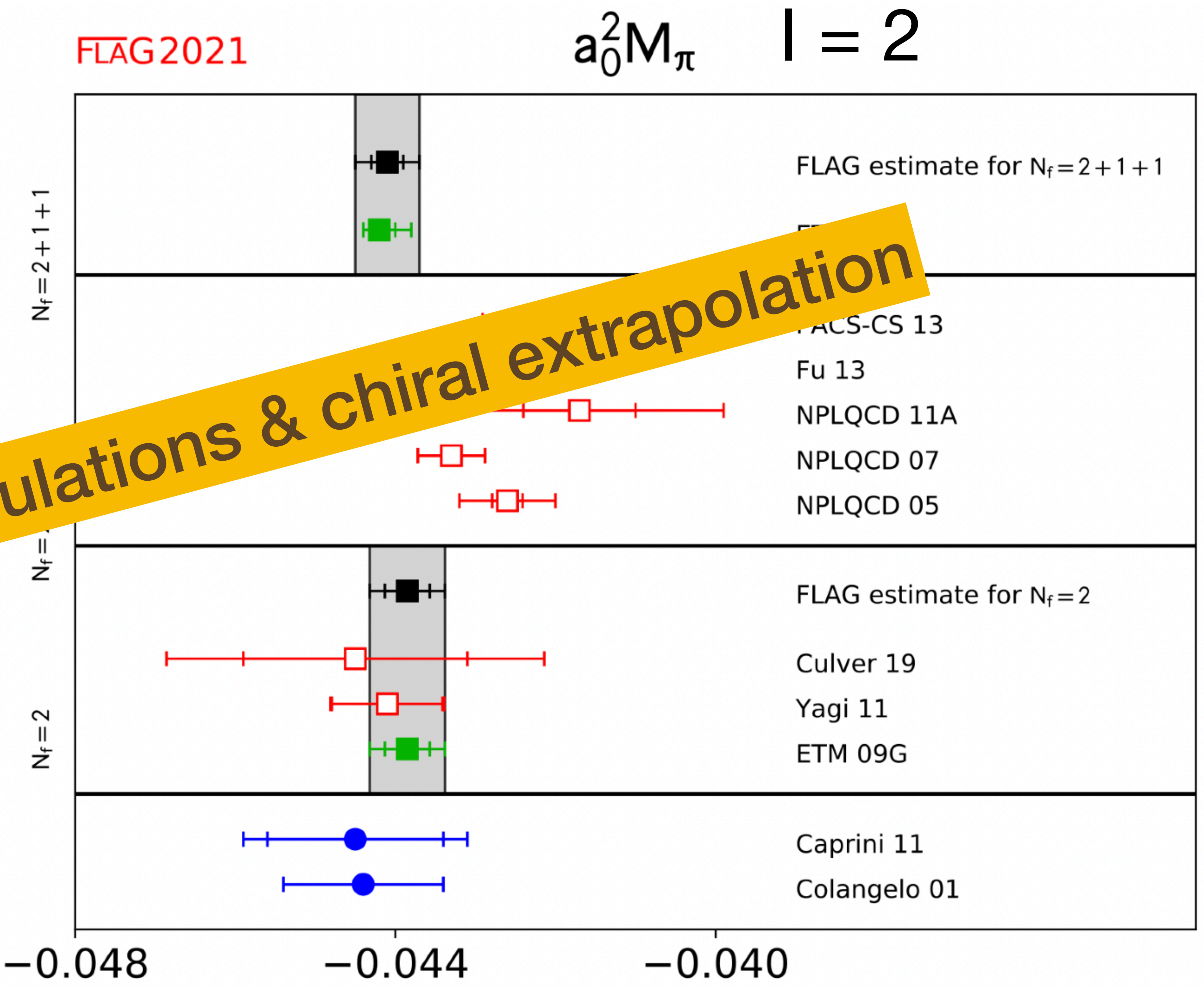



$N_f$ = 2+1

$N_f$ = 2

ChPT

$a_0^2 M_\pi$ I = 2




$N_f$ = 2+1+1

$N_f$ = 2+1

$N_f$ = 2

ChPT

**All plotted are from unphysical $m_\pi$ simulations & chiral extrapolation

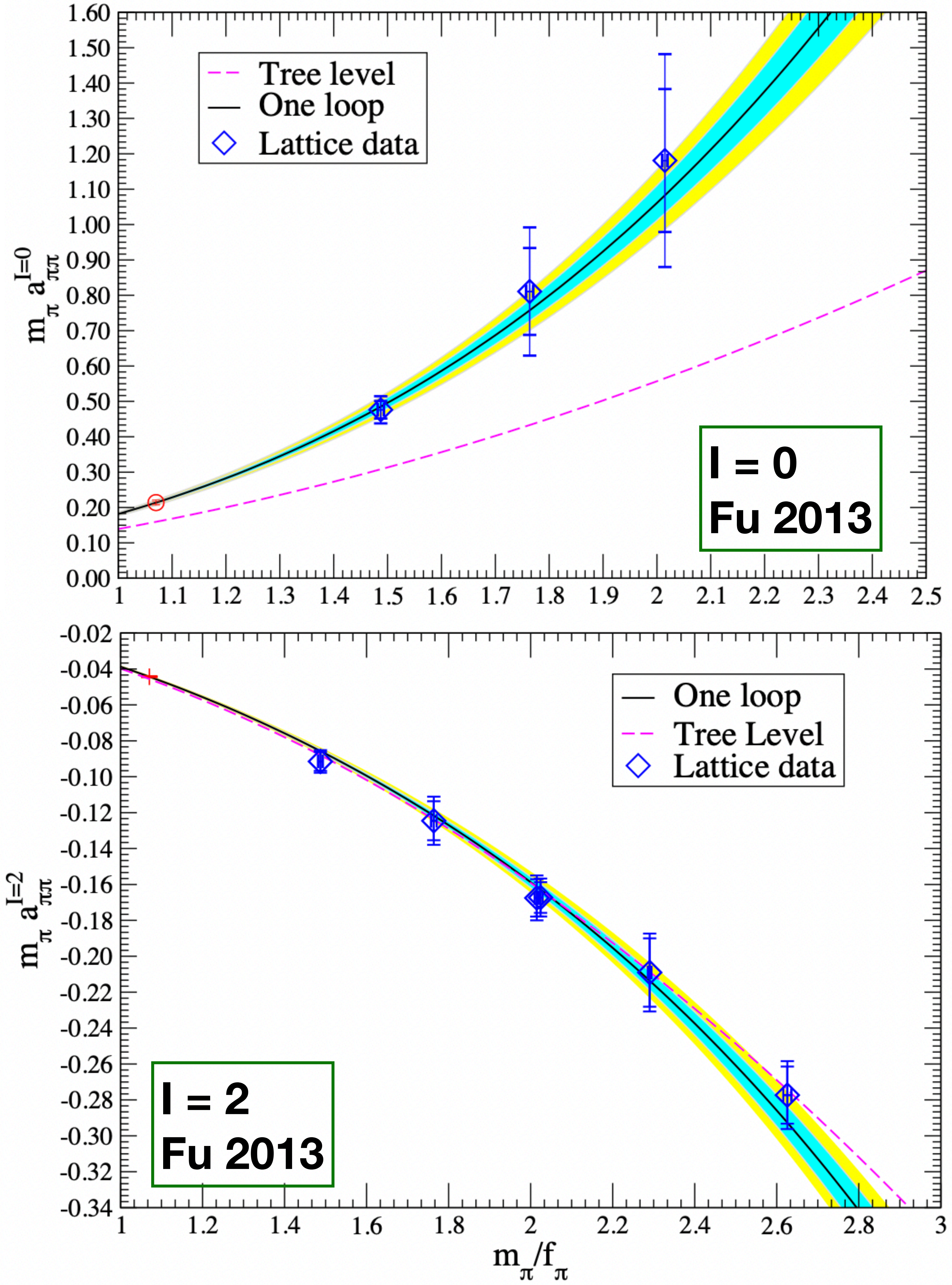
# Chiral extrapolation of $a_0^I m_\pi$

- Fit functions (in earlier works using ChPT)

$$m_\pi a_0^0 = \frac{7m_\pi^2}{16\pi f_\pi^2}\left\{1 - \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[9\ln\frac{m_\pi^2}{f_\pi^2} - 5 - l_{\pi\pi}^0\right]\right\}$$

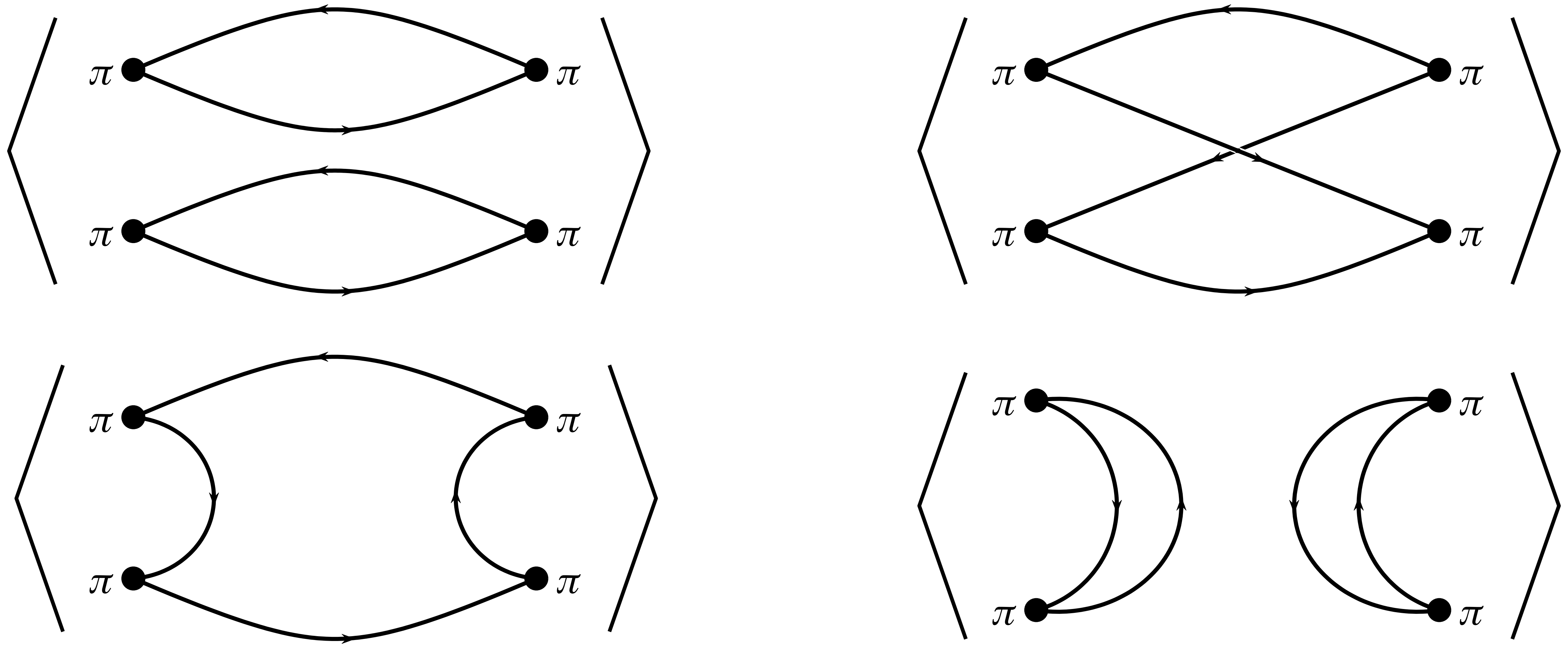
$$m_\pi a_0^2 = -\frac{m_\pi^2}{8\pi f_\pi^2}\left\{1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\frac{m_\pi^2}{f_\pi^2} - 1 - l_{\pi\pi}^2\right]\right\}$$

  - with only $l_{\pi\pi}^I$ as the free parameter
  - input $m_\pi/f_\pi$ gives precise LO
  - lattice data only contribute to NLO
- Result from physical $m_\pi$ simulations meaningful
- Ambitious for physical $m_\pi$ simulations to try to surpass the precision



# Non-interacting ππ 2pt func

- Interacting ππ correlators



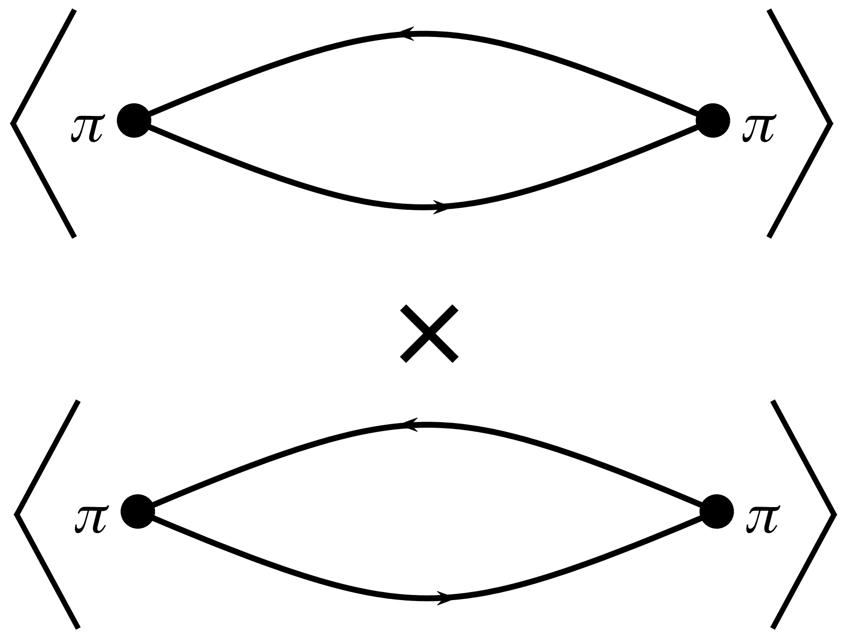
I = 2

I = 0

GEVP

$$\sim e^{-E_{\pi\pi,n}t}$$

- Non-interacting ones



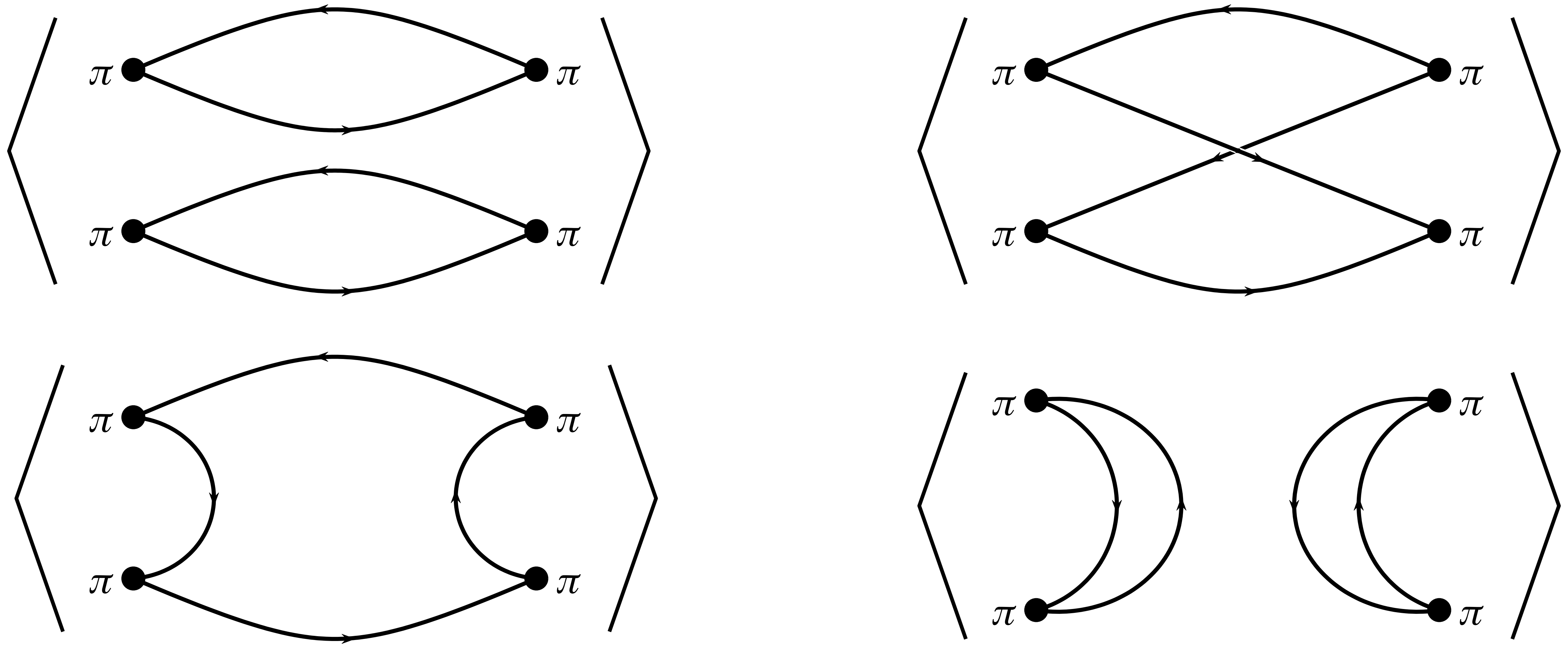
$$\sim e^{-E^{(0)}_{\pi\pi,n}t}$$

$\|$ same value, but
statistical correlation not maximized

$$2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

# Non-interacting ππ 2pt func

- Interacting ππ correlators



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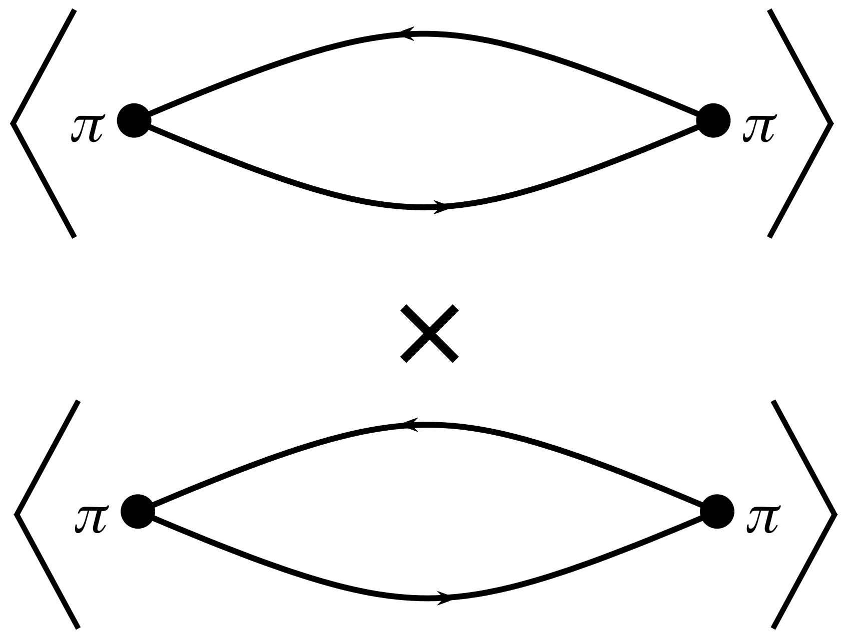


$$\sim e^{-E^{(0)}_{\pi\pi,n}t}$$

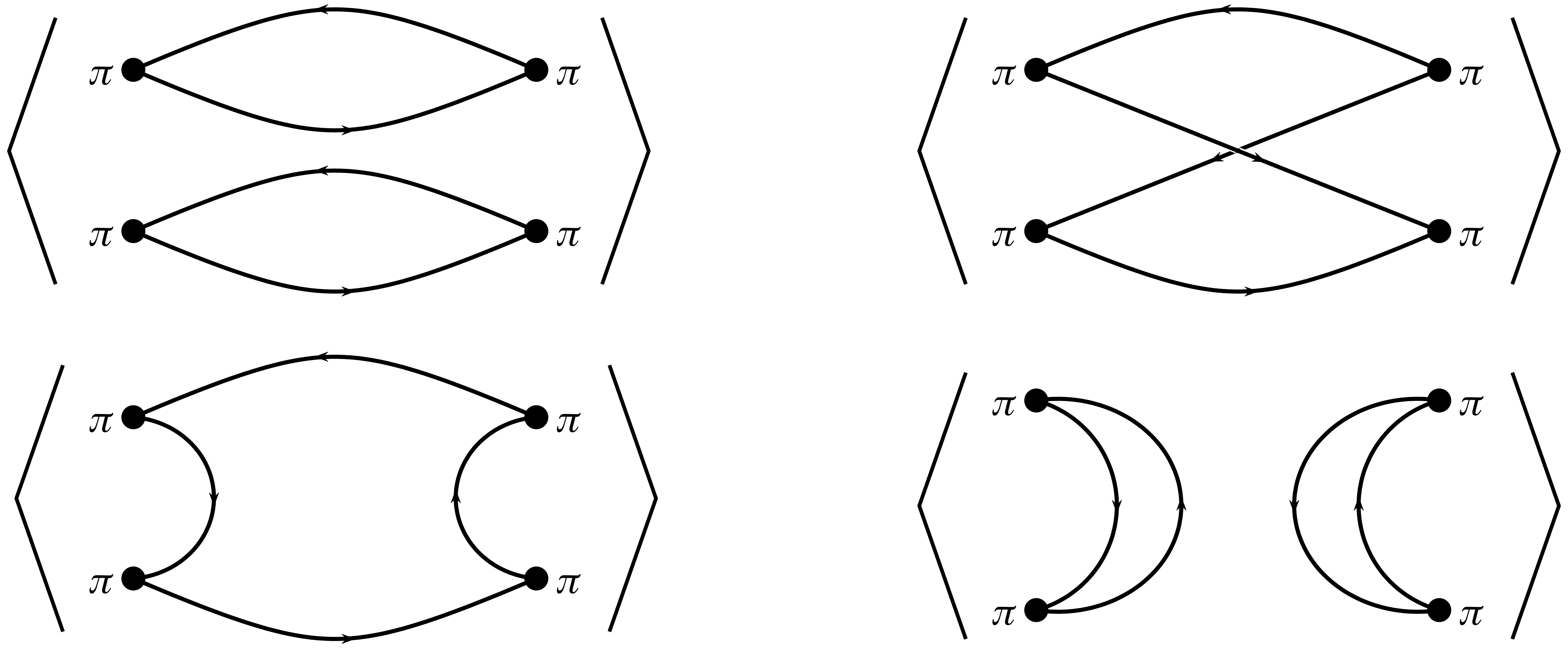
same value, but
statistical correlation not maximized

$$2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

similar values
significant correlation

# Non-interacting ππ 2pt func

- Interacting ππ correlators

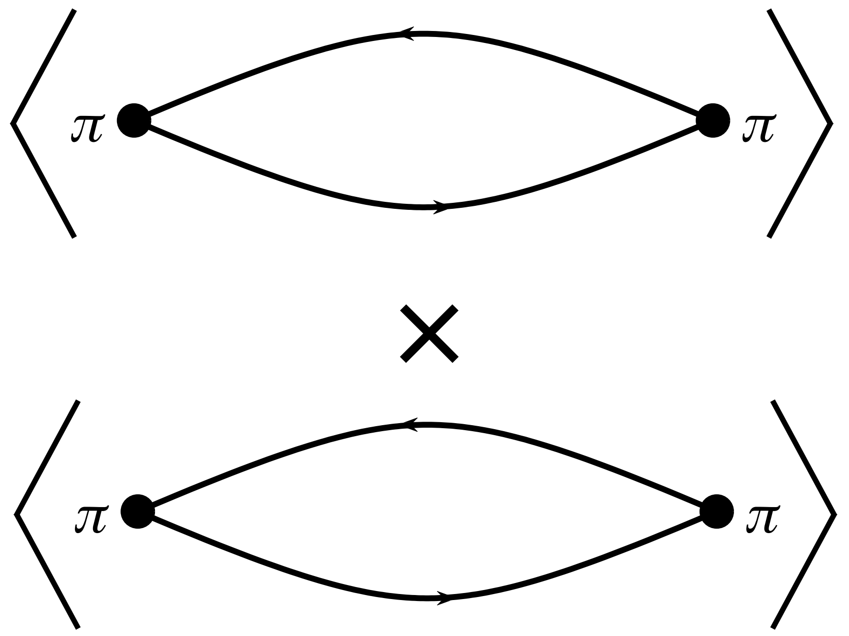



I = 2

I = 0

GEVP $\sim e^{-E_{\pi\pi,n}t}$

- Non-interacting ones




$$\sim e^{-E^{(0)}_{\pi\pi,n}t}$$

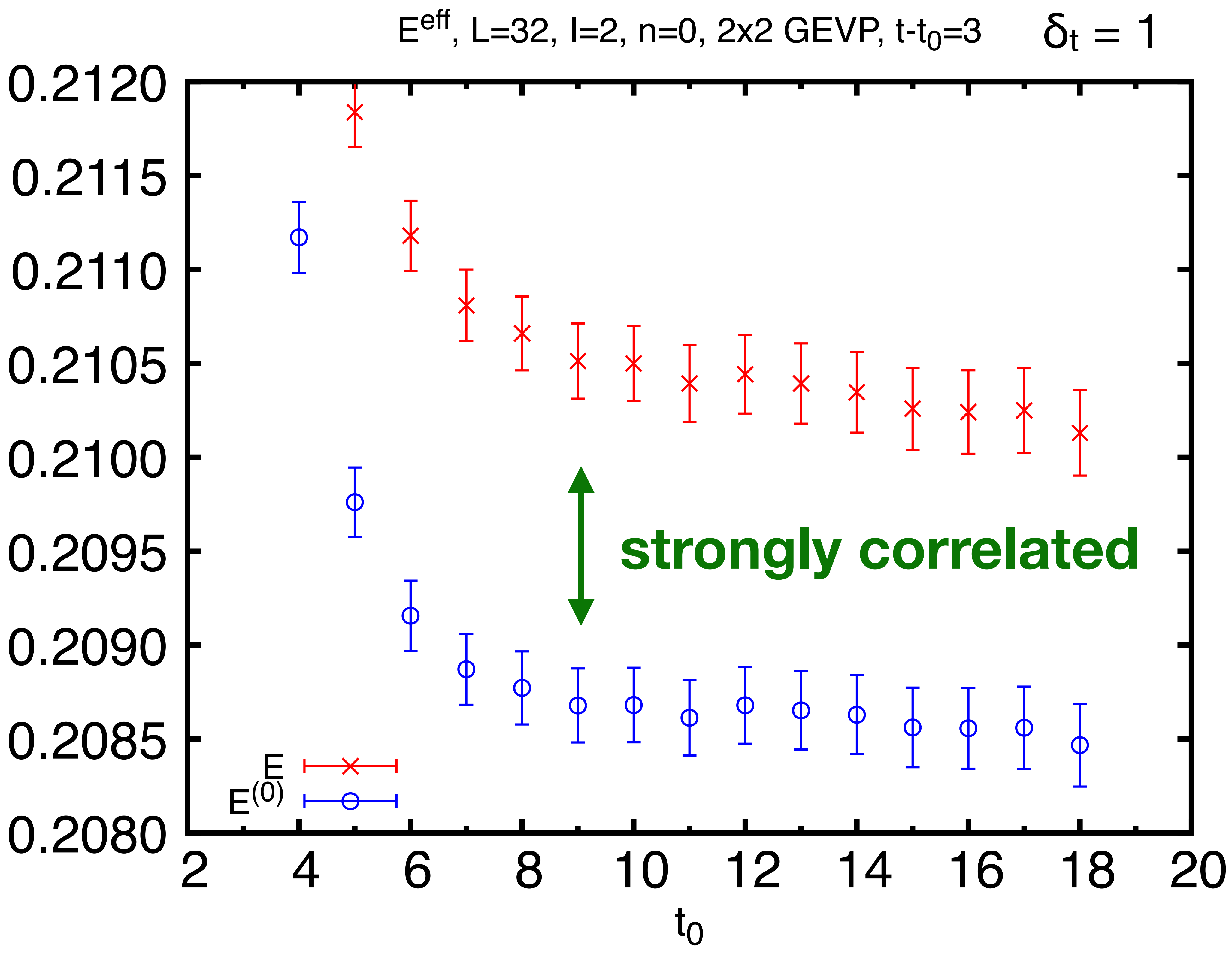
$\|$ same value, but
statistical correlation not maximized

$$2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

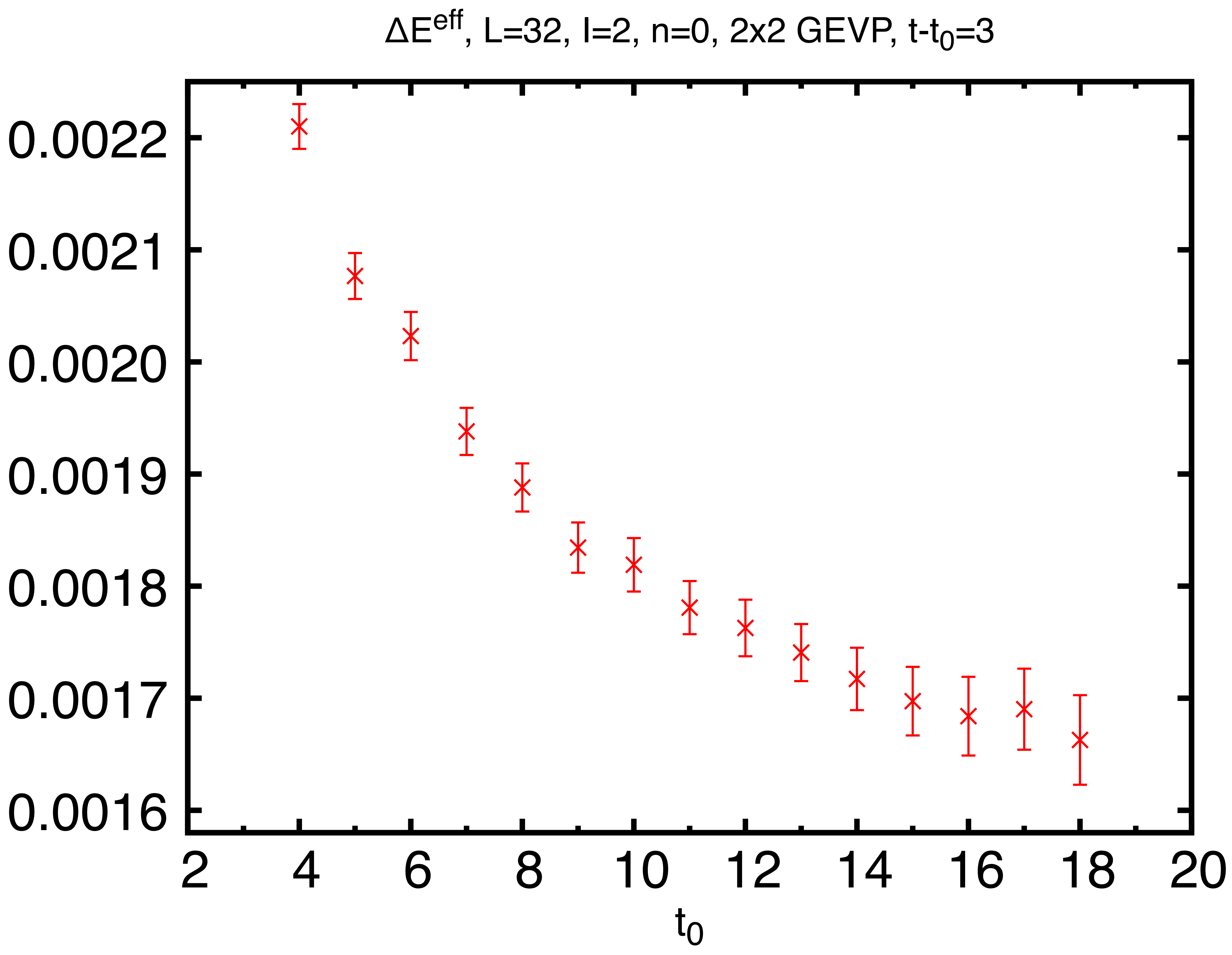
similar values
significant correlation

→ **ratio $\sim e^{-\Delta E_{\pi\pi,n}t}$**
**more precise**

# $\Delta E_{\pi\pi,n=0}$ vs $E_{\pi\pi,n=0}$



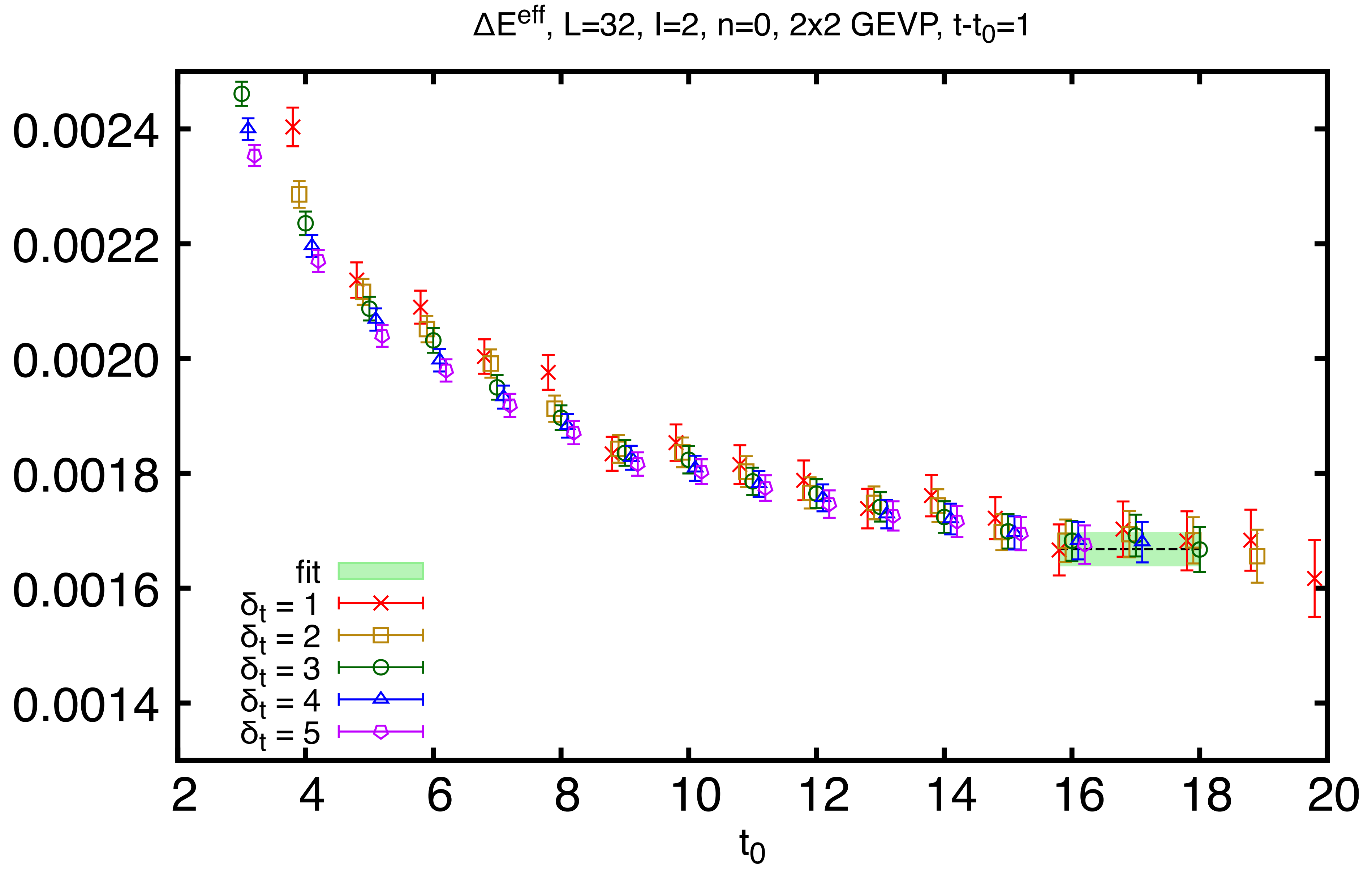

difference




- Error drastically decreased
- Plateau from $t_0 = 16$ (see also next slide)

# Preliminary result for $a_0^I m_\pi$

| | | |
|---|---|---|
| $\Delta E_0$ | fit | 0.001668(30) |
| $E_0 = 2m_\pi + \Delta E_0$ | | 0.21025(19) |
| phase shift & scattering length | Lüscher formalism | $\delta_0 = -0.3315(89)^\circ$<br>$a_0^2 m_\pi = -0.04566(81)$ |




- I = 0 needs more investigation (signal loses before $t_0 = 16$)
- I = 2 reaching the FLAG precision of 2%
- need investigation of systematic error
- may need scaling correction wrt $(m_\pi / f_\pi)^2$

# $K \to \pi\pi$ calculation

# Matrix elements

- For extraction of ground-state ME

$$M^{\mathrm{eff}}(t_2, t_1) = C^{(3)}(t_2, t_1) \left[ \frac{e^{E^{\pi\pi} t_2} e^{E^K t_1}}{C^{\pi\pi}(t_2) C^K(t_1)} \right]^{1/2} \xrightarrow{\text{large } t_1 \ \& \ t_2} M$$

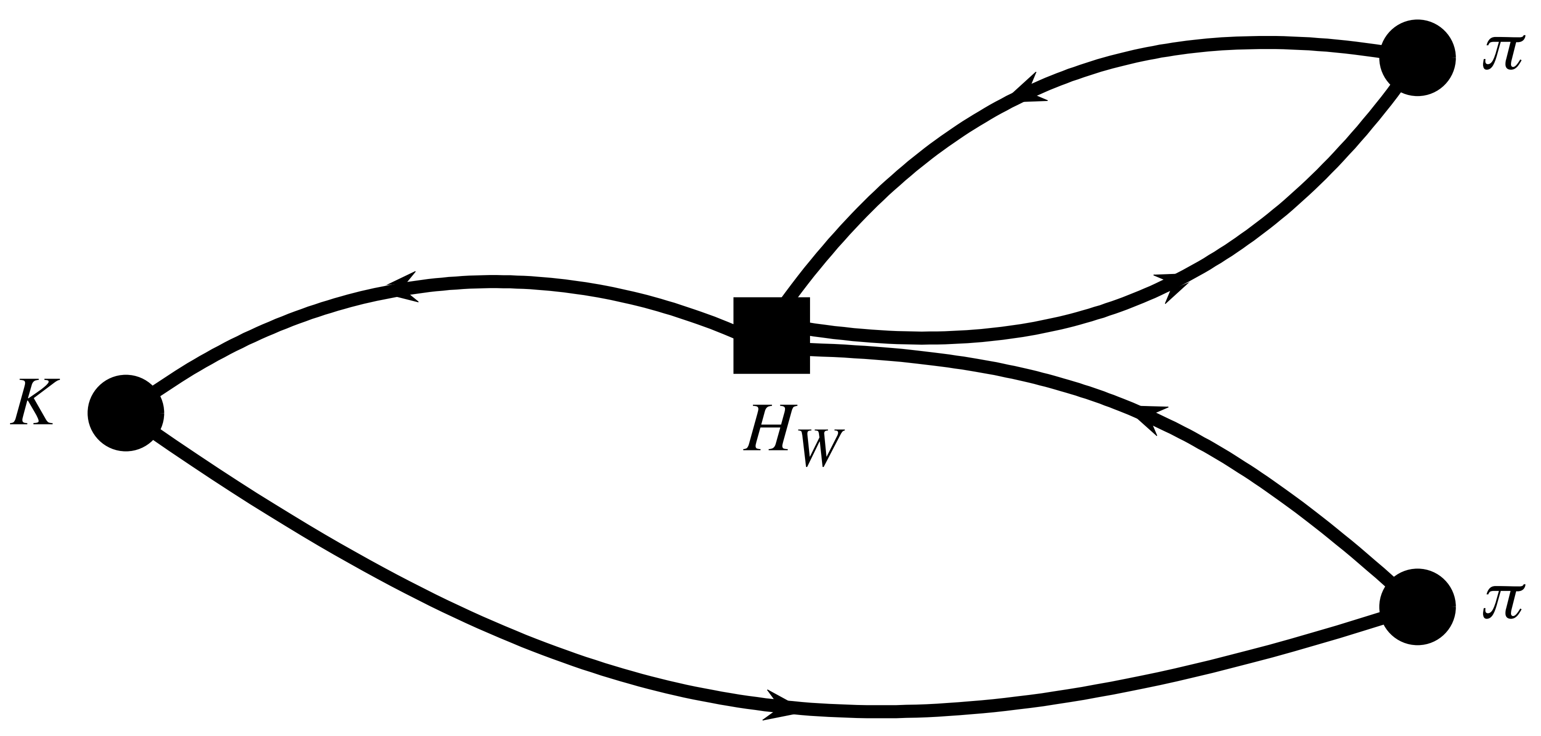
- Excited (n-th) ππ state needed for on-shell kinematics with PBC

$$\boxed{M_n^{\mathrm{eff}}(t_2, t_1) = C_n^{(3)}(t_2, t_1) \left[ \frac{e^{E_n^{\pi\pi} t_2} e^{E^K t_1}}{C_n^{\pi\pi}(t_2) C^K(t_1)} \right]^{1/2}} \xrightarrow{\text{large } t_1 \ \& \ t_2} M_n$$

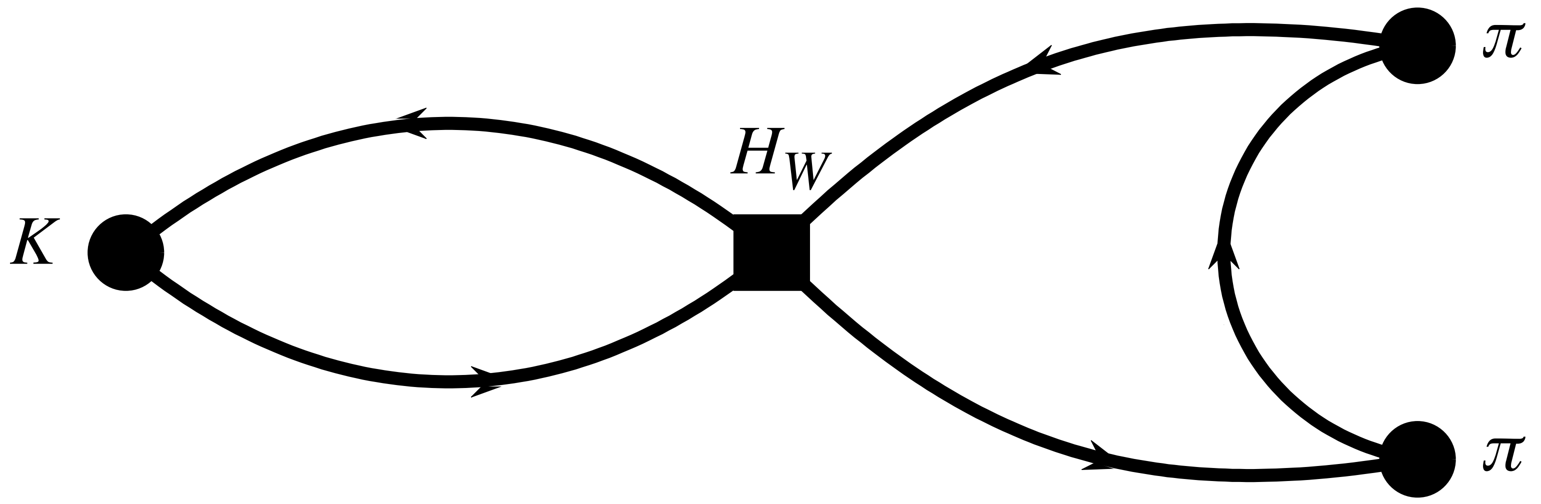
$C_n^{\pi\pi}$ : 2-pt function of ππ operators diagonalized by GEVP

$C_n^{(3)}$ : K → ππ 3-pt function with ππ operator used in $C_n^{\pi\pi}$

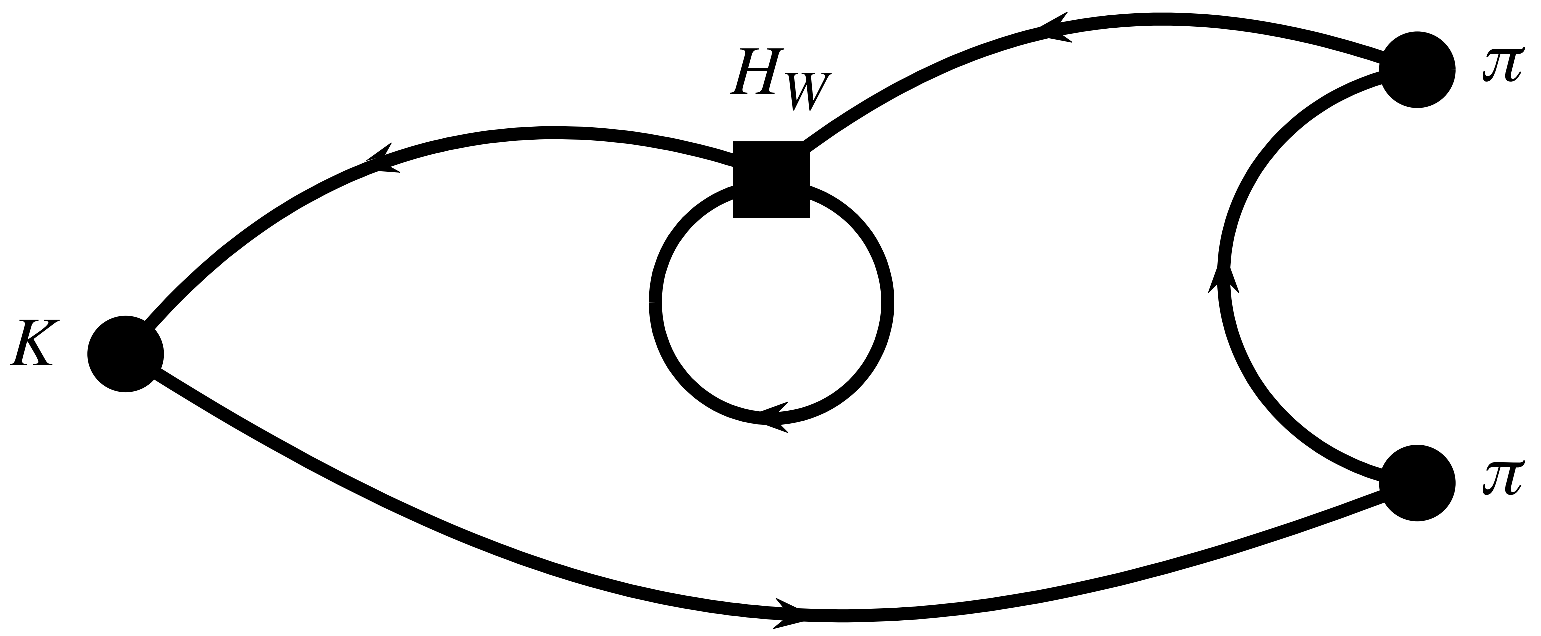
# Diagrams for K → ππ 3pt functions



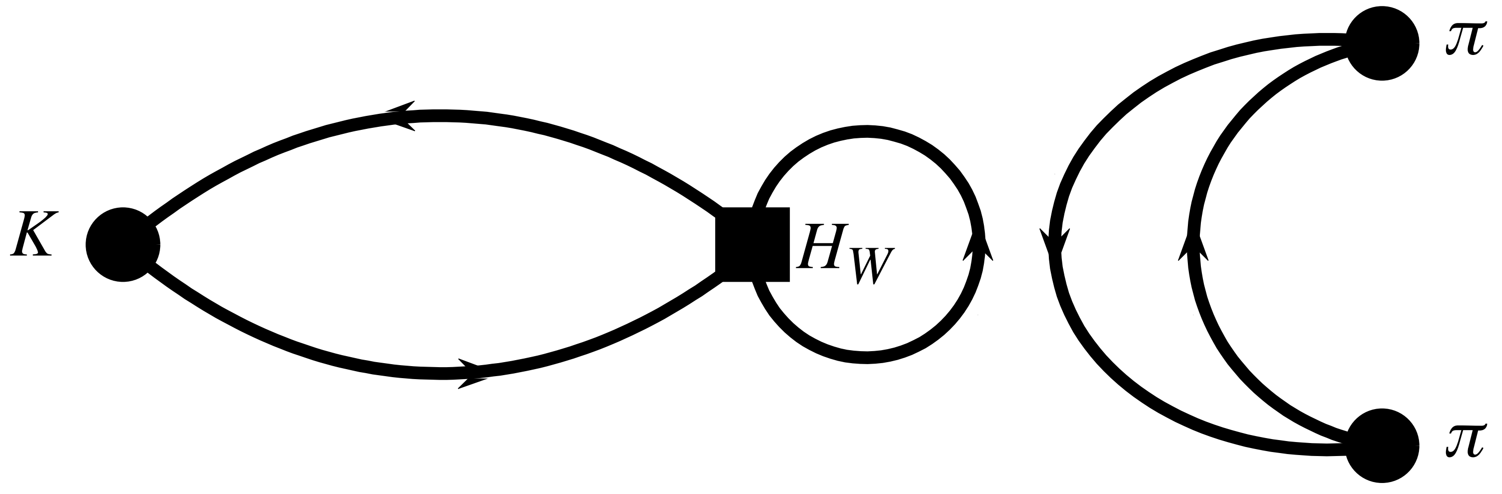

**type 1**




**type 2**




**type 3**



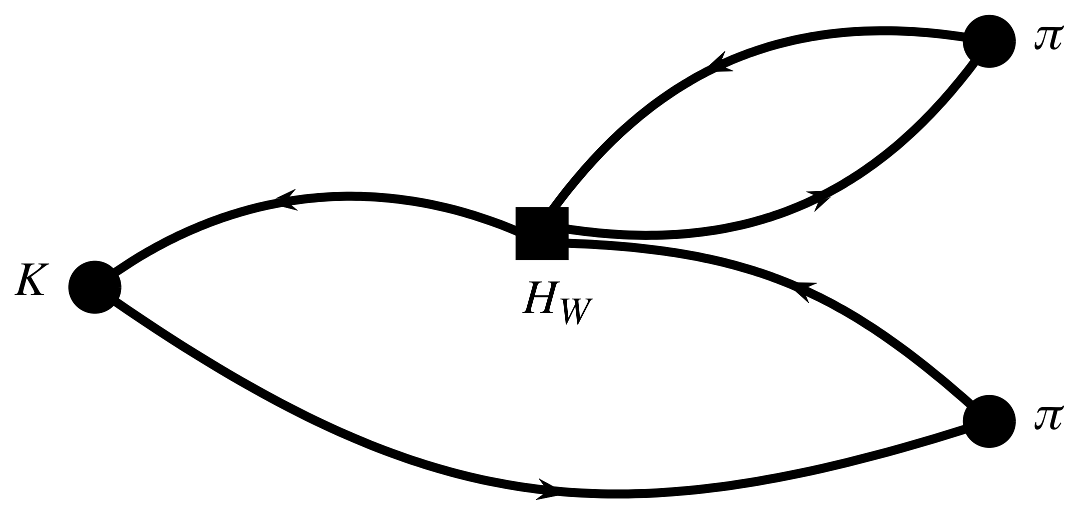

**type 4**

# type 4 dominates stat. error

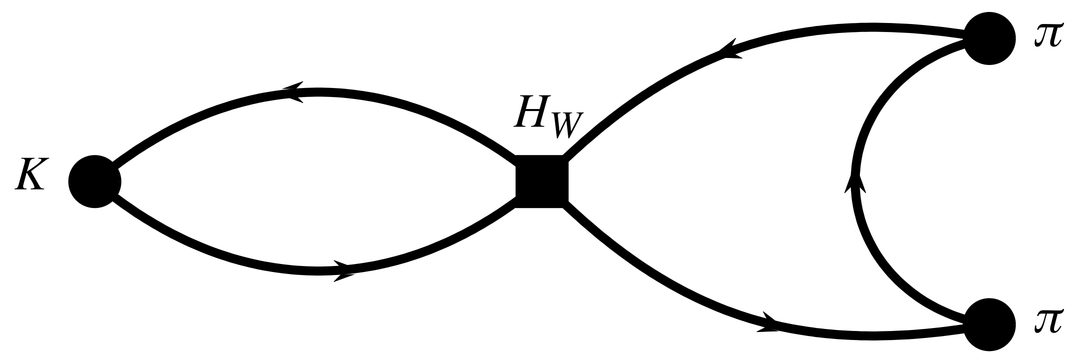
- Previous G-parity calculation
  - types 1,2: averaged over every 8 time translations
  - types 3,4: averaged over every time translation

- types 1,2 still expensive but no need of such precision

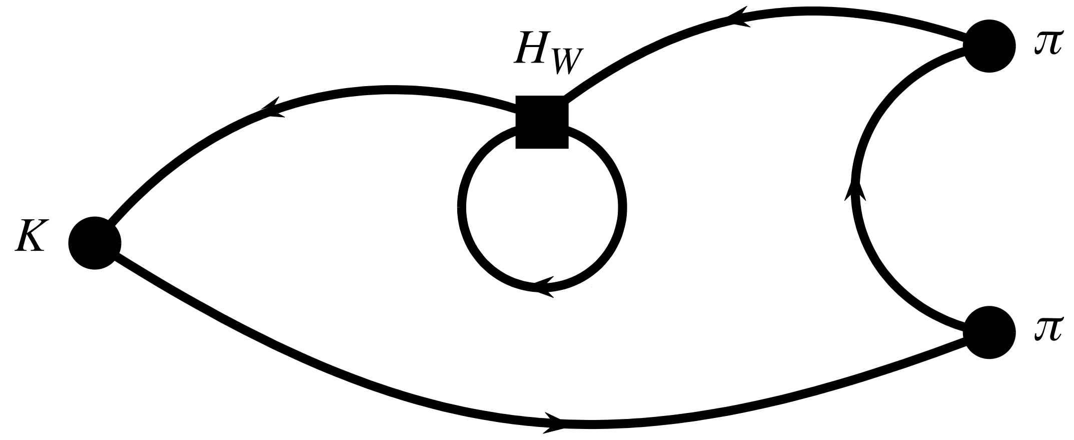
  → cost reduction?



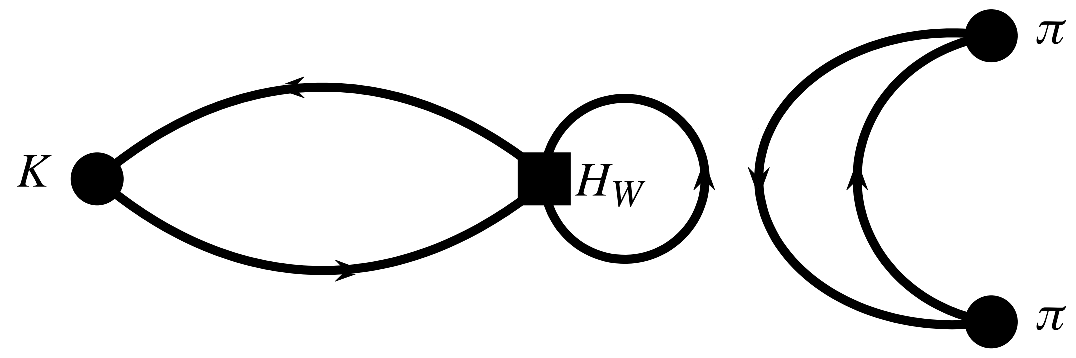

type 1



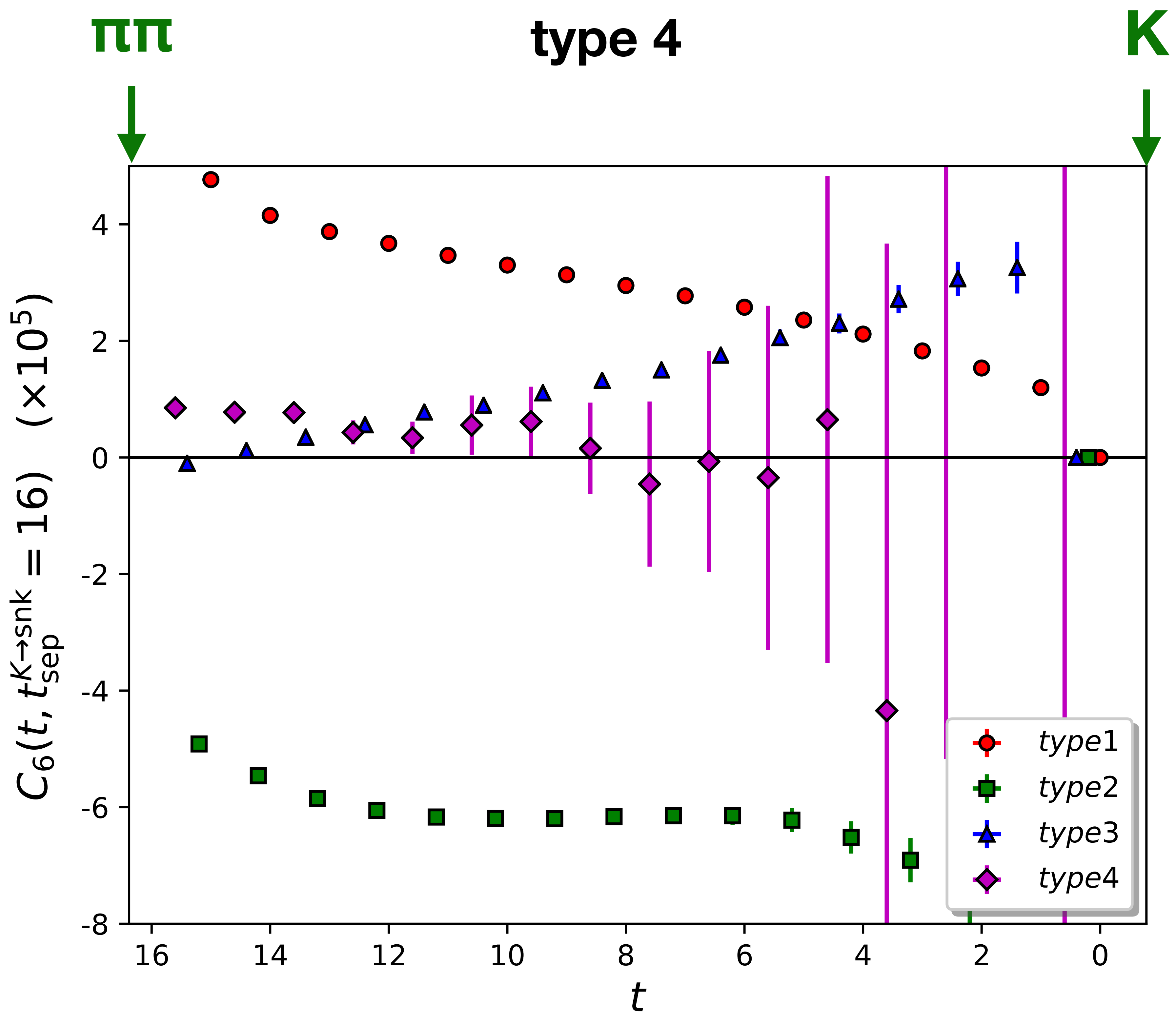

type 2




type 3



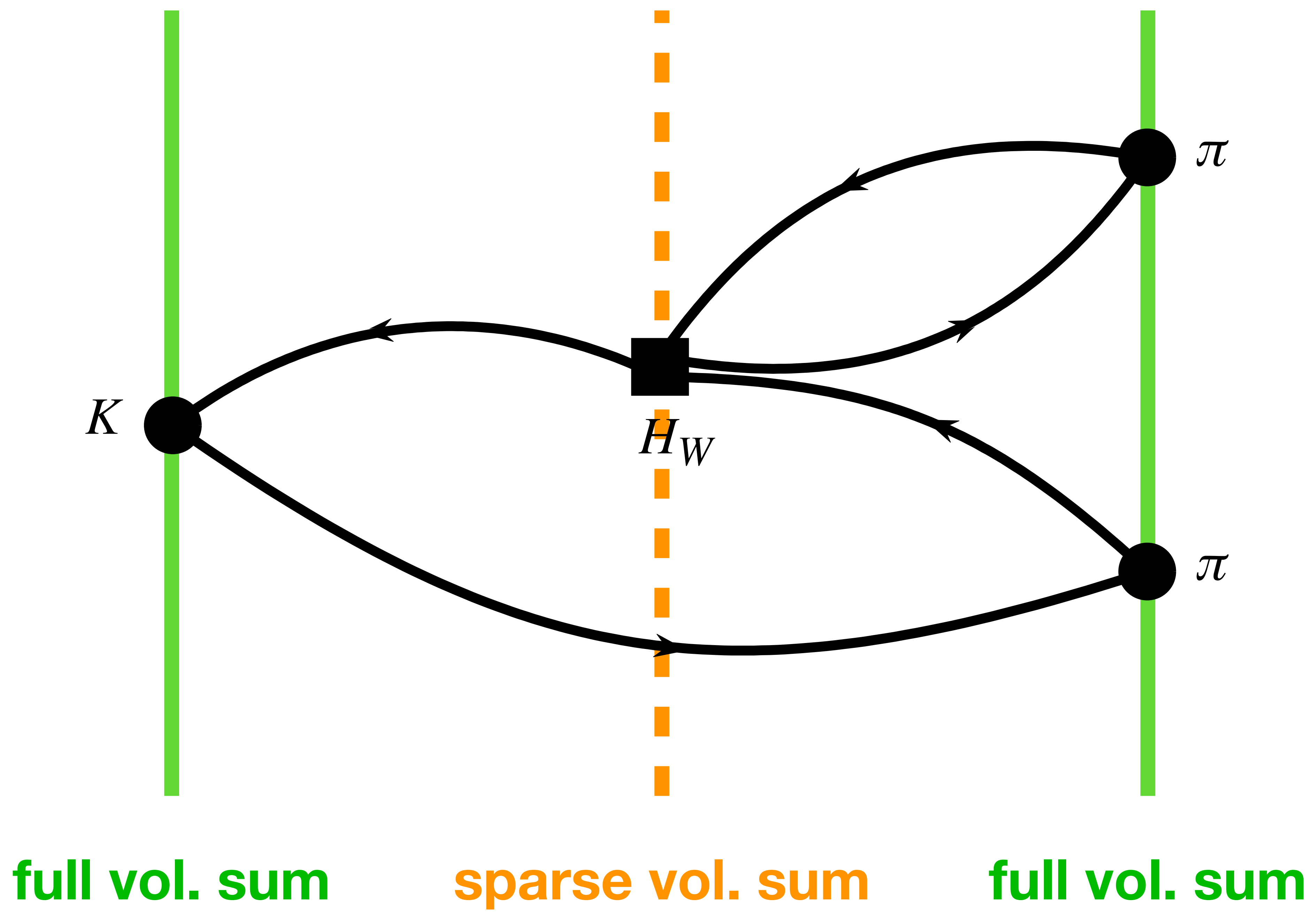

type 4



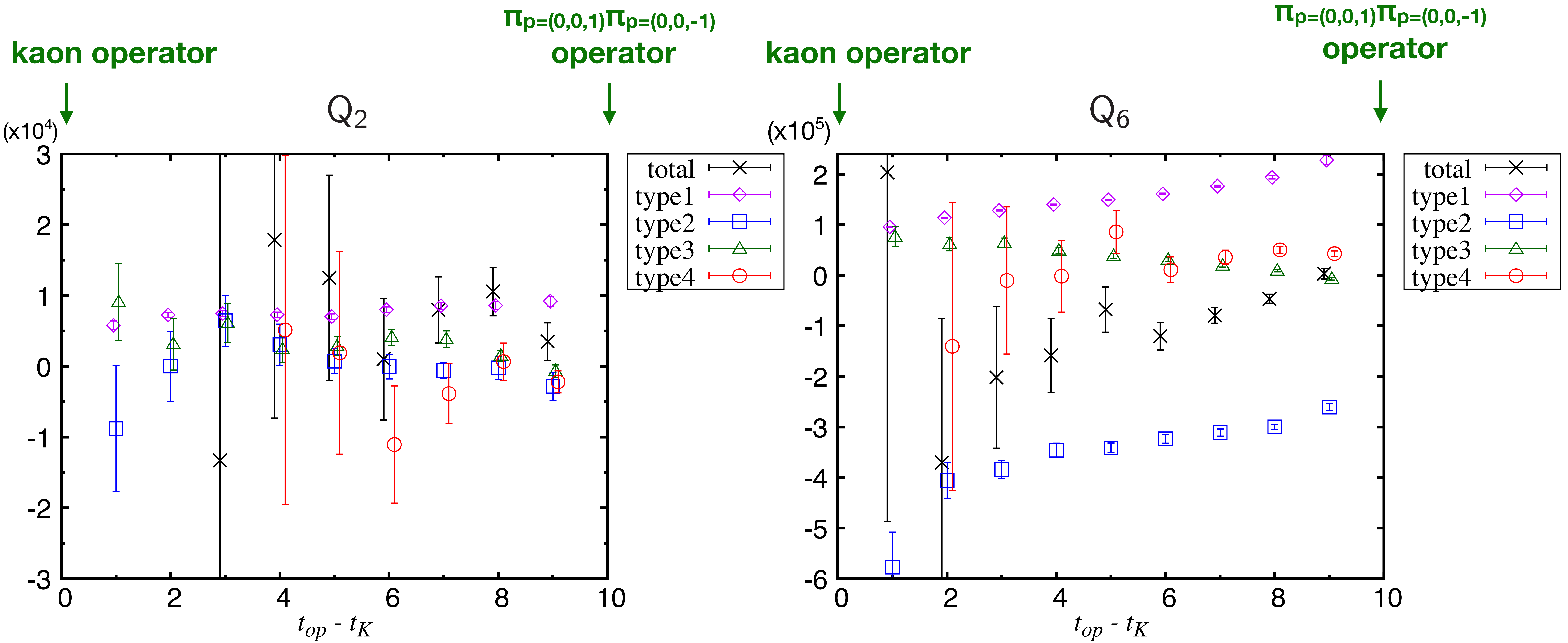

RBC/UKQCD, PRD 102,054509

# Sparsening $H_W$

- Contraction cost mostly proportional to volume of $H_W$
- G-parity calculation: summed $H_W$ over full 3D volume
- This calculation: volume of $H_W$ ($24^3 \rightarrow 8^3$: 27x speed up) for types 1 & 2



# I = 0 correlation functions

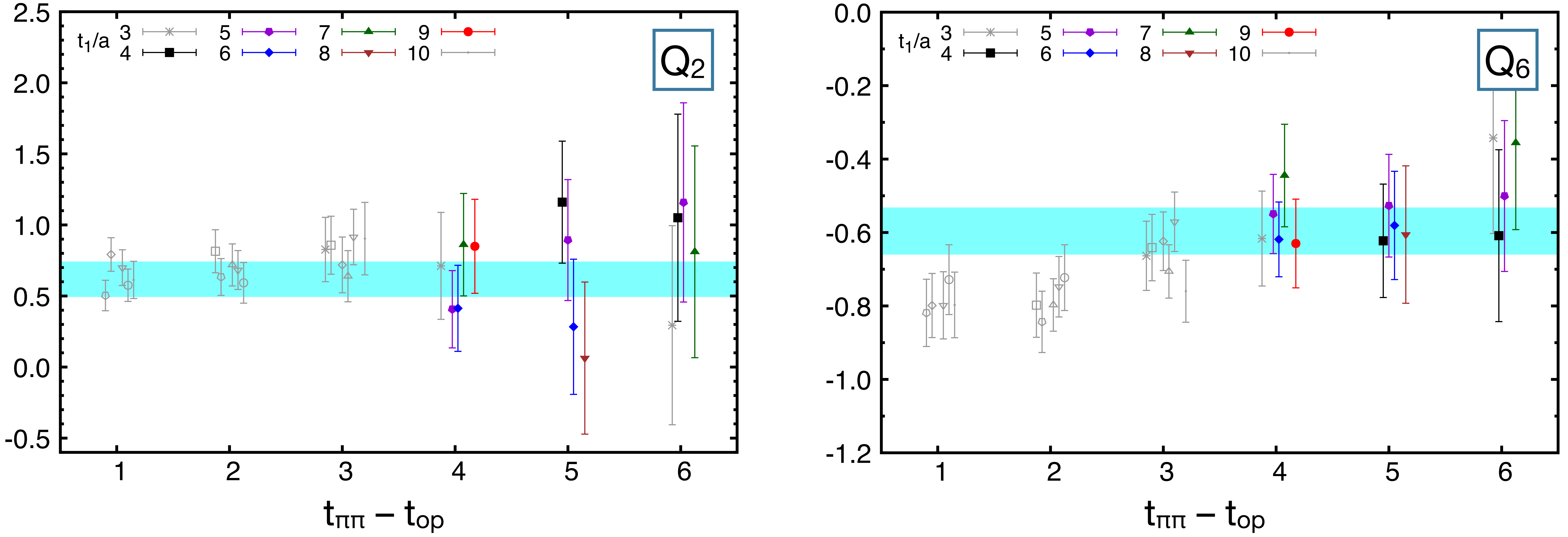



- Sparsen $H_W$ for types1,2 – still more precise than type4
- Precision of type4 disconnected diagram

# Effective matrix elements ($\Delta I = 1/2$)

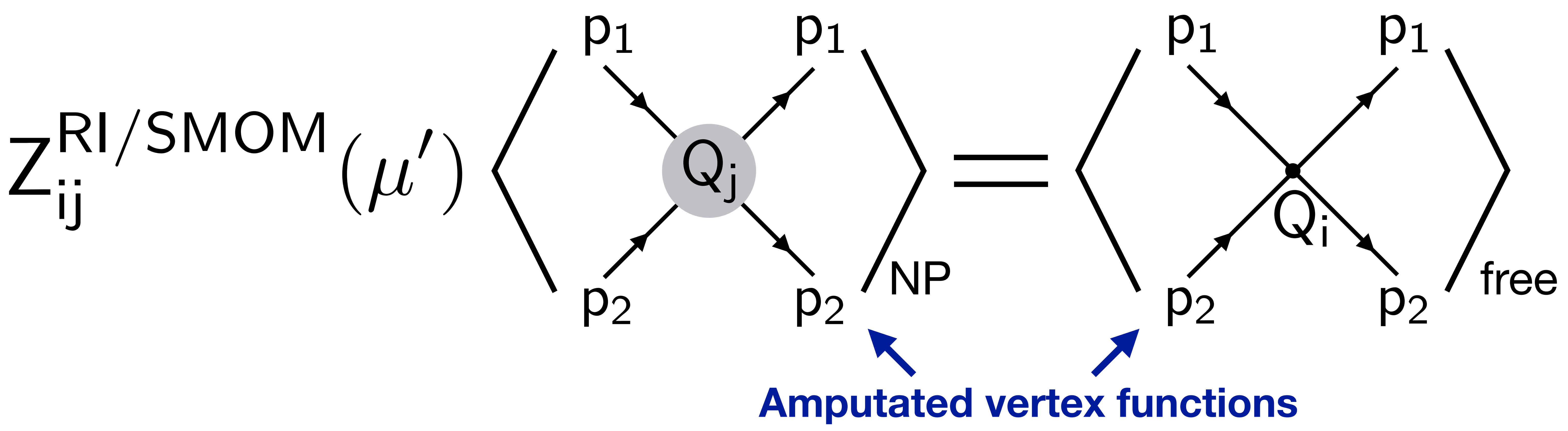
- $24^3$ x 64
- Plateau appears
- : Correlated fit result with

$t_1 = t_{op} - t_K \geq 4$ && $t_2 = t_{\pi\pi} - t_{op} \geq 4$ (colored filled data points)



# Translating to more physical ME

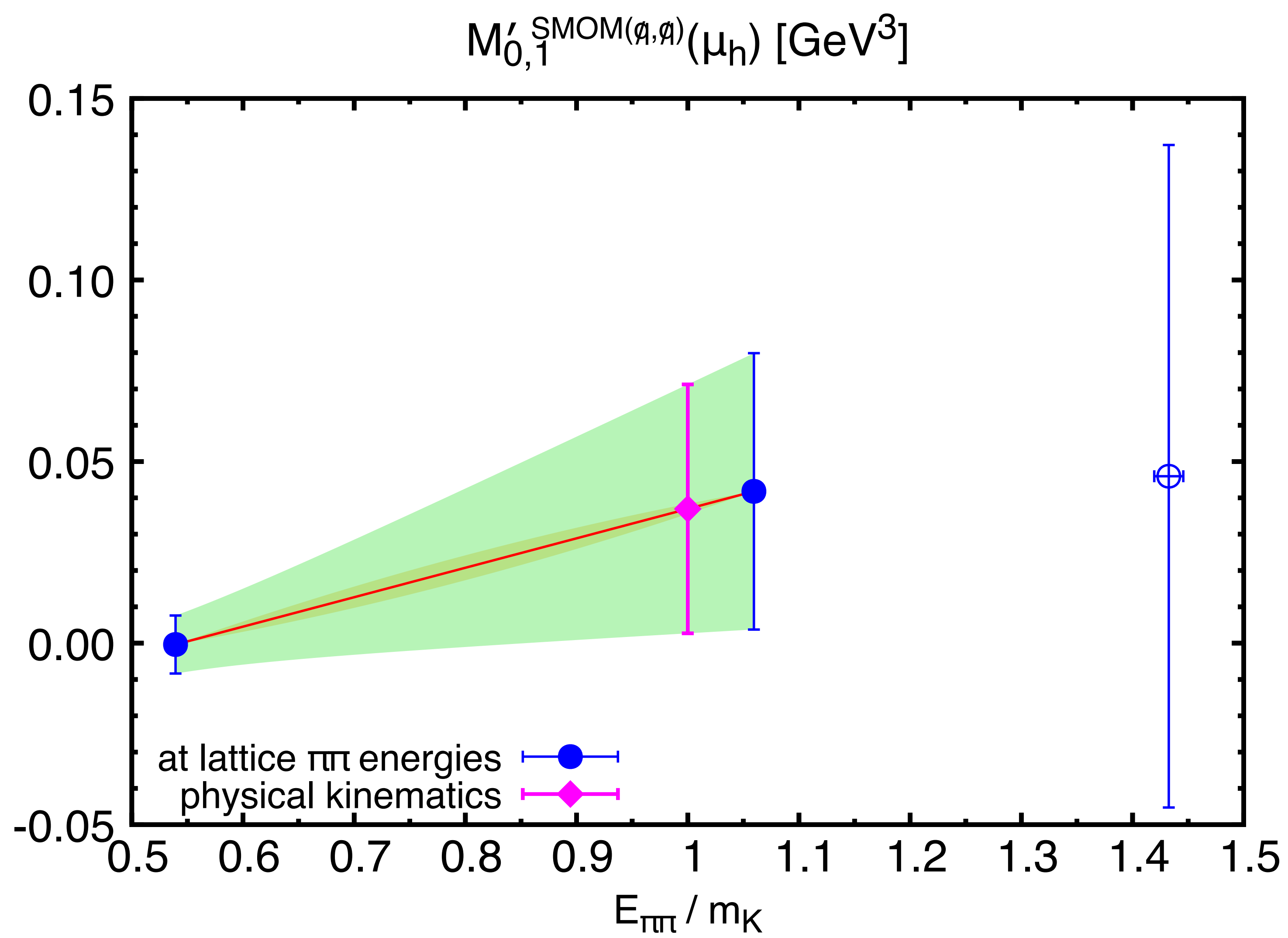
- Renormalization (RI/SMOM scheme)

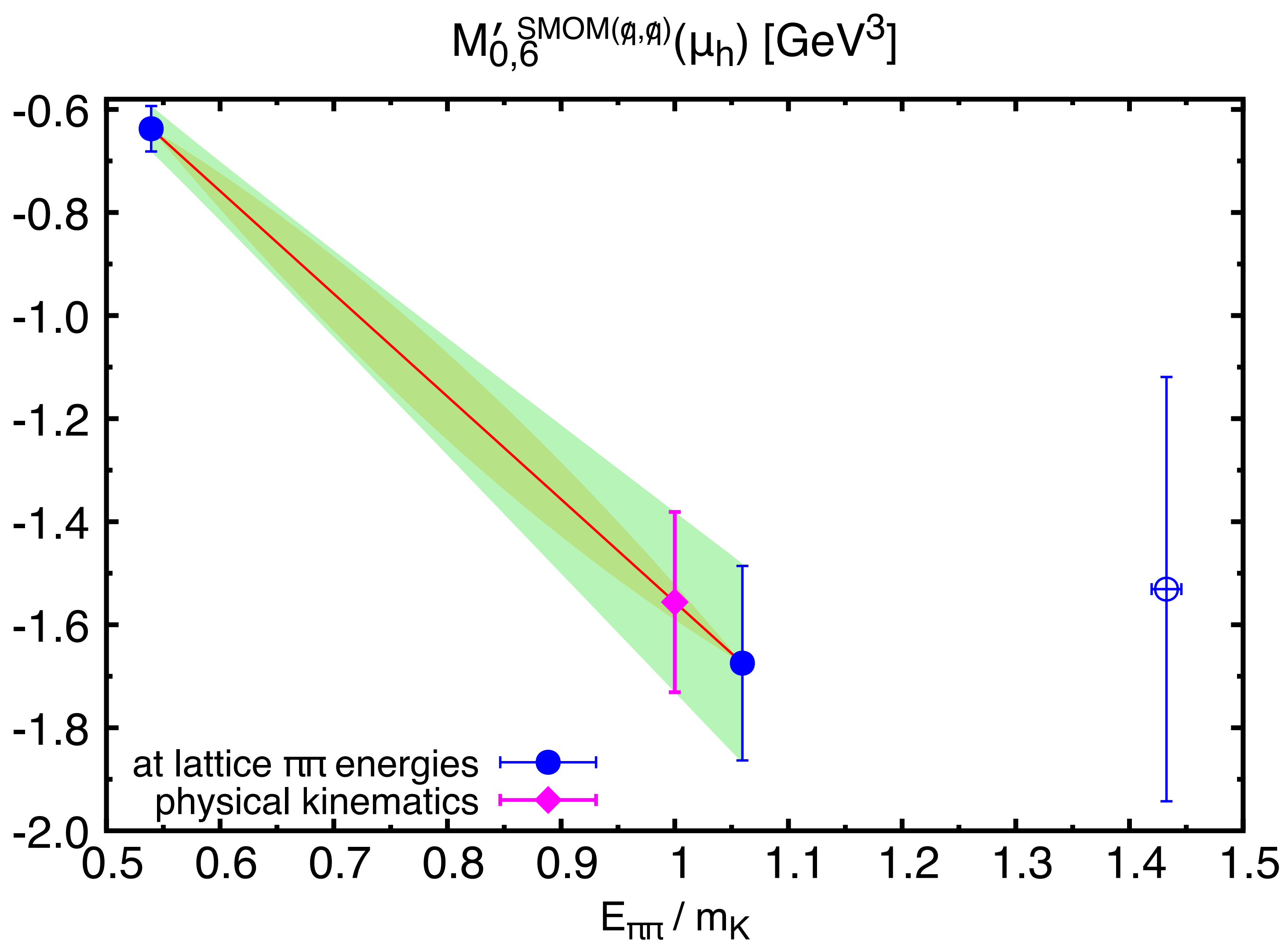



$$\mu'^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- Interpolation

  Examples of interpolation of renormalized ME
  - Linear & quadratic in $E_{\pi\pi}/m_K$
  - Systematic error estimated as lin vs quad is small as 1st excited st. close to on-shell



# Results for $A_0$ & $\varepsilon'$

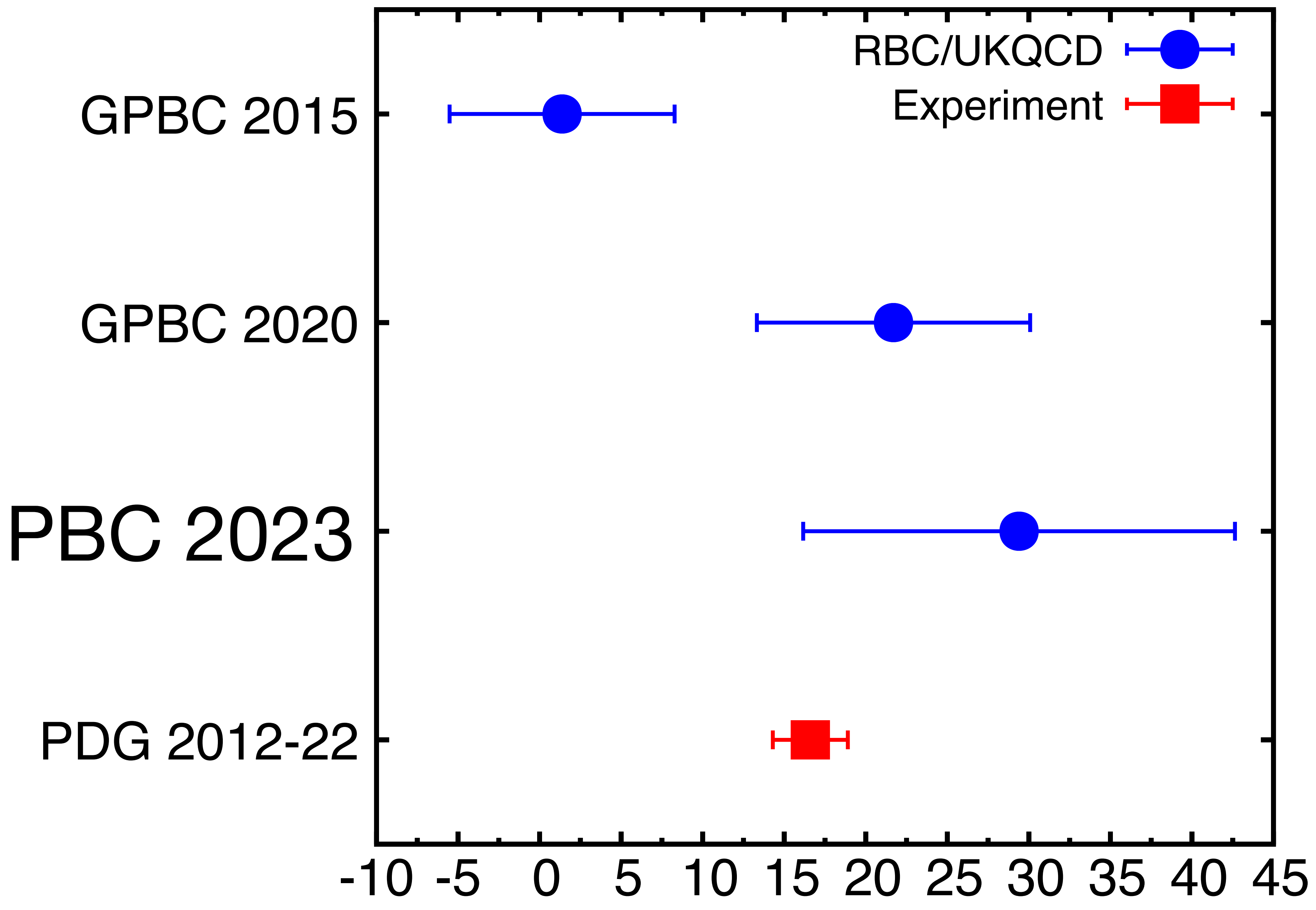
- $A_0$

| | 2020 (GPBC) $a^{-1} \approx 1.4$ GeV | PBC 2023 $a^{-1} \approx 1.0$ GeV |
|---|---|---|
| $\mathrm{Re}(A_0)$ [x$10^{-7}$ GeV] | $2.99(32)_{stat}(59)_{sys}$ | $2.84(57)_{stat}(87)_{sys}$ |
| $\mathrm{Im}(A_0)$ [x$10^{-11}$ GeV] | $-6.98(62)_{stat}(1.44)_{sys}$ | $-8.7(1.2)_{stat}(2.6)_{sys}$ |
| Systematic errors on $\mathrm{Im}(A_0)$ | | |
| finite lat spacing | 12% | 22% |
| Wilson coefs. | 12% | 12% |
| Others | 12% | 16% |

**NPR error became significant on the coarse lattice**

- $\mathrm{Re}(\varepsilon'/\varepsilon)$
  - This work: 0.00294(52)(111)(50) (stat, sys, EM/IB)
  - 2020 (GPBC): 0.00217(26)(62)(50)
  - Experiment: 0.00166(23)

**$\mathrm{Re}(\varepsilon'/\varepsilon)$ [x$10^{-4}$]**



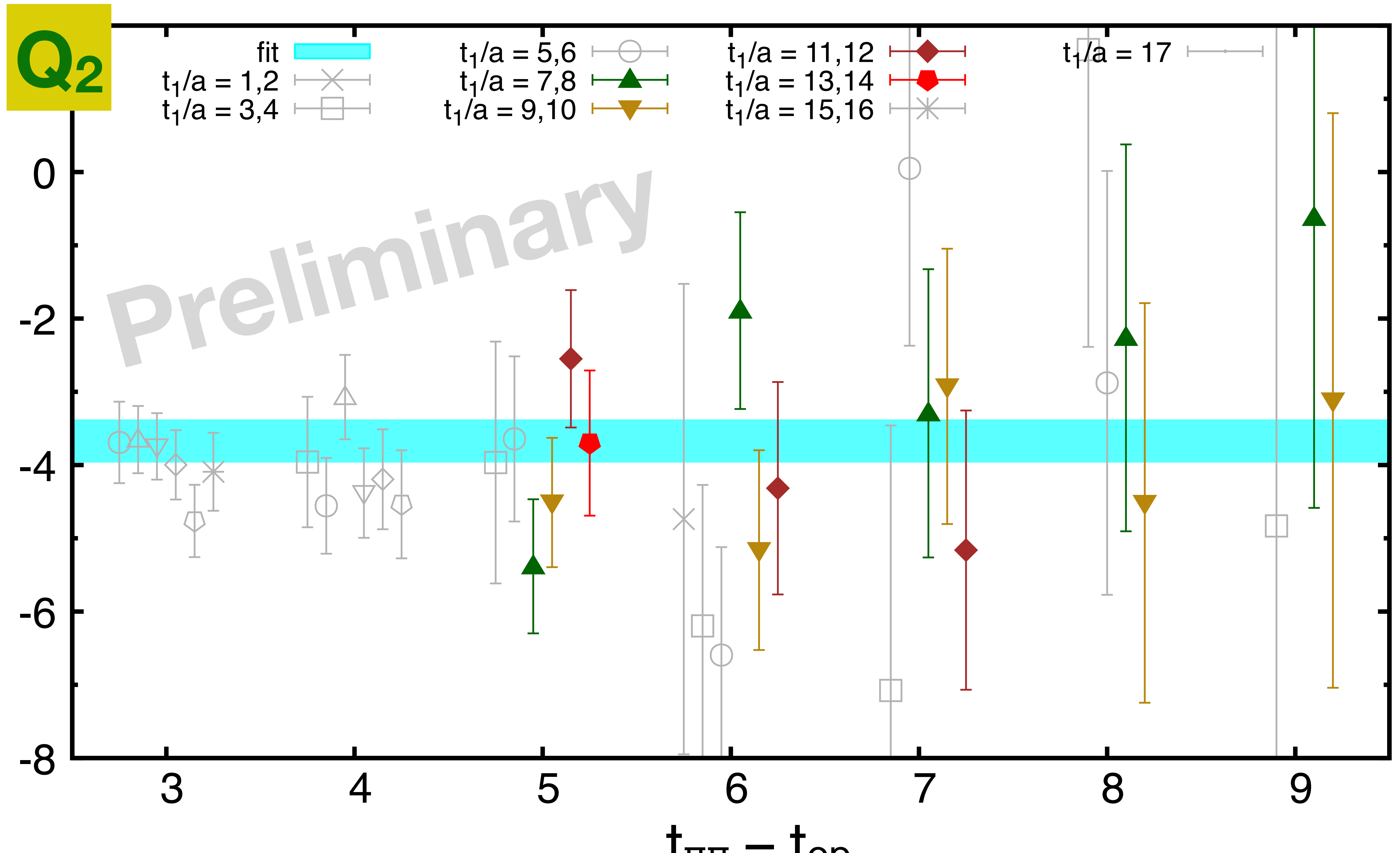
# PBC achieving a better precision

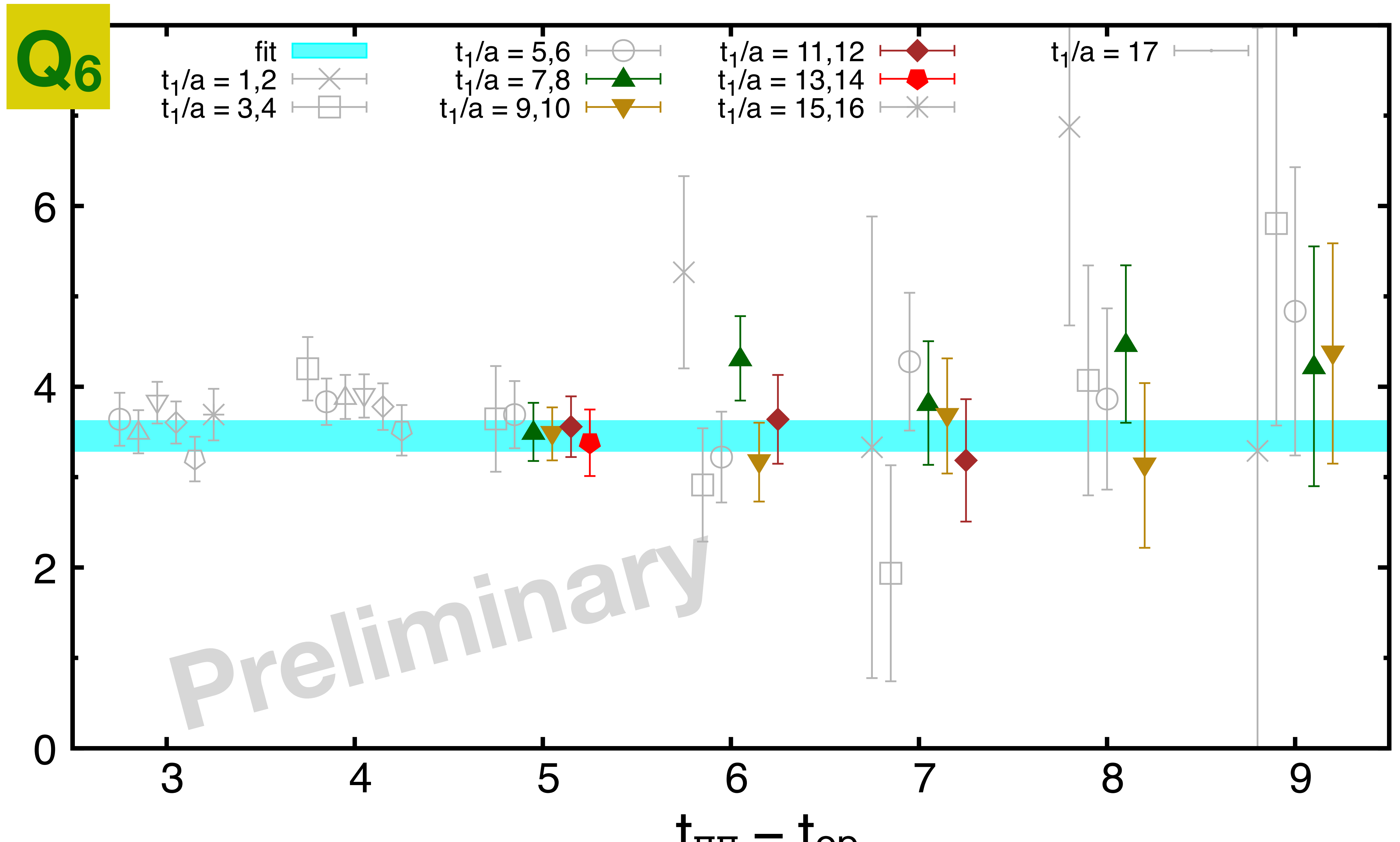
| | $32^3$ G-parity BC (2020) | $24^3$ Periodic BC | $32^3$ Periodic BC |
|---|---|---|---|
| # of configurations | 741 | 258 → 440 | 107 → 470 |
| $\Delta I = 1/2$ ME via $Q_2^{lat}$ | 10% | 14% → (11%) | 14% → (6.7%) |
| $\Delta I = 1/2$ ME via $Q_6^{lat}$ | 6.5% | 8.9% → (6.8%) | 11% → (5.3%) |

Stat. Err (%)

expectations of on-going analysis in ( )

**matrix elements with 1st-excited ππ, I = 0 (x $10^3$)**



# Summary & Outlook

- Main sources of systematic errors at the moment
  - Finite lattice spacing - *Easier to take continuum limit with PBC as we already have lattice ensembles*
  - Wilson coefficients - *Going to be tested with existing data*
  - QED/IB effects - *Theoretical approach being developed [Christ et al, PRD106, 014508 (2022)] with PBC*
- We are successful in
  - Extracting excited-state signals of the challenging $\Delta I = 1/2$ process
  - Good precision performance of PBC approach
- Precision will compete with experiment in the near future
  - Could attract a big attention from lattice, pheno & exp!

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Also … $\pi\pi$ scattering length at physical $m_\pi$ very competitive precision to other results from unphysical $m_\pi$