Randomized higher-order tensor renormalization group for higher dimension.

[D. Kadoh, K.N. arXiv:1912.02414]
[K.N. arXiv:2307.14191]

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2023/07/31@Online.

Photo: RIKEN, Japan, 2022

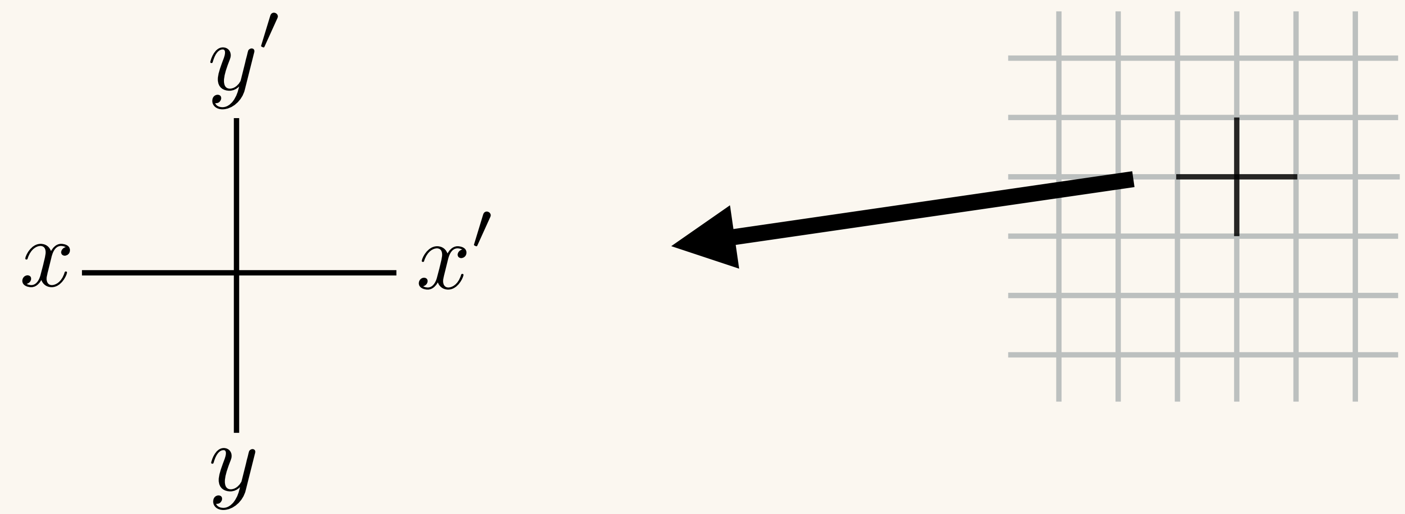
◉ Tensor renormalization group (TRG)

[M. Levin, C. P. Nave. arXiv:cond-mat/0611687]

◇ TRG calculate the physical quantity as trace of tensors.

$$Z = \mathrm{Tr} \sum_{i \in \mathrm{lattice}} A_{x_i y_i x'_i y'_i}$$

$$A_{xyx'y'} =$$



○Sign problem

○Another representation

×High cost ($\dim \geq 3$)

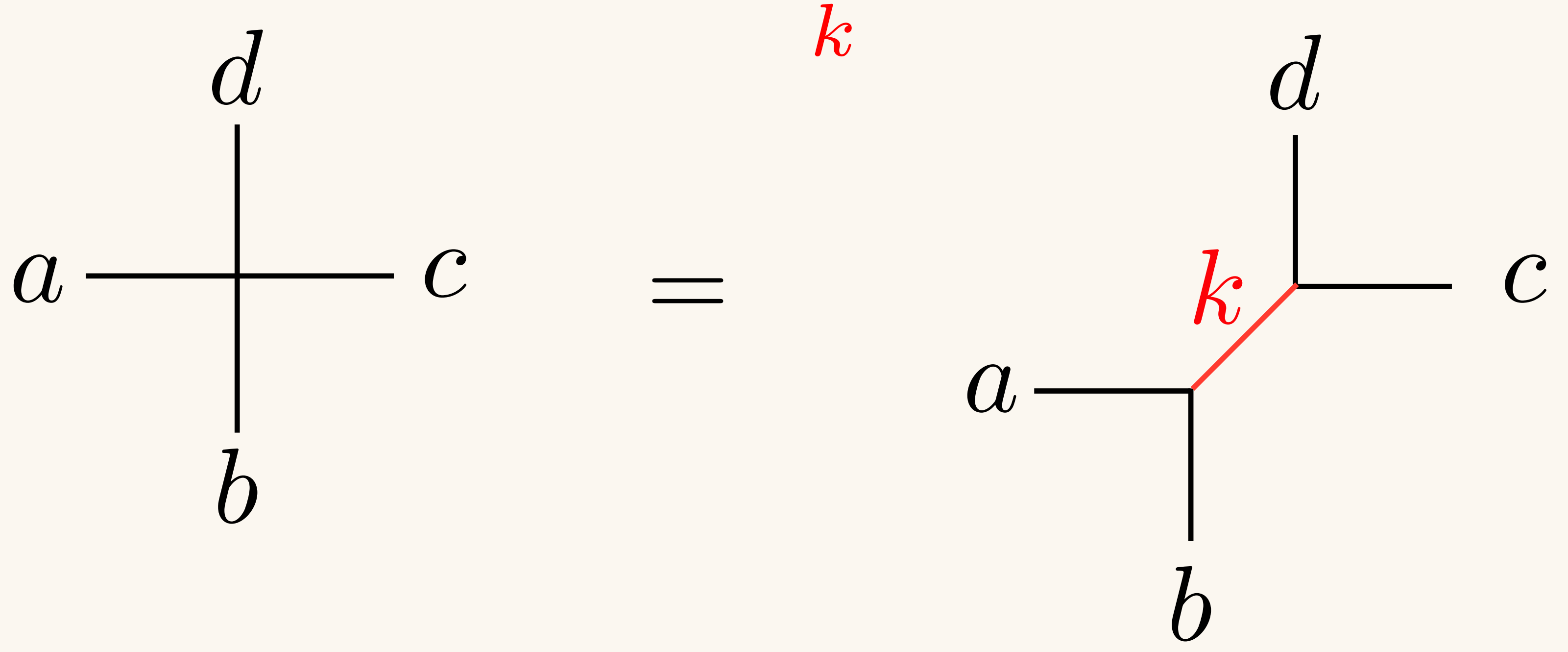
△Systematic error

How can we take whole contraction approximately?

→ Singular value decomposition (Frobenius norm)

Singular Value Decomposition (SVD)

$$T_{abcd} = \sum_{k}^{D} A_{ab}{}^{k} \lambda^{k} B_{cd}{}^{k}$$



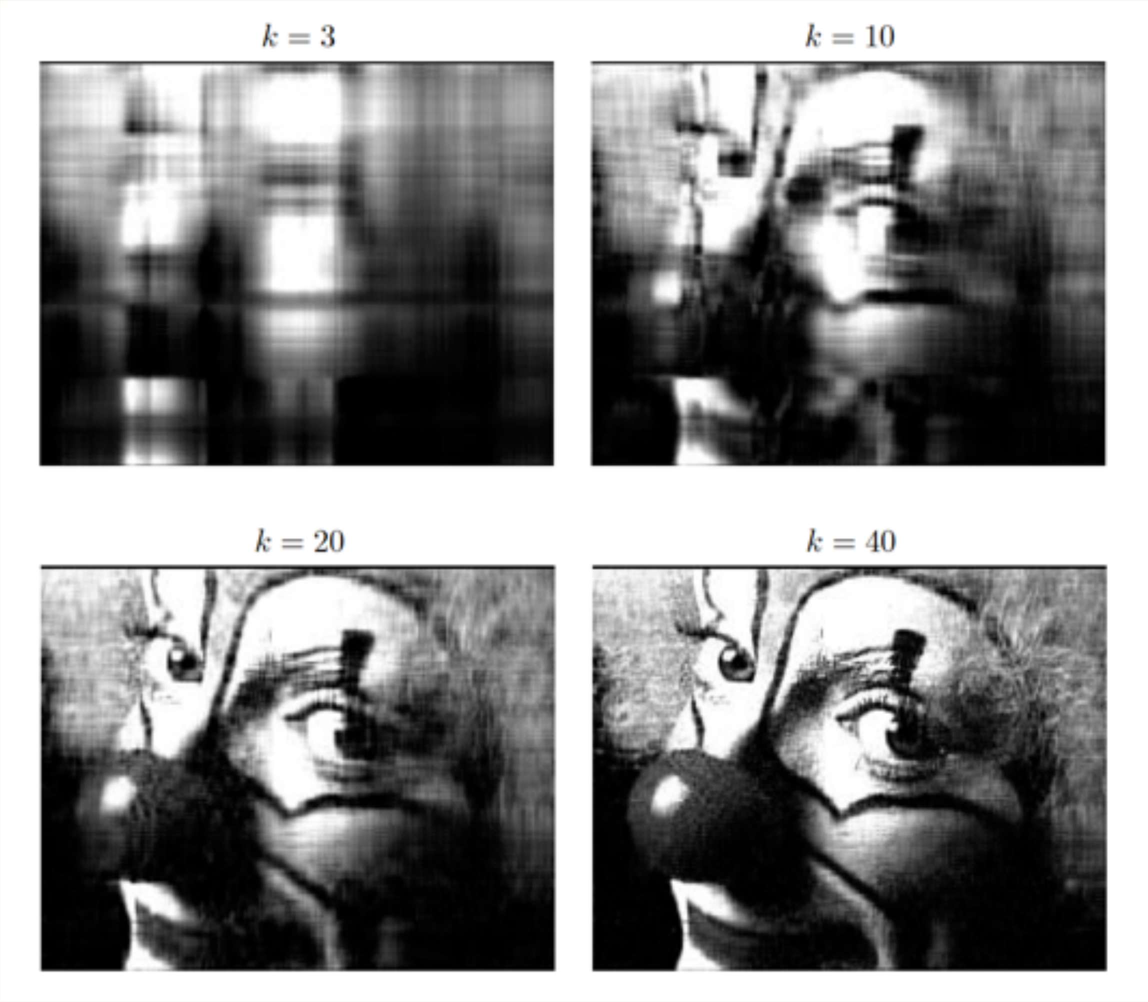
◇ Larger singular values λ have much “information” of T

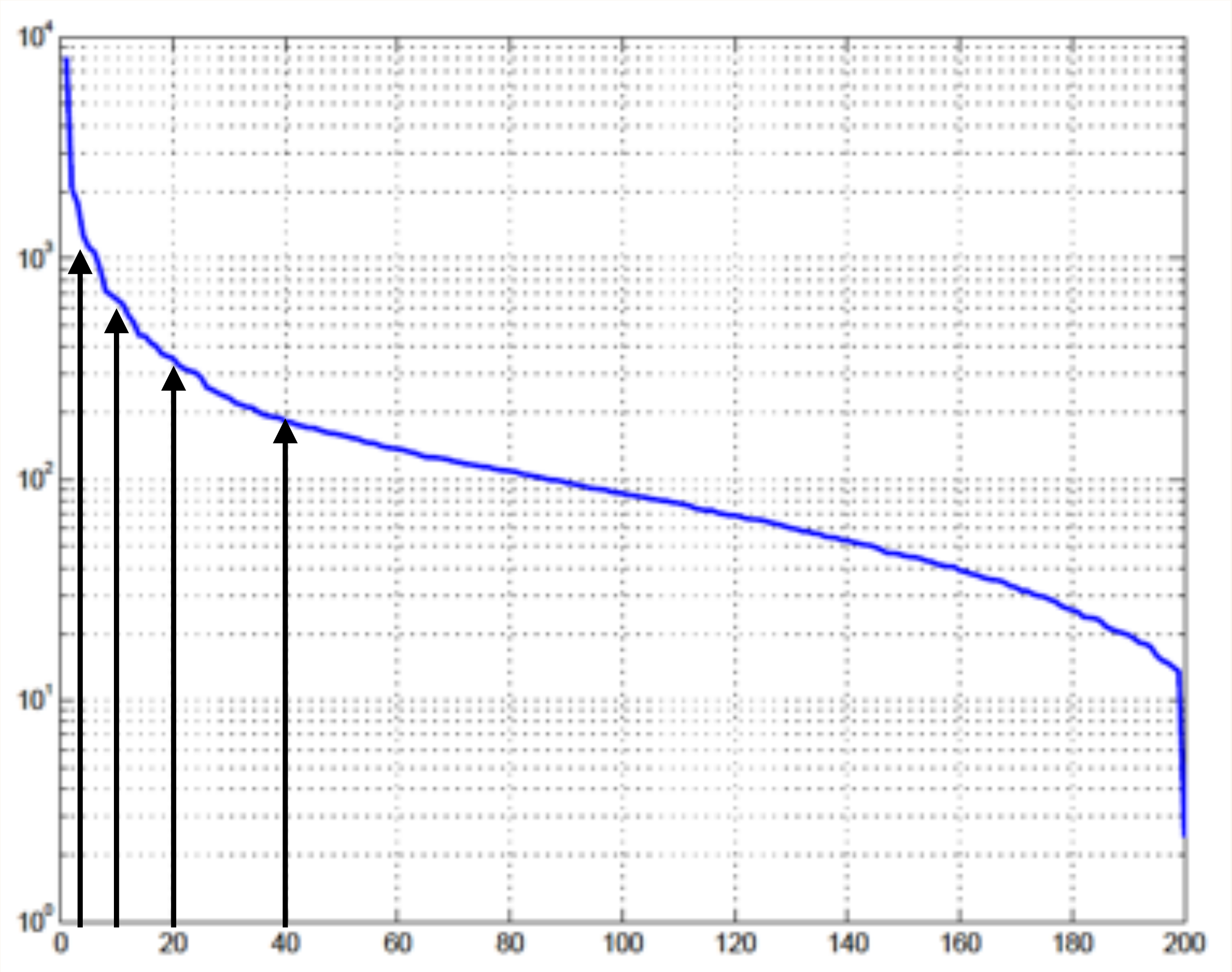
→ (Frobenius norm)

→ We can approximate the matrix by the cutoff of index k

$$\dim(k) = \dim(a)\dim(b) \to D$$

◉ SVD for a coarse graining (e.g. Image)





[http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT]

◉ Motivation: The origin of the systematic error

◇ HOTRG [Z.Y. Xie, J. Chen, et al. arXiv:1201.1144]

d :dimension

D :truncated bond size

Systematic error: isometry,

Cost: $O(D^{4d-1})$

◇ Anisotropic TRG(ATRG) [D. Adachi, T.Okubo, S. Todo. arXiv:1906.02007]

Systematic error: (isometry), decomposition, R-SVD

Cost: $O(D^{2d+1})$

◇ TriadTRG (TTRG) [D. Kadoh, K.N. arXiv:1912.02414]

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→ How about HOTRG with Randomized-SVD?

→ Can we reduce the systematic error from decomposition?

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→ How about HOTRG with Randomized-SVD?

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◉ HOTRG with R-SVD

	with R-SVD	w/o R-SVD
◇ HOTRG	?	$O(D^{4d-1})$
◇ ATRG	$O(D^{2d+1})$	$O(D^{3d})$
◇ TTRG	$O(D^{d+3})$	$O(D^{d+4})$

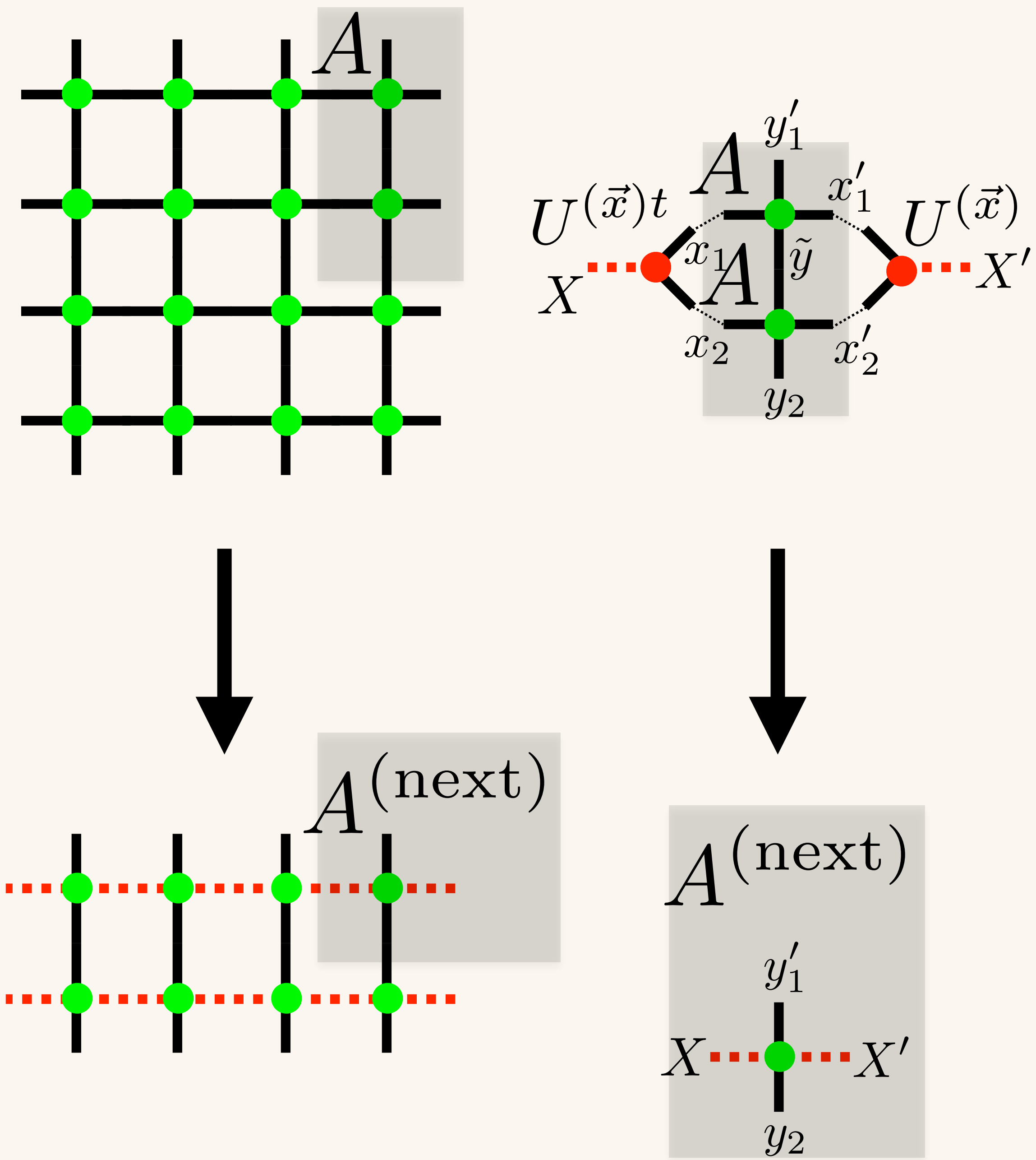
→ We propose HOTRG with R-SVD.

HOTRG with randomized SVD

◉ Higher-Order TRG (HOTRG)

[Z.Y. Xie, J. Chen, et al. arXiv:1201.1144]

◇ Contraction by projection operator U (isometry)



$$\Gamma^{(AA)} = AA \to A^{(\rm next)}$$

$\to U^{(\vec{x})}$ is made by SVD of $\Gamma\Gamma^t$

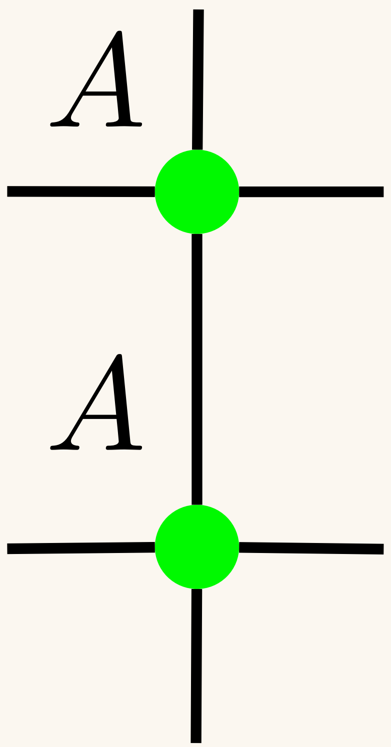
$$[\Gamma\Gamma^t]_{[x_1x_2][x_1^t x_2^t]} = \sum_{k=1}^{D^2} U^{(x)}_{[x_1x_2]k} \lambda_k U^{(x)}_{[x_1^t x_2^t]k}$$

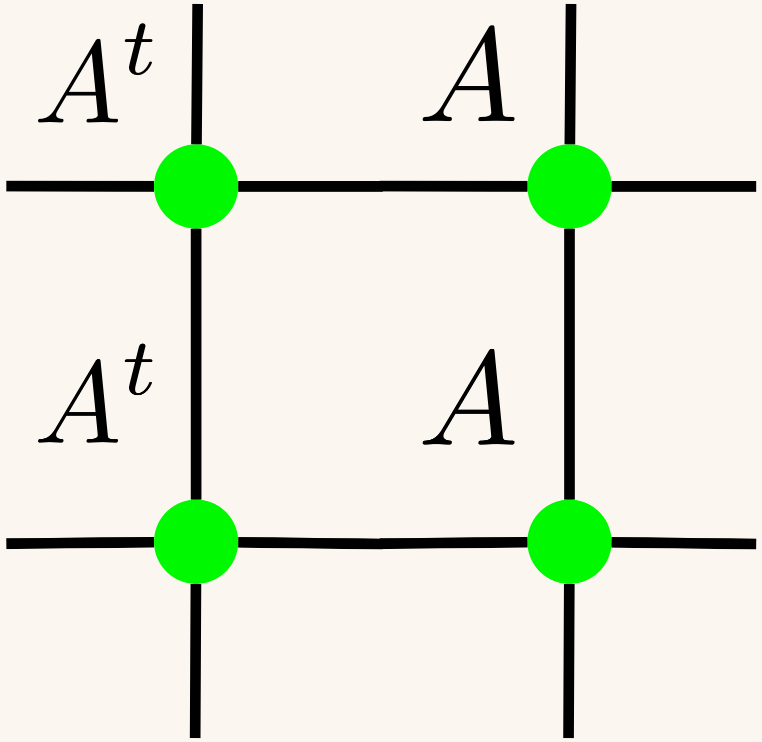
SVD Cutoff: $D^2 \to D$

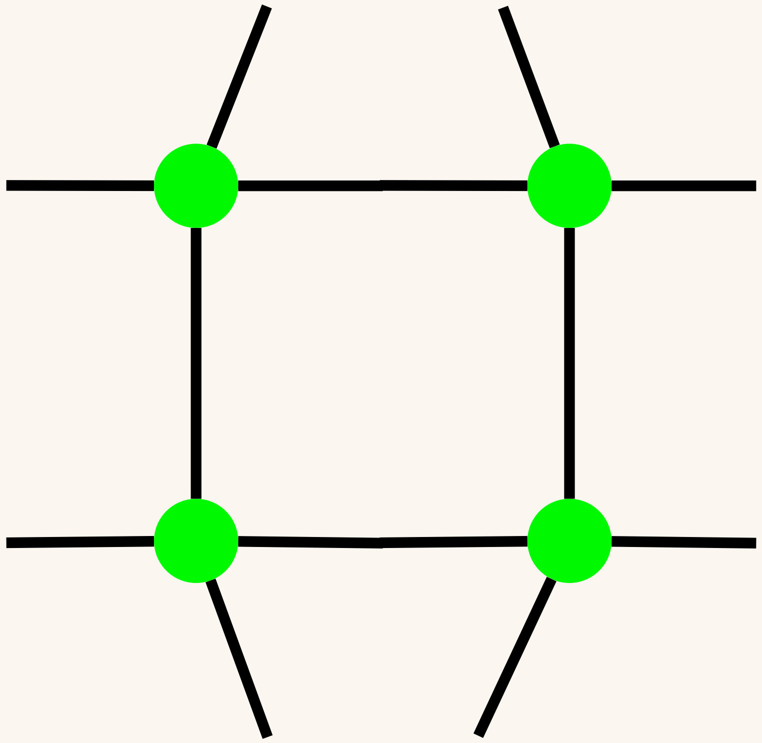
Cost: $O(D^6)$

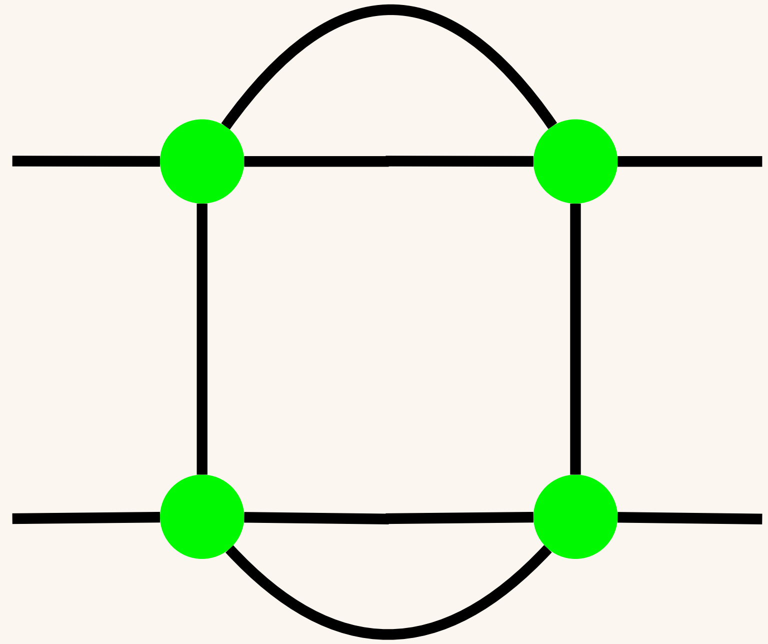
$$U^t AAU = A^{(\rm next)}$$

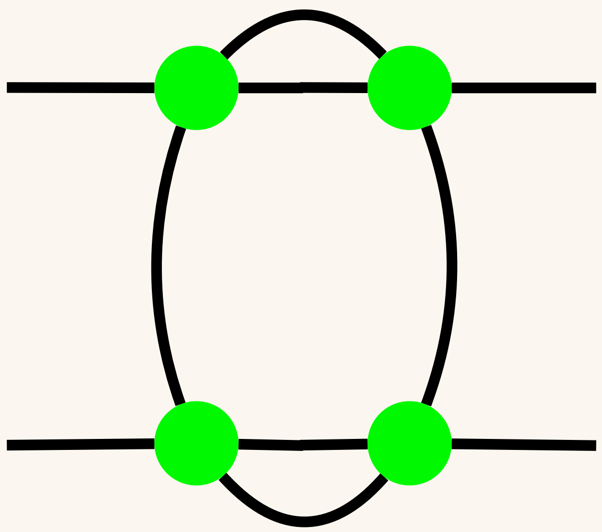
Contraction Cost: $O(D^7)$

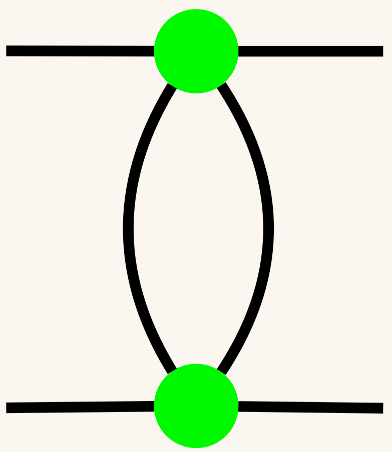

A
A

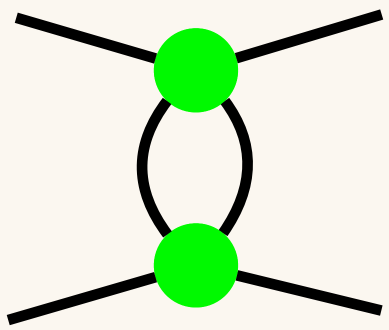

A^t
A
A^t
A

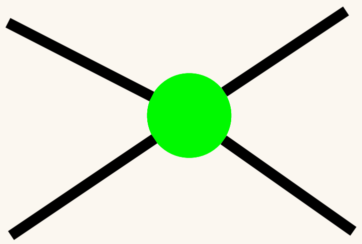


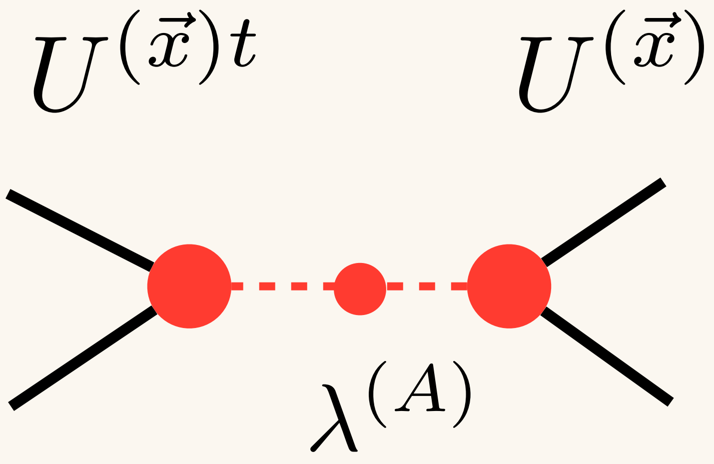




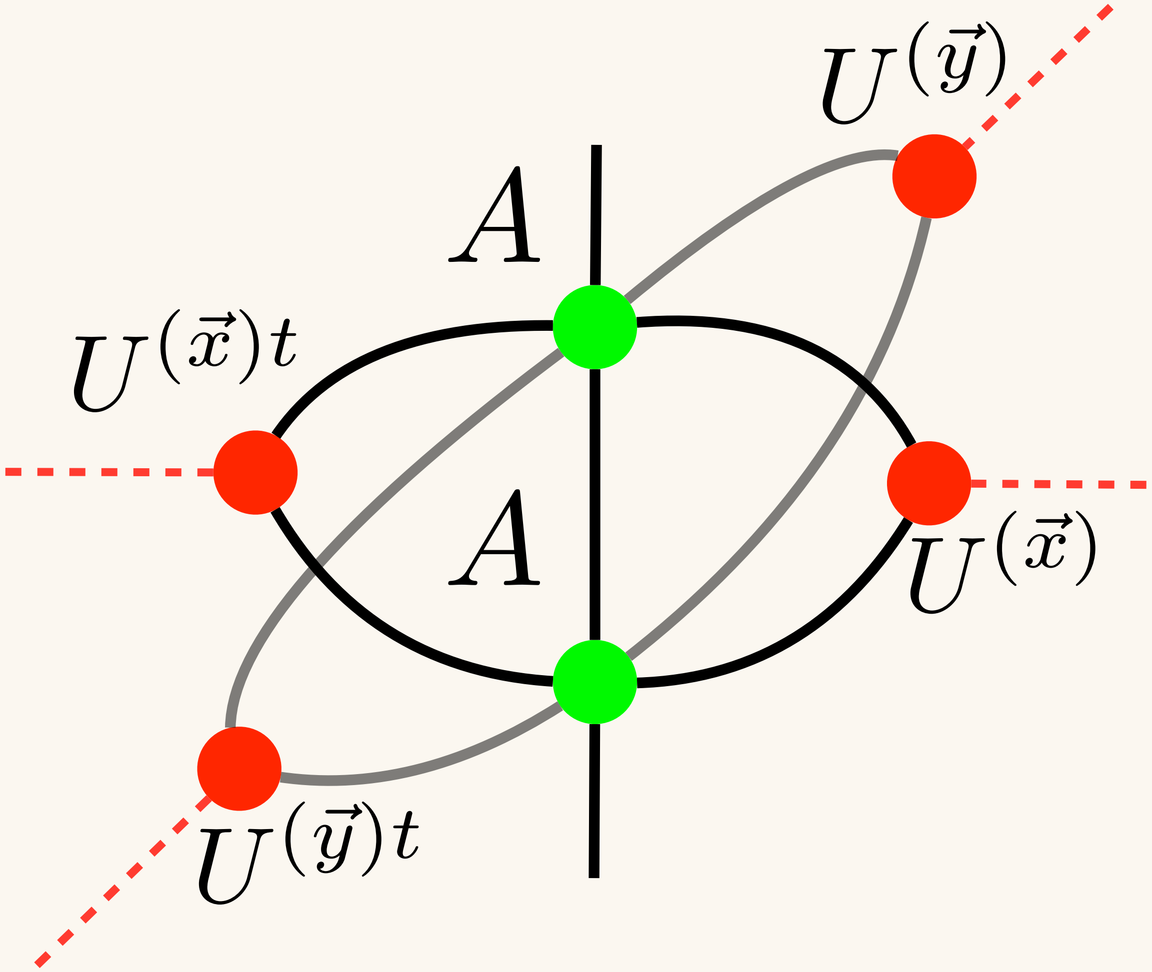


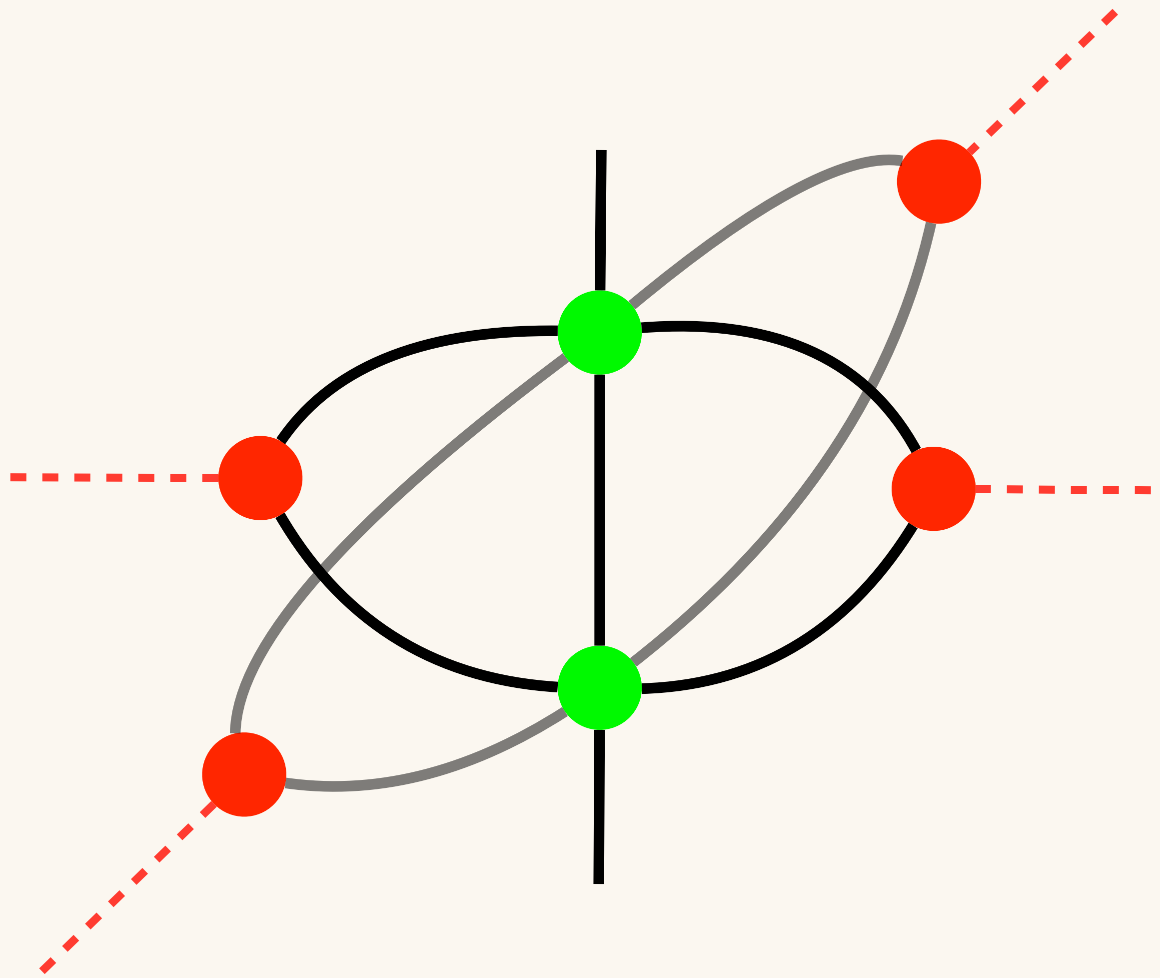


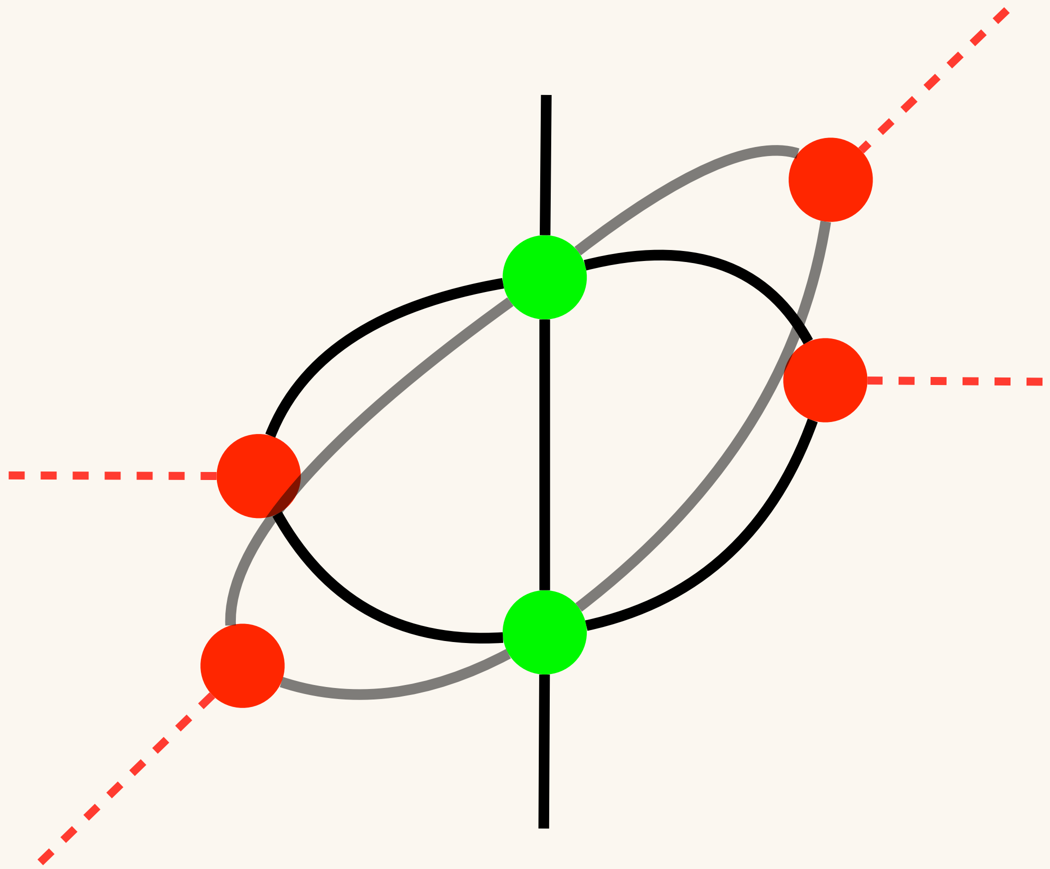


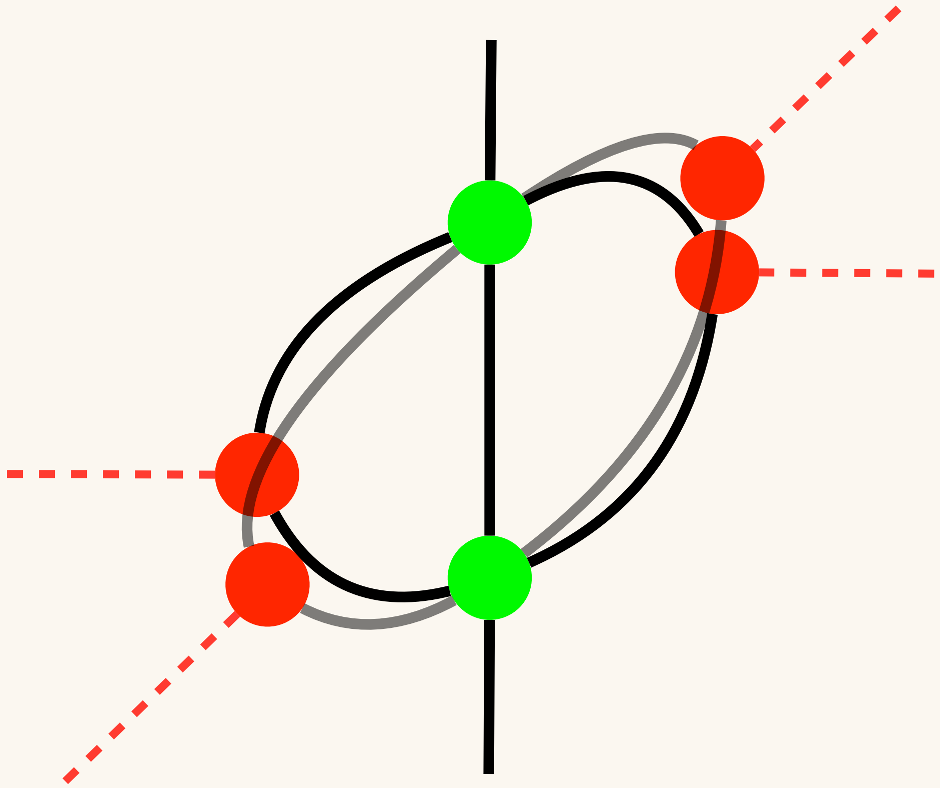


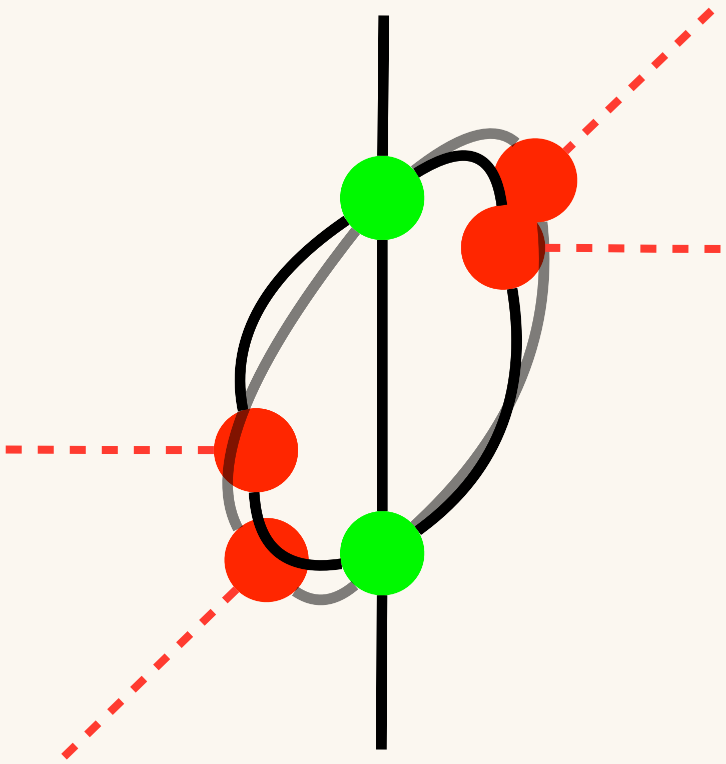
$U^{(\vec{x})t}$
$U^{(\vec{x})}$
$\lambda^{(A)}$

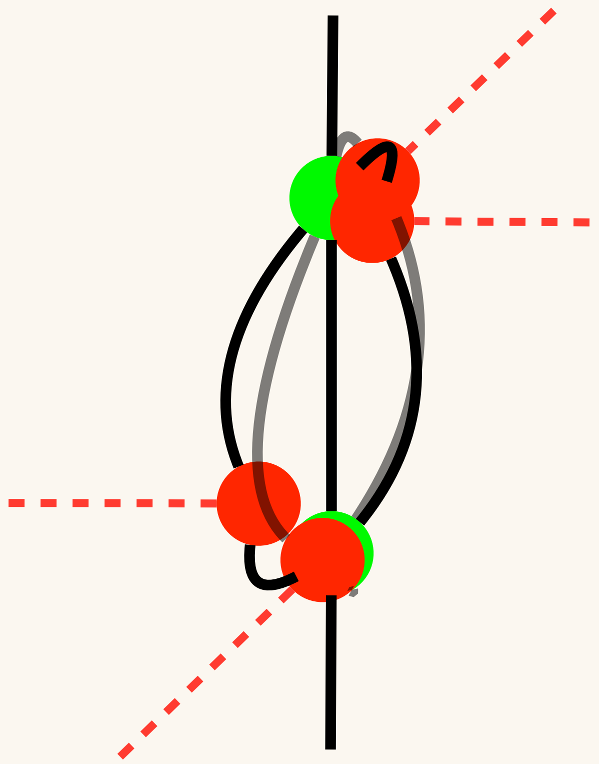

$U^{(\vec{y})}$
A
$U^{(\vec{x})t}$
A
$U^{(\vec{x})}$
$U^{(\vec{y})t}$

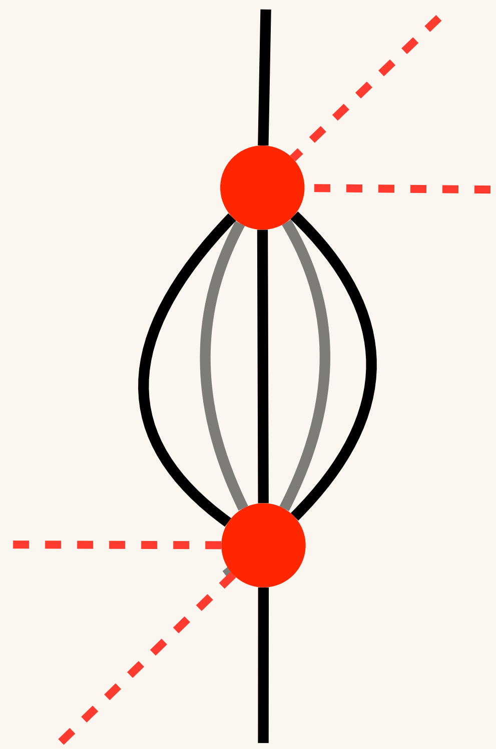


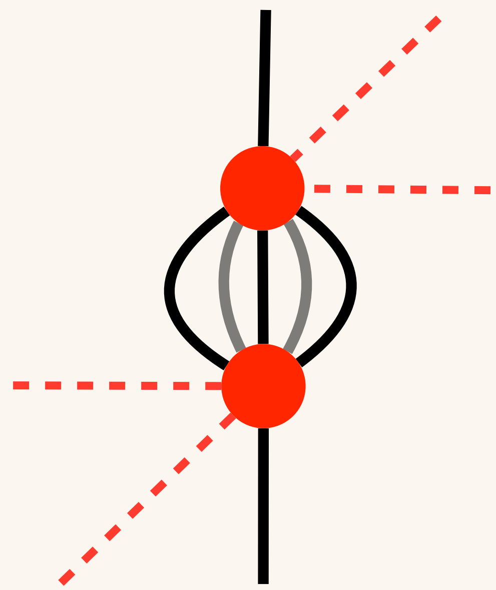




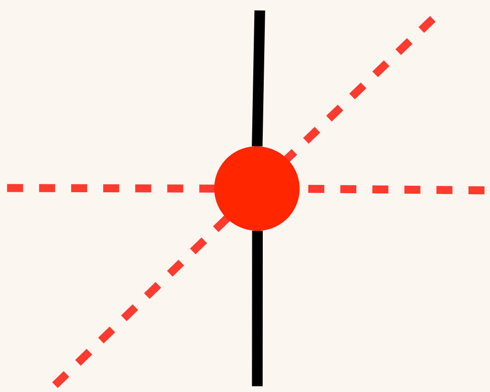








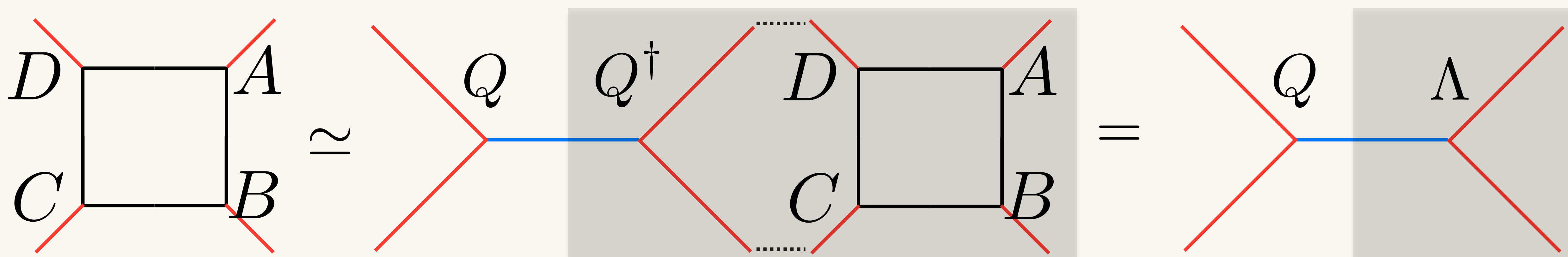
$A^{(\mathrm{next})}$



◉ Randomized-SVD

[N. Halko, et al. arXiv:0909.4061]
[S. Morita, et al. arXiv:1712.01458]

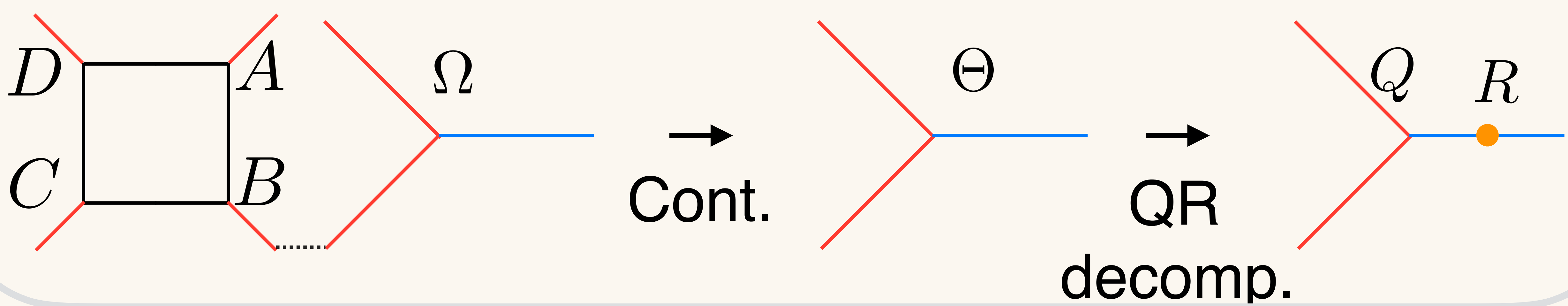
◇ Approximated contraction by orthogonal matrix Q



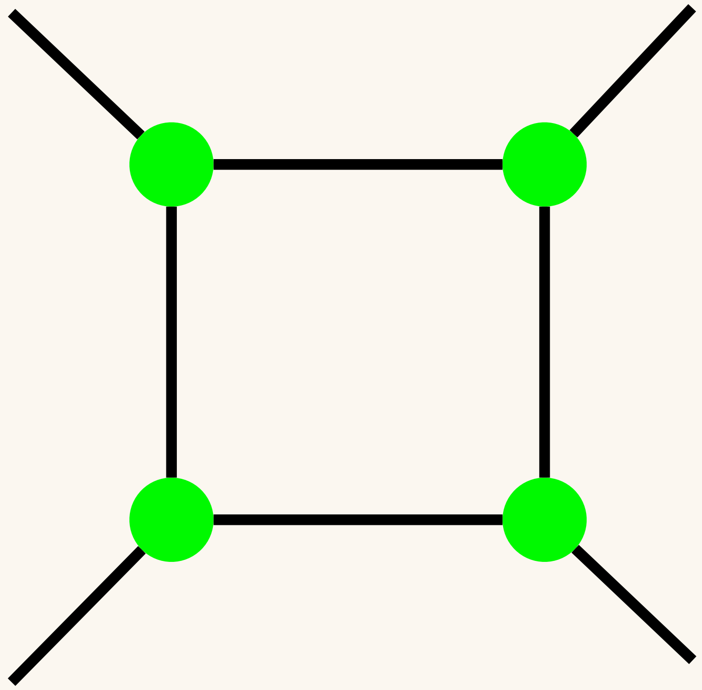
◇ SVD of $\Lambda \equiv Q^\dagger ABCD$.

◇ To prepare Q, we use randomized method & QR decomp.

◇ Contraction with the random tensor Ω



◇ Step 1: $O(D^5)$



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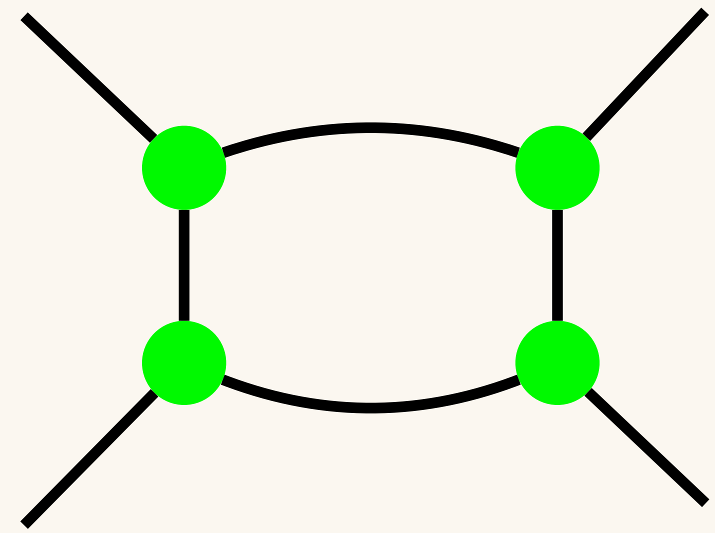

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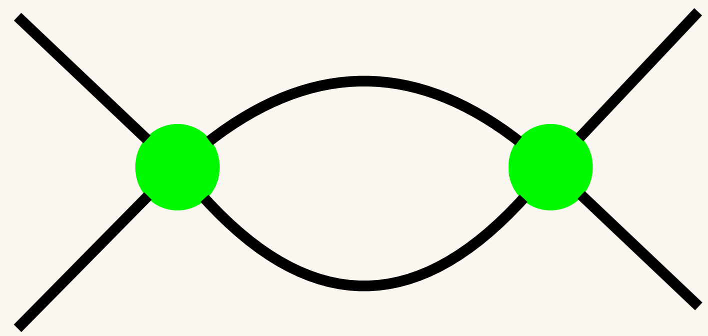
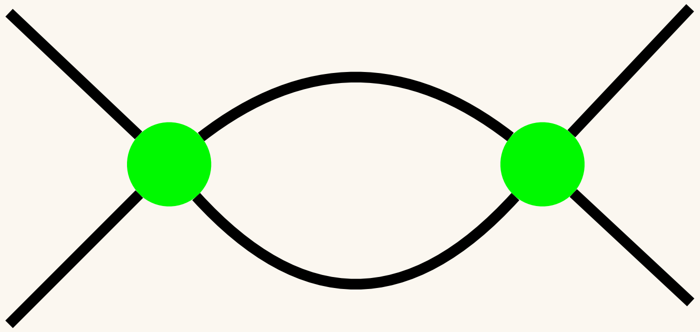


◇ Step 2: $O(D^6)$



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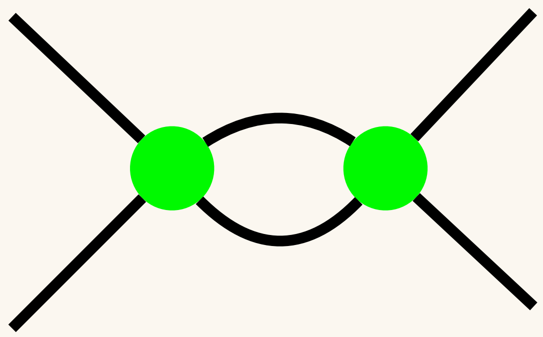

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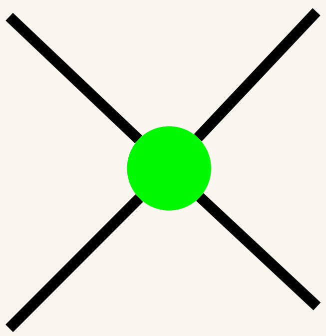

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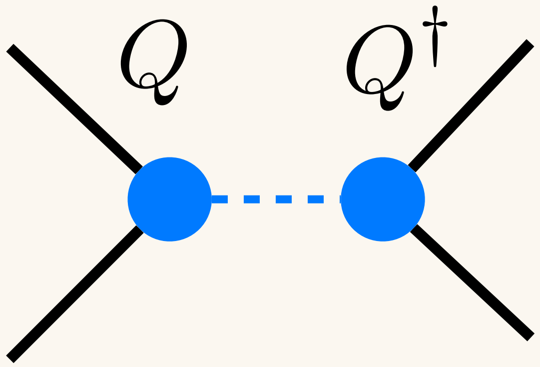

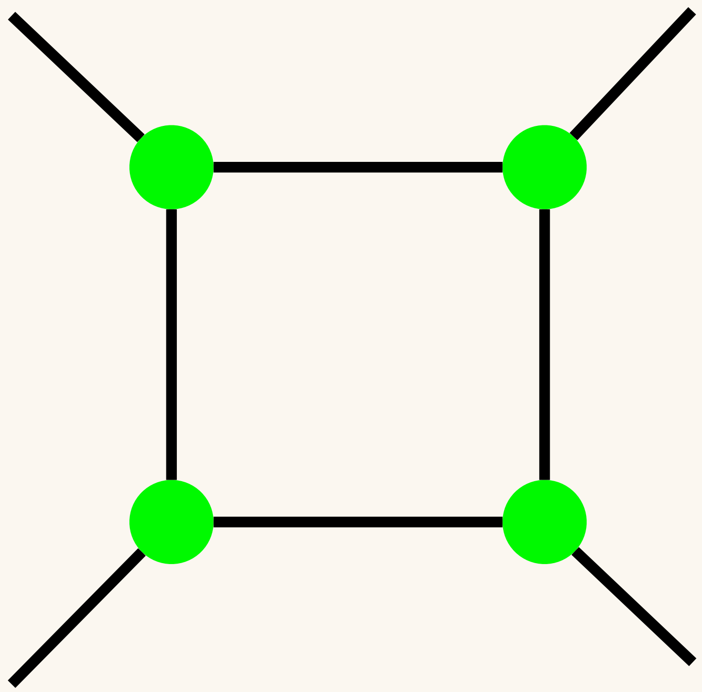

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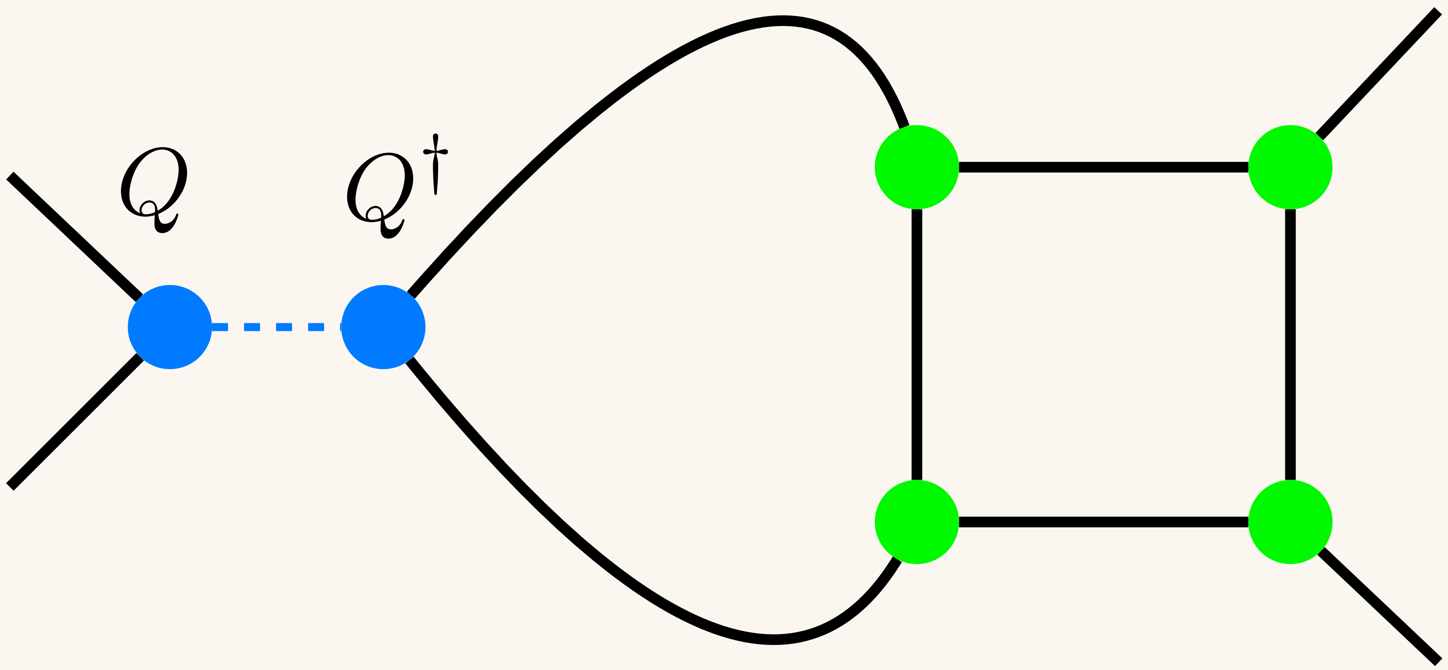

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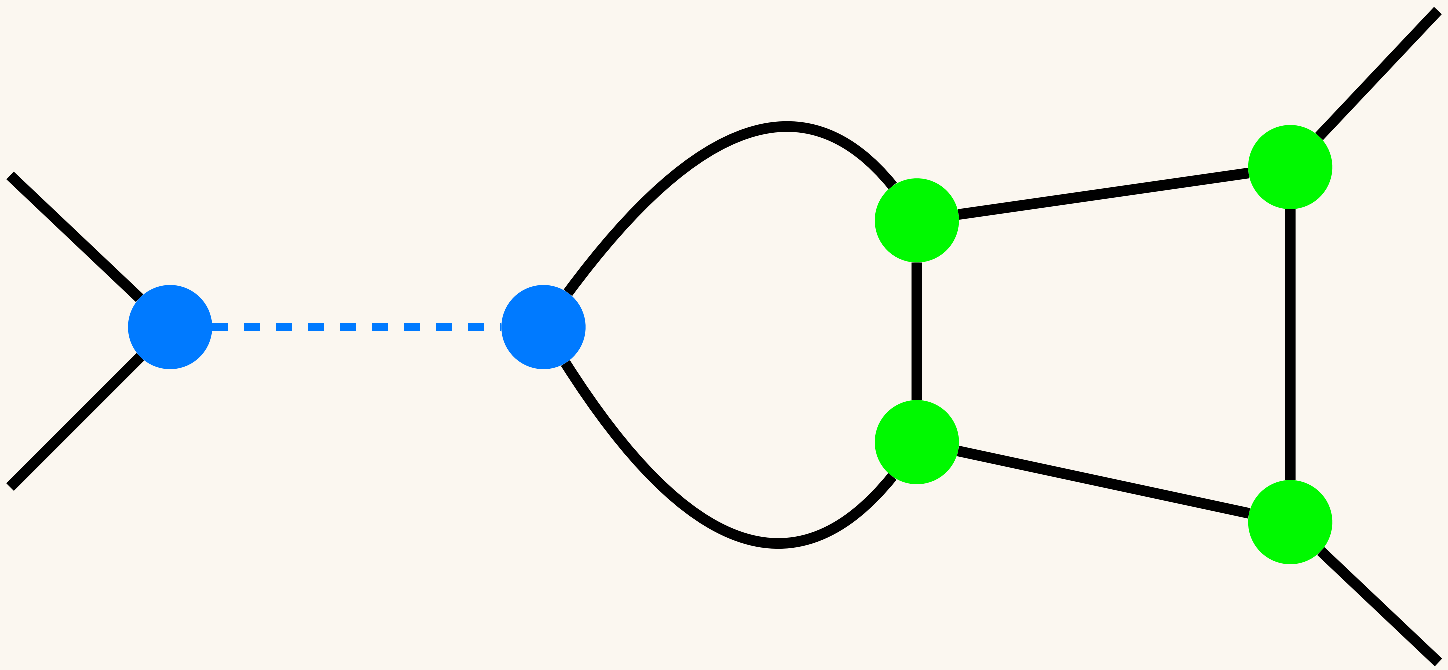

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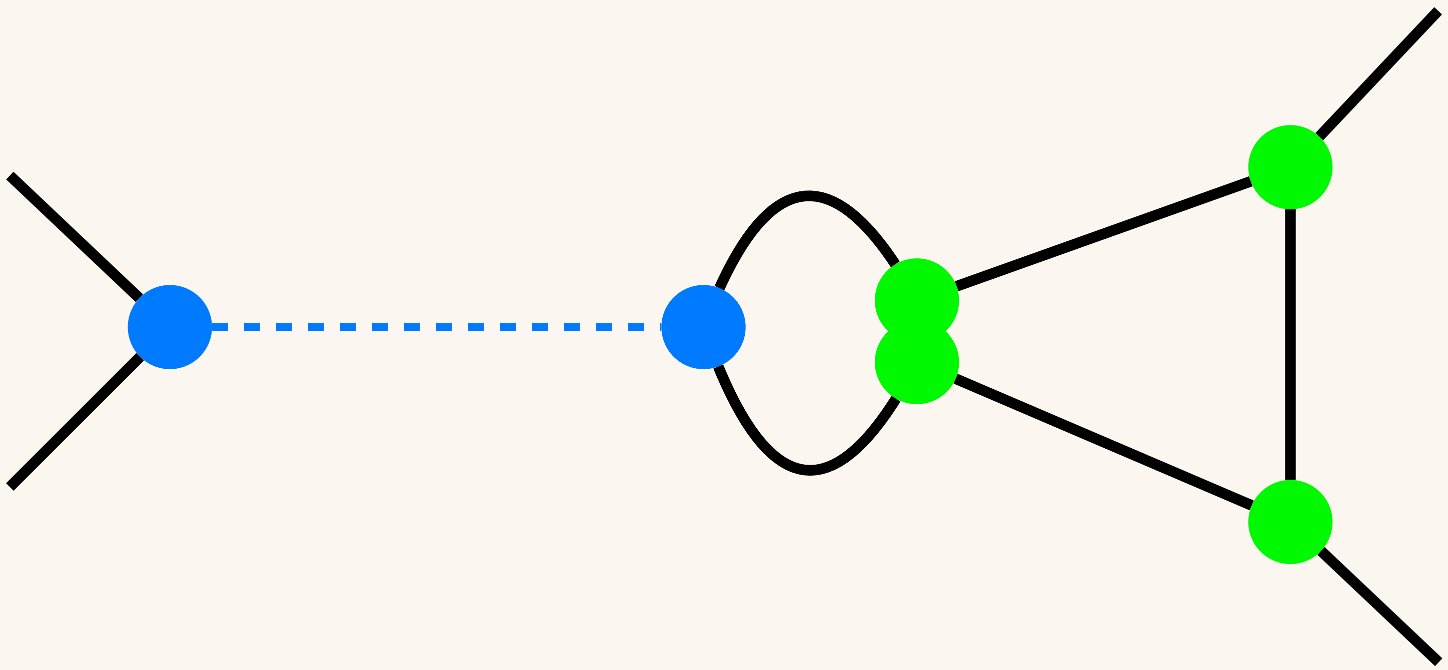

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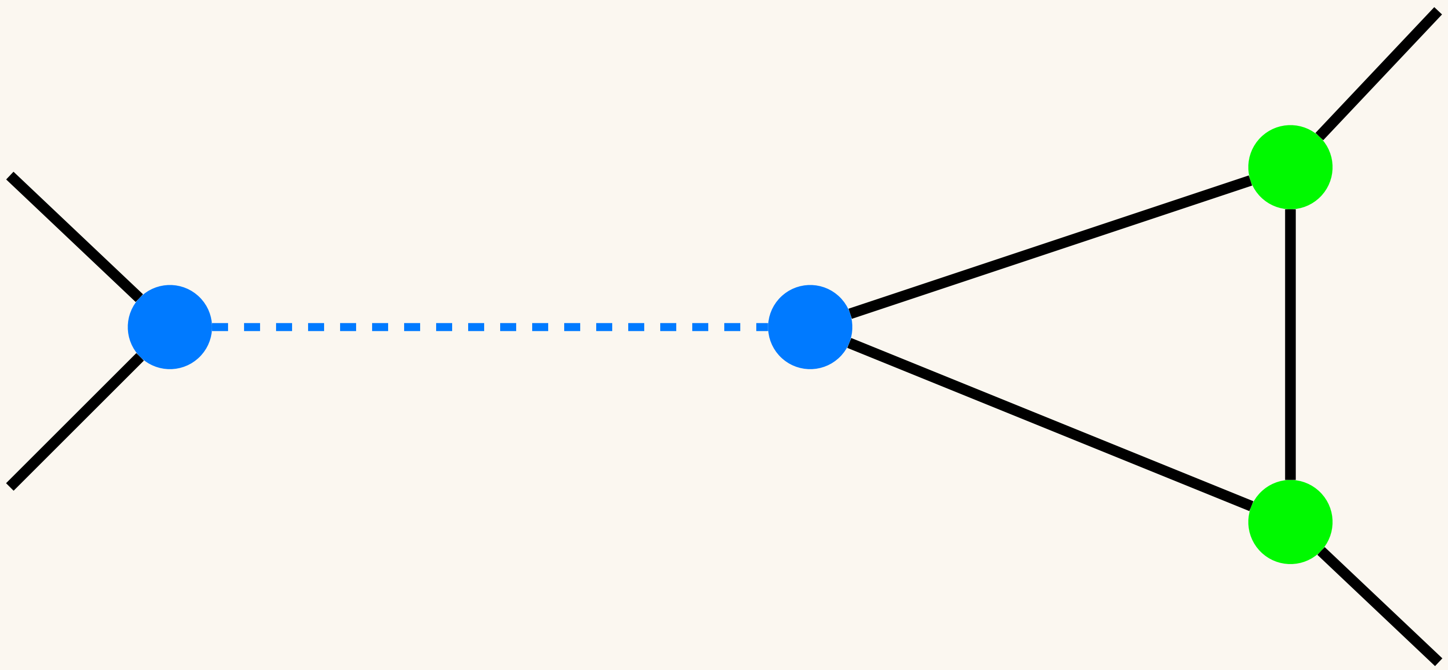

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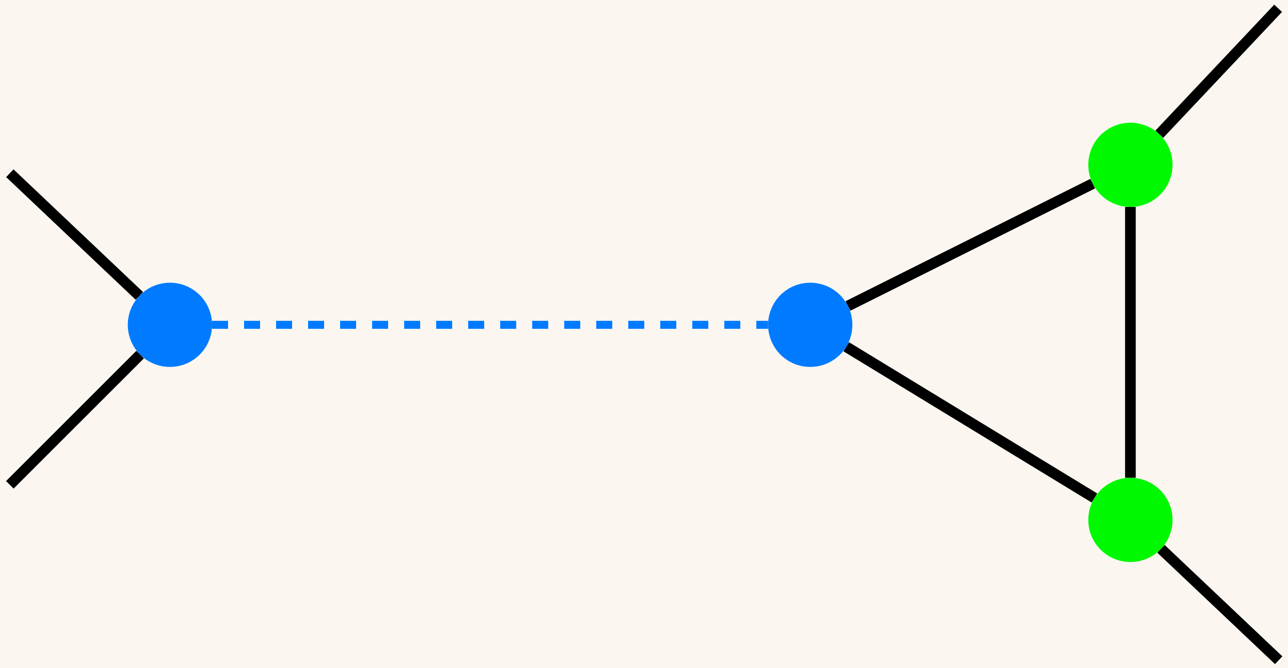

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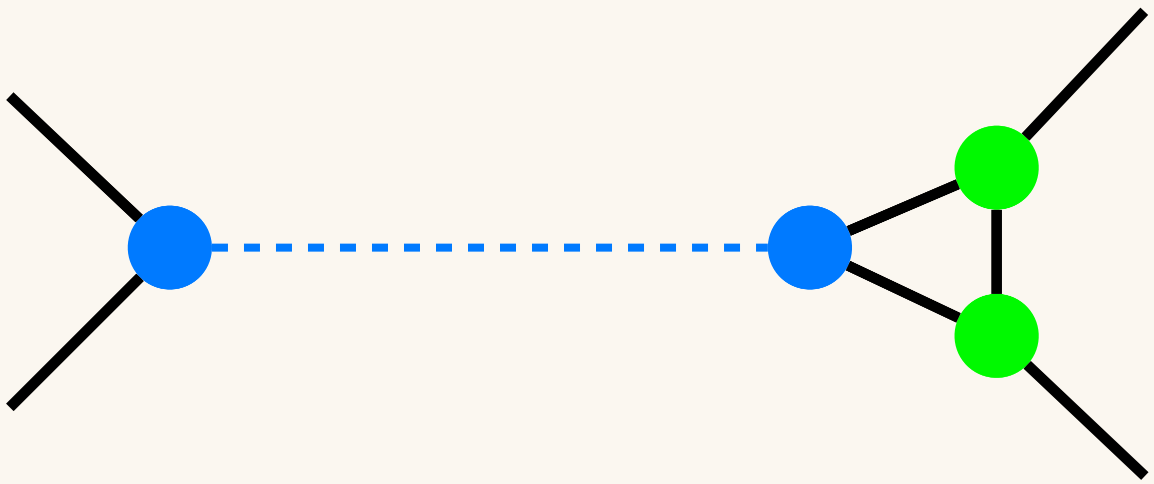
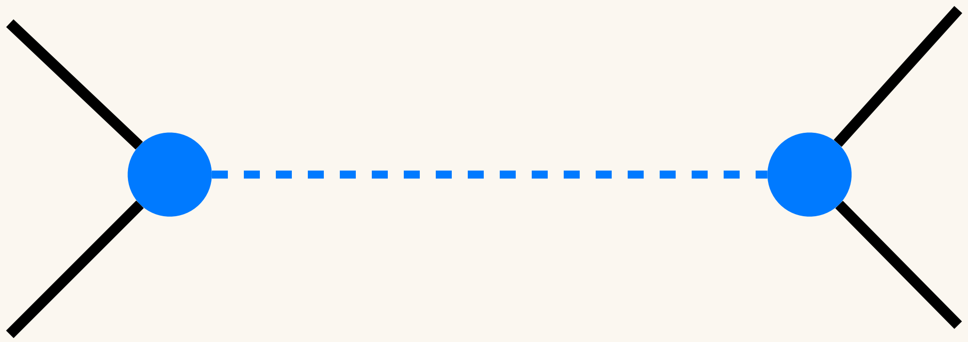


◇ Step 2: $O(D^5)$

◇ Step 3: $O(D^5)$



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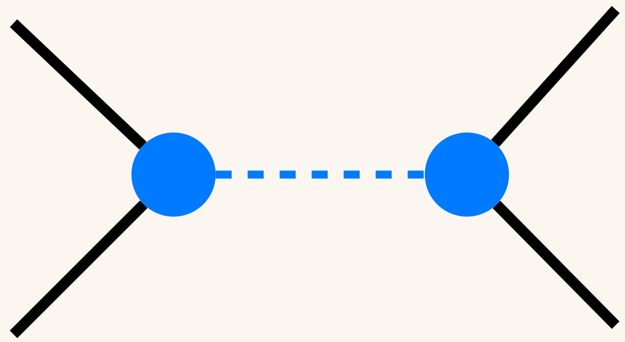

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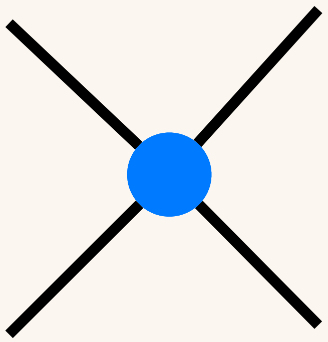
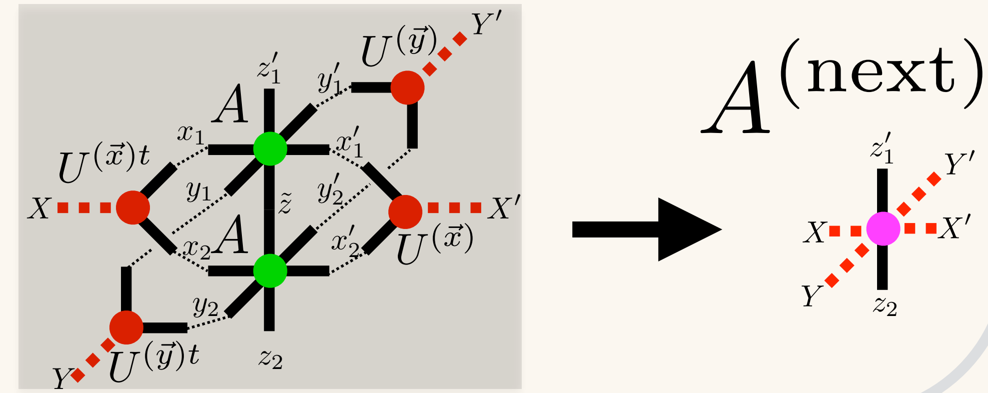


◉ HOTRG [Z.Y. Xie, J. Chen, et al. arXiv:1201.1144]

◇ Contraction with U

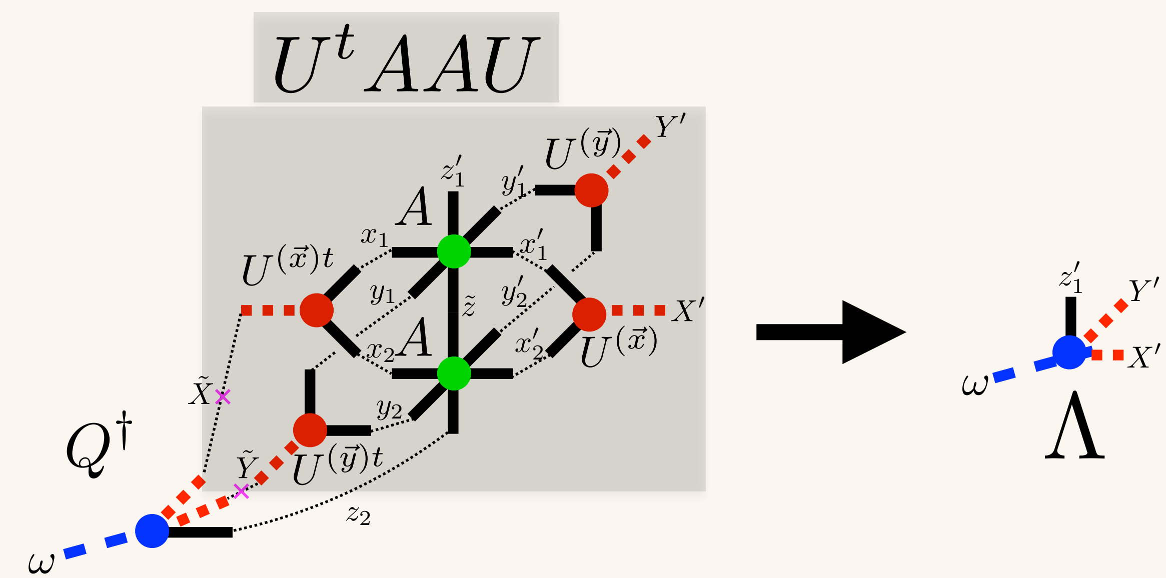
$U^{(\vec{y})t}U^{(\vec{x})t}AAU^{(\vec{x})}U^{(\vec{y})} \to A^{(\mathrm{next})}$



◉ HOTRG with R-SVD [K.N. arXiv:2307.14191]

◇ Contraction with U, Q

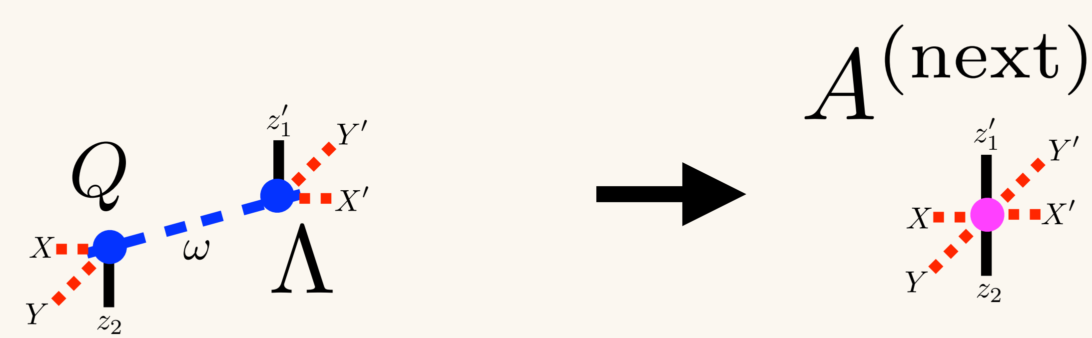
$QQ^{\dagger}U^{(\vec{y})t}U^{(\vec{x})t}AAU^{(\vec{x})}U^{(\vec{y})} \simeq A^{(\mathrm{next})}$

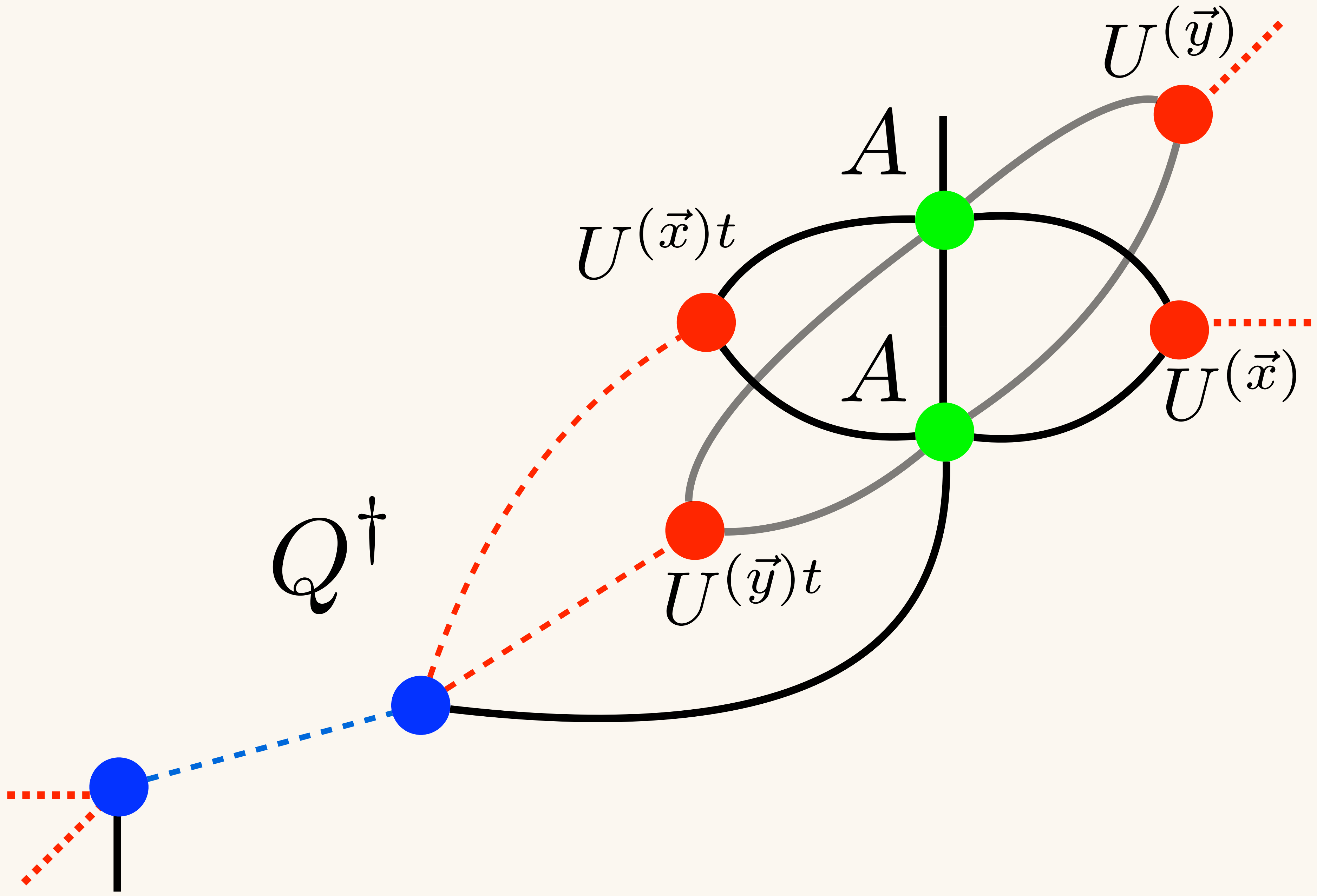


◇ Cost reduction

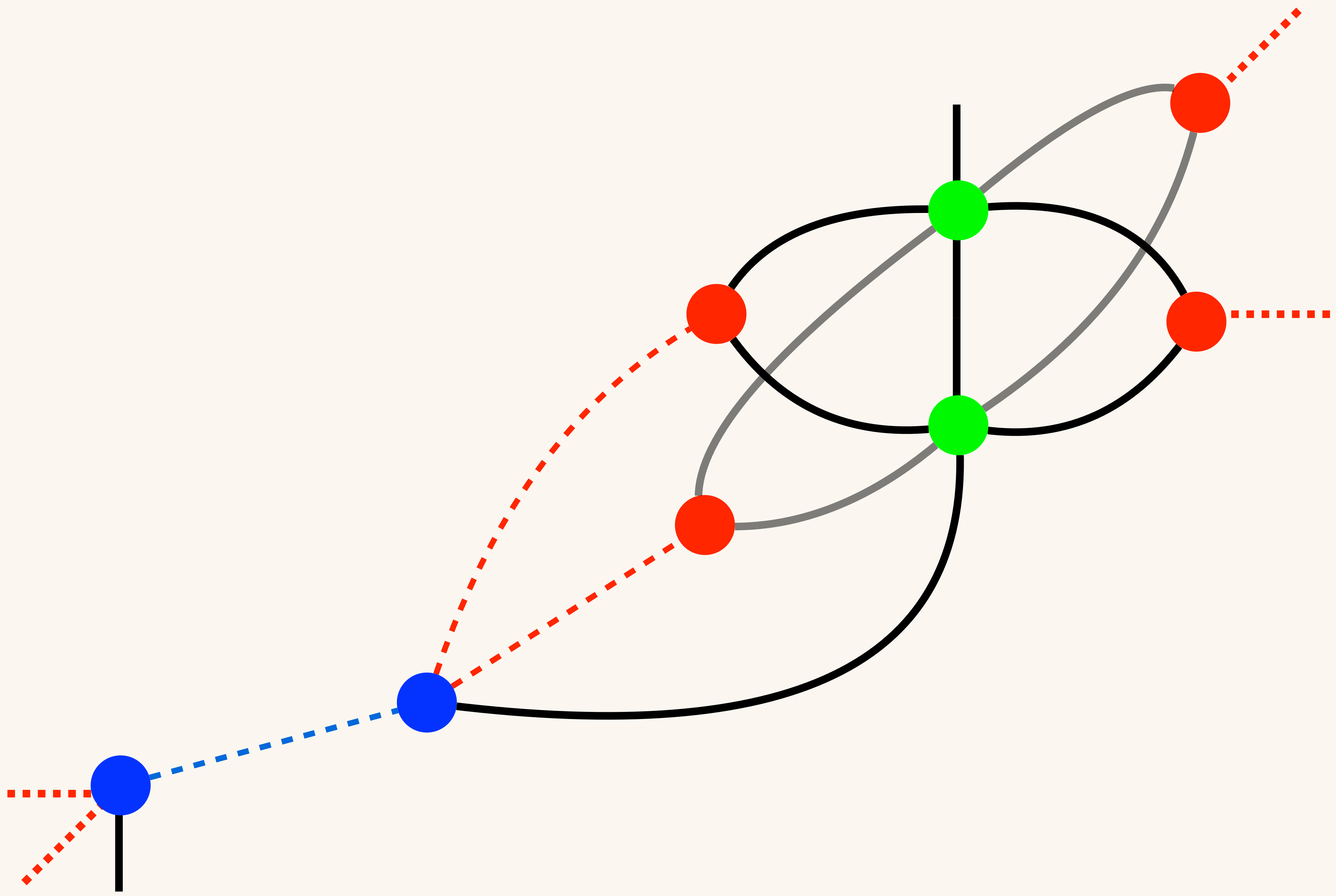
$$O(D^{4d-1}) \to O(D^{3d})$$

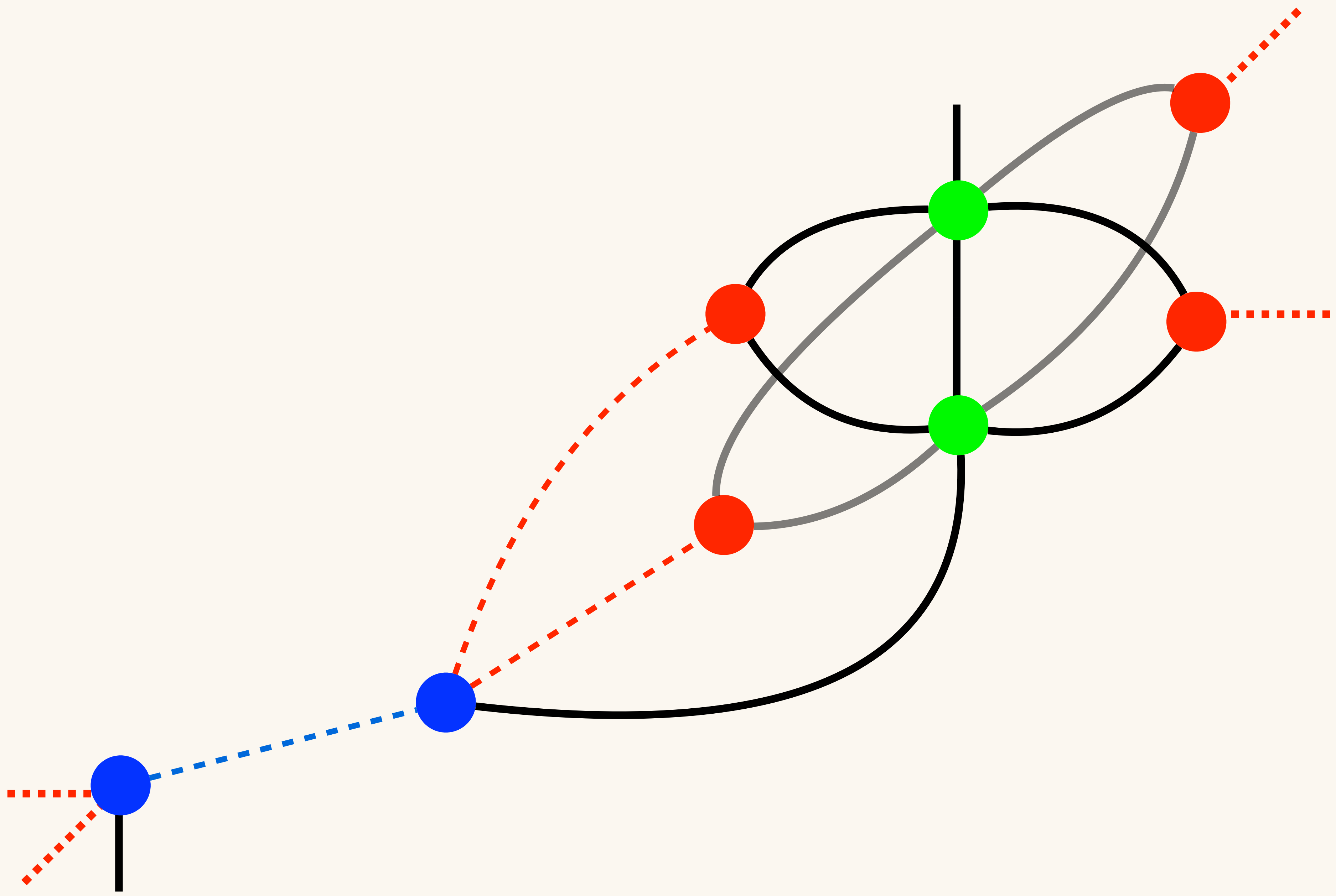
→ Further cost reduction?

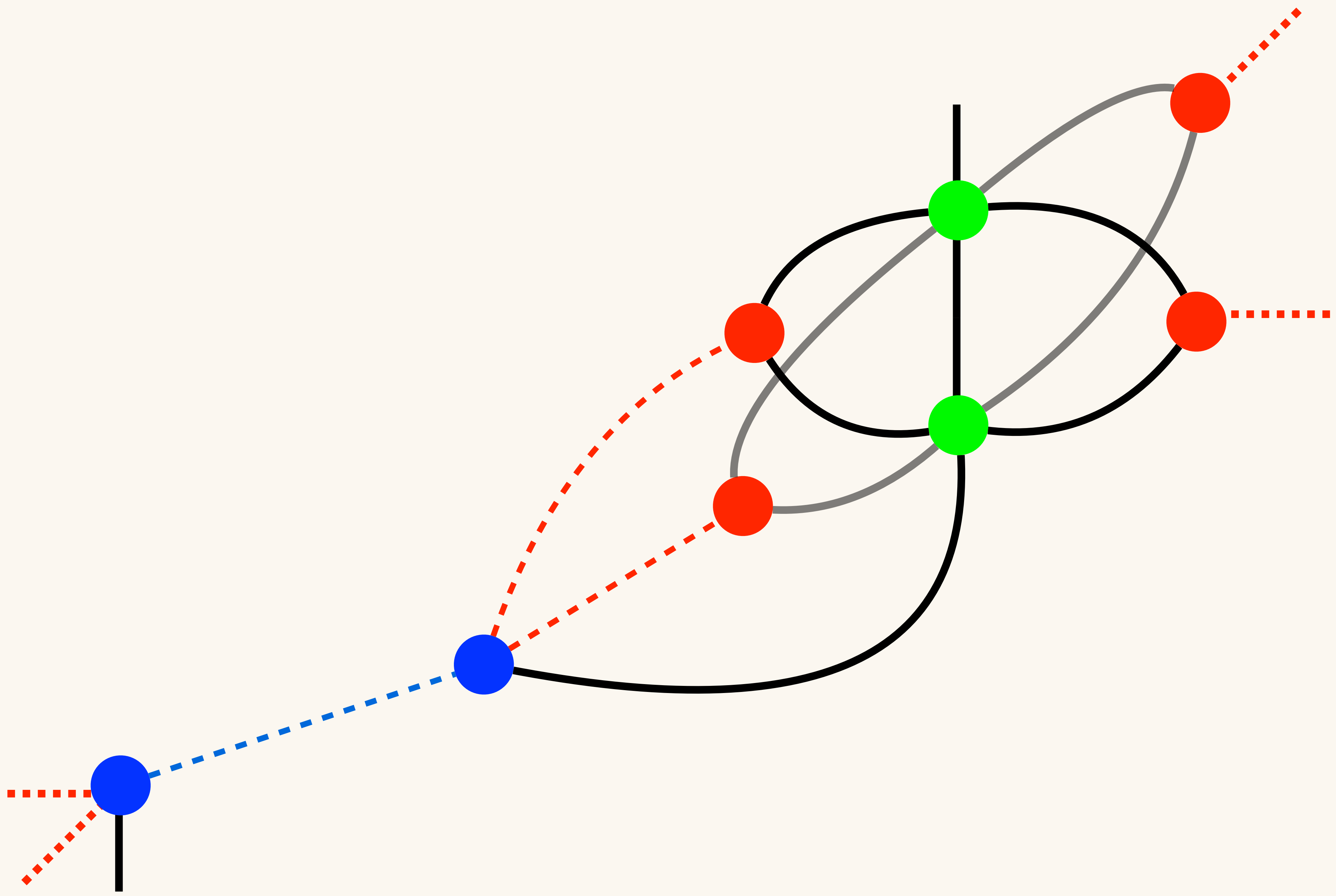


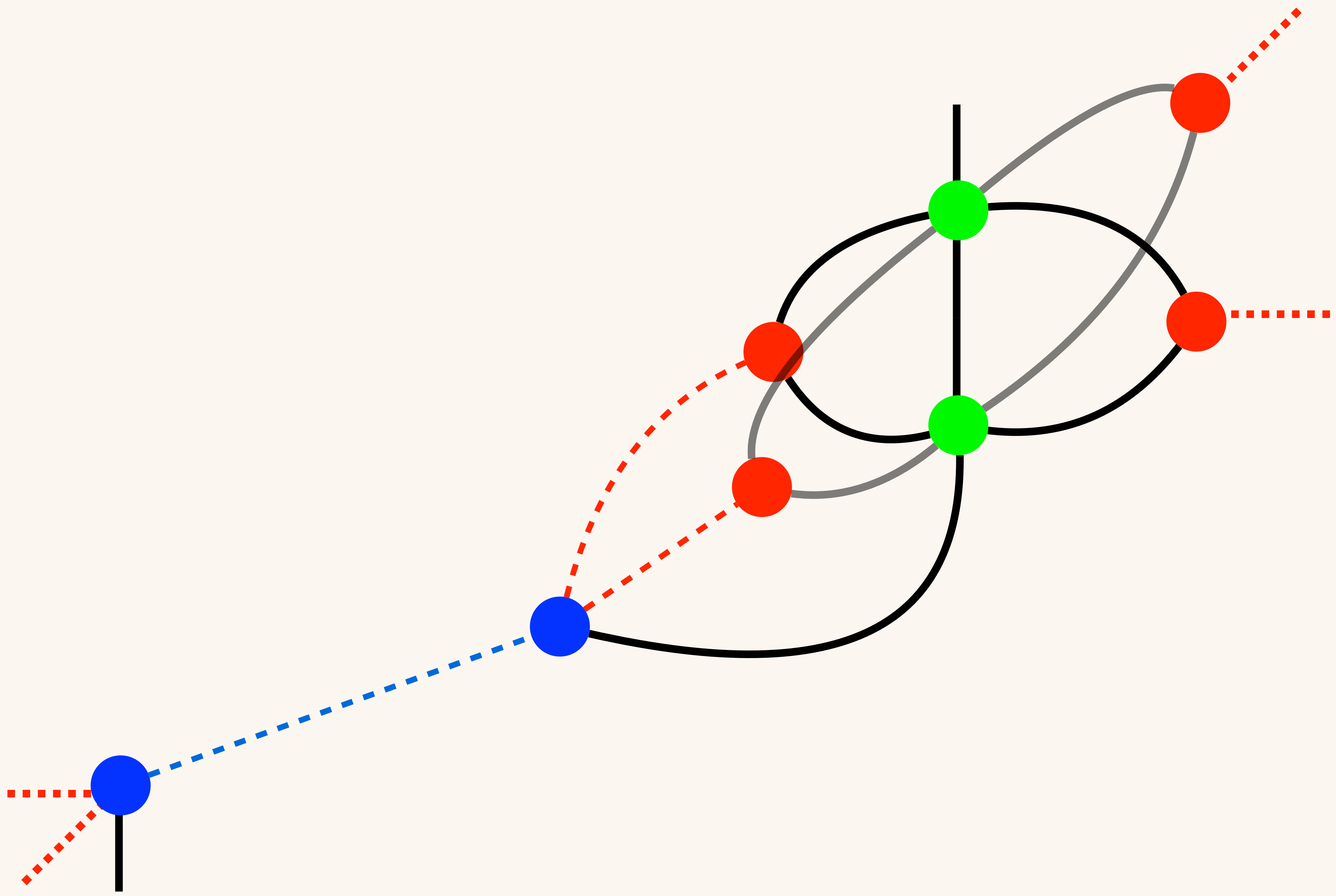


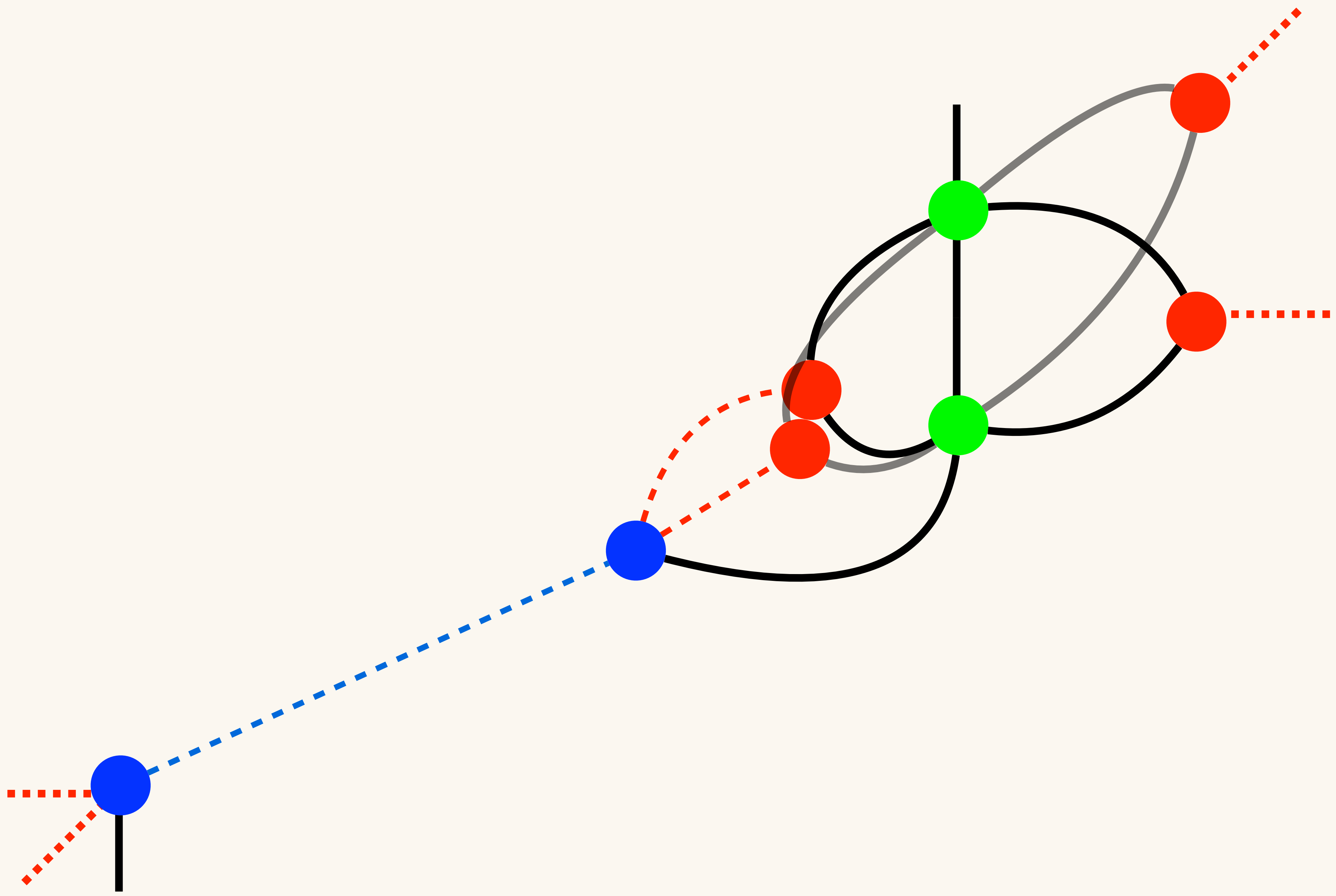
$U^{(\vec{y})}$
A
$U^{(\vec{x})t}$
A
$U^{(\vec{x})}$
$Q^\dagger$
$U^{(\vec{y})t}$

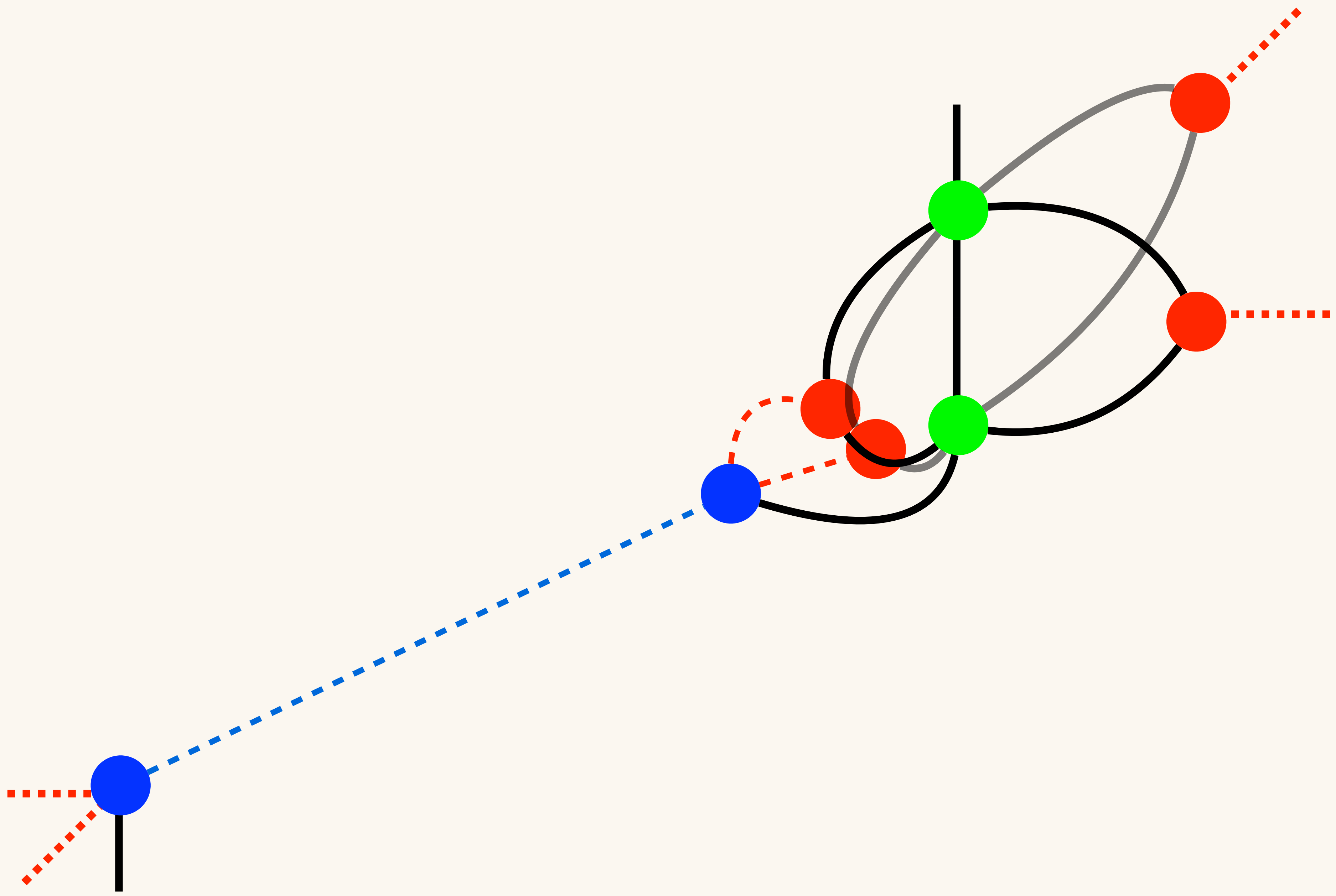


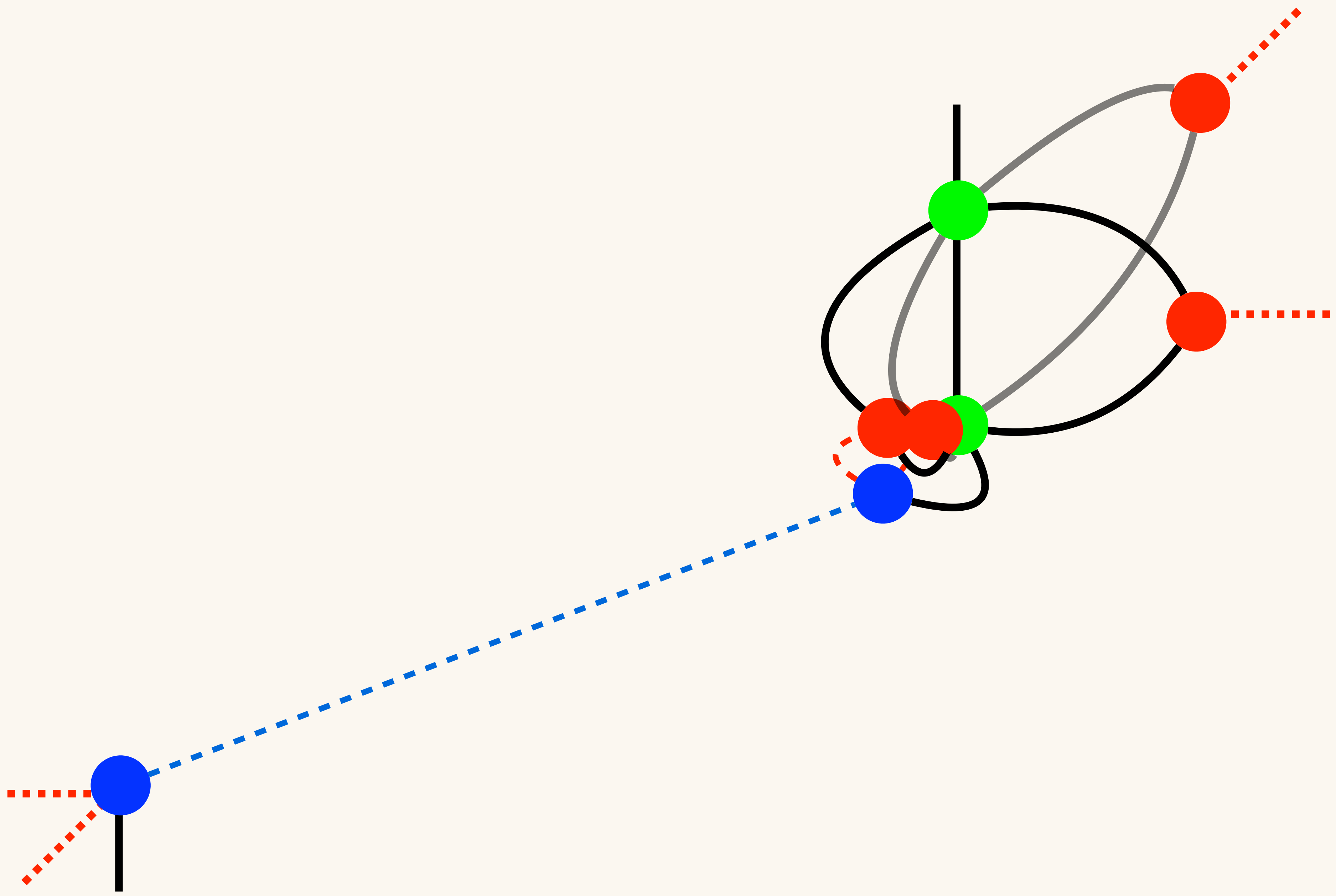


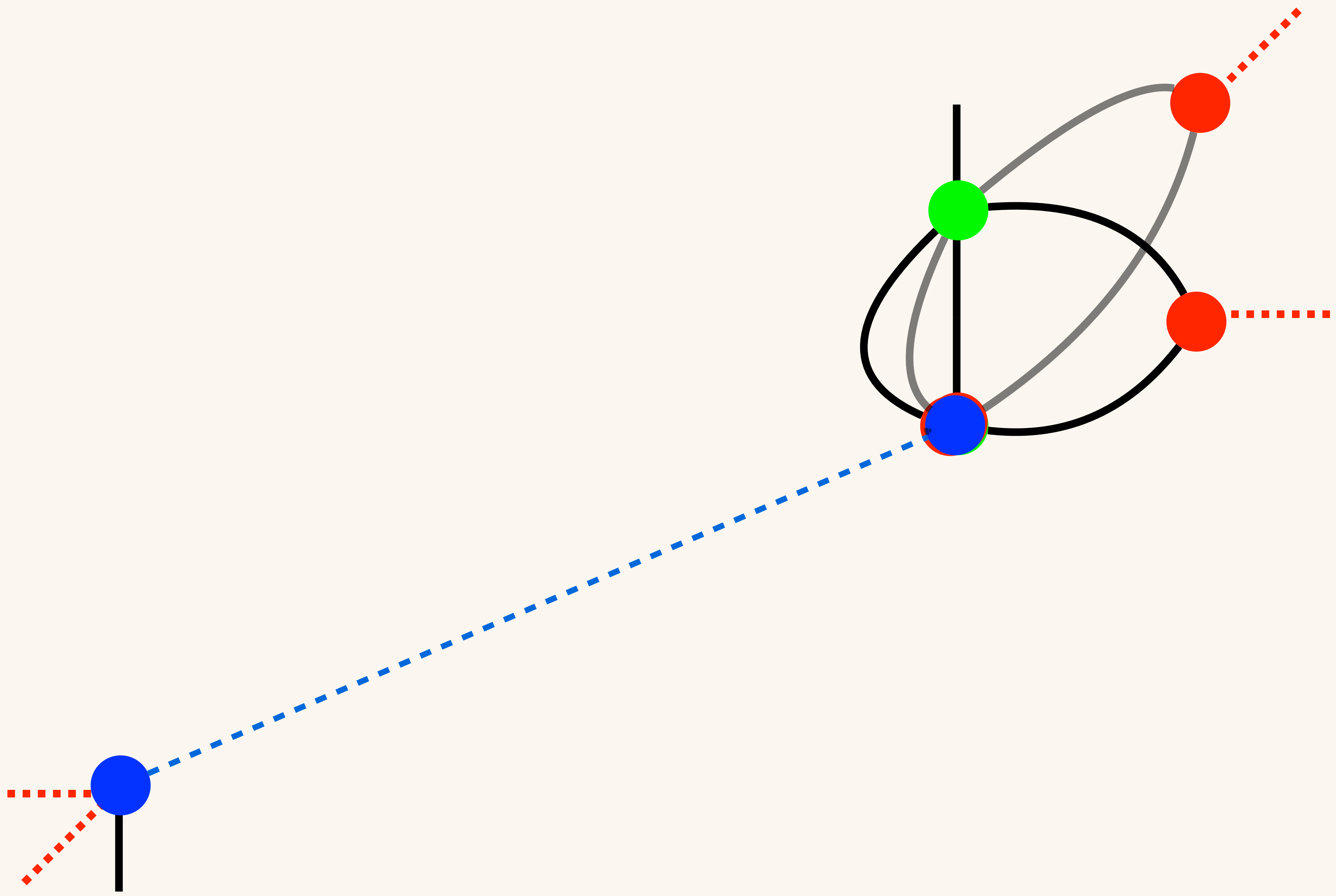


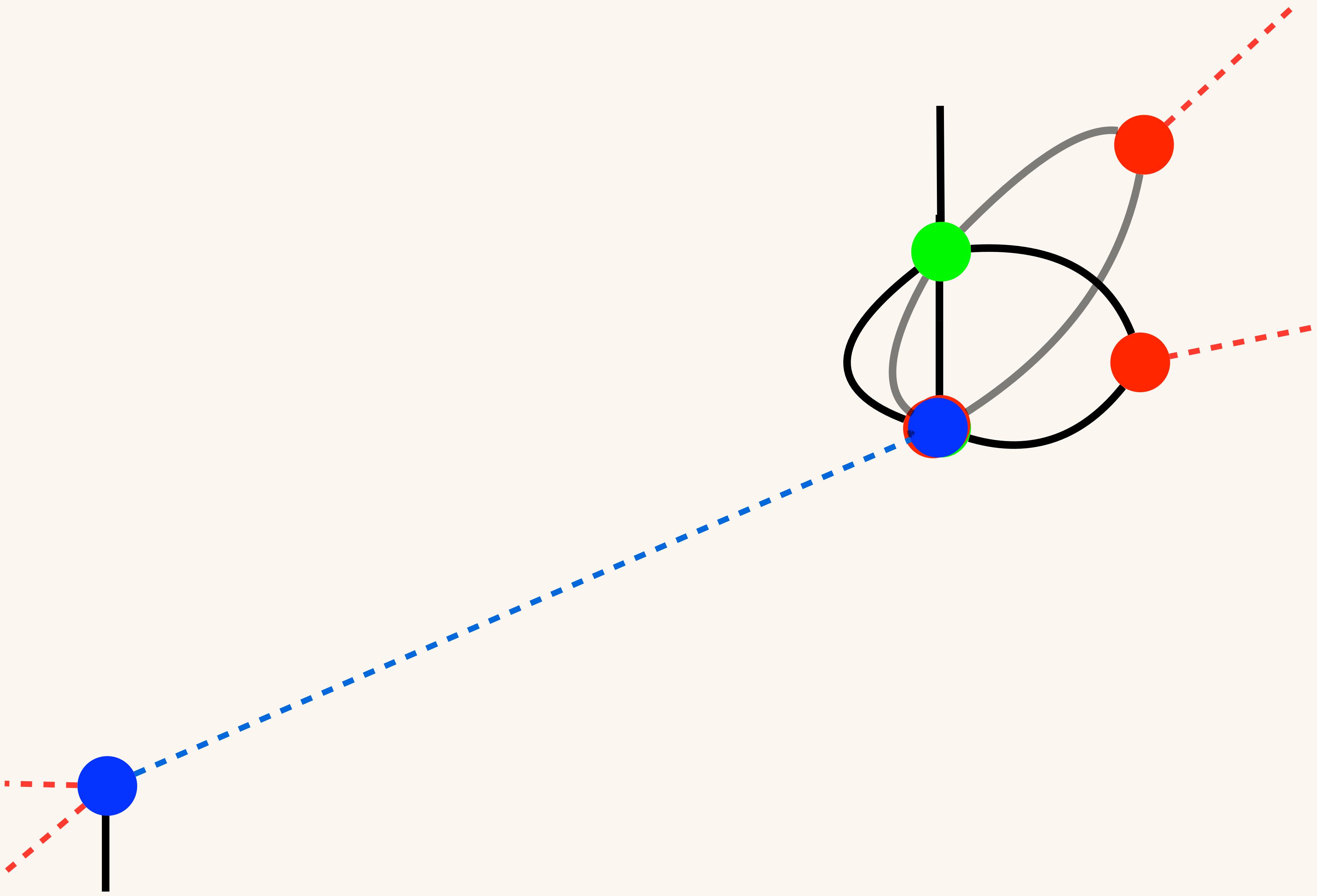


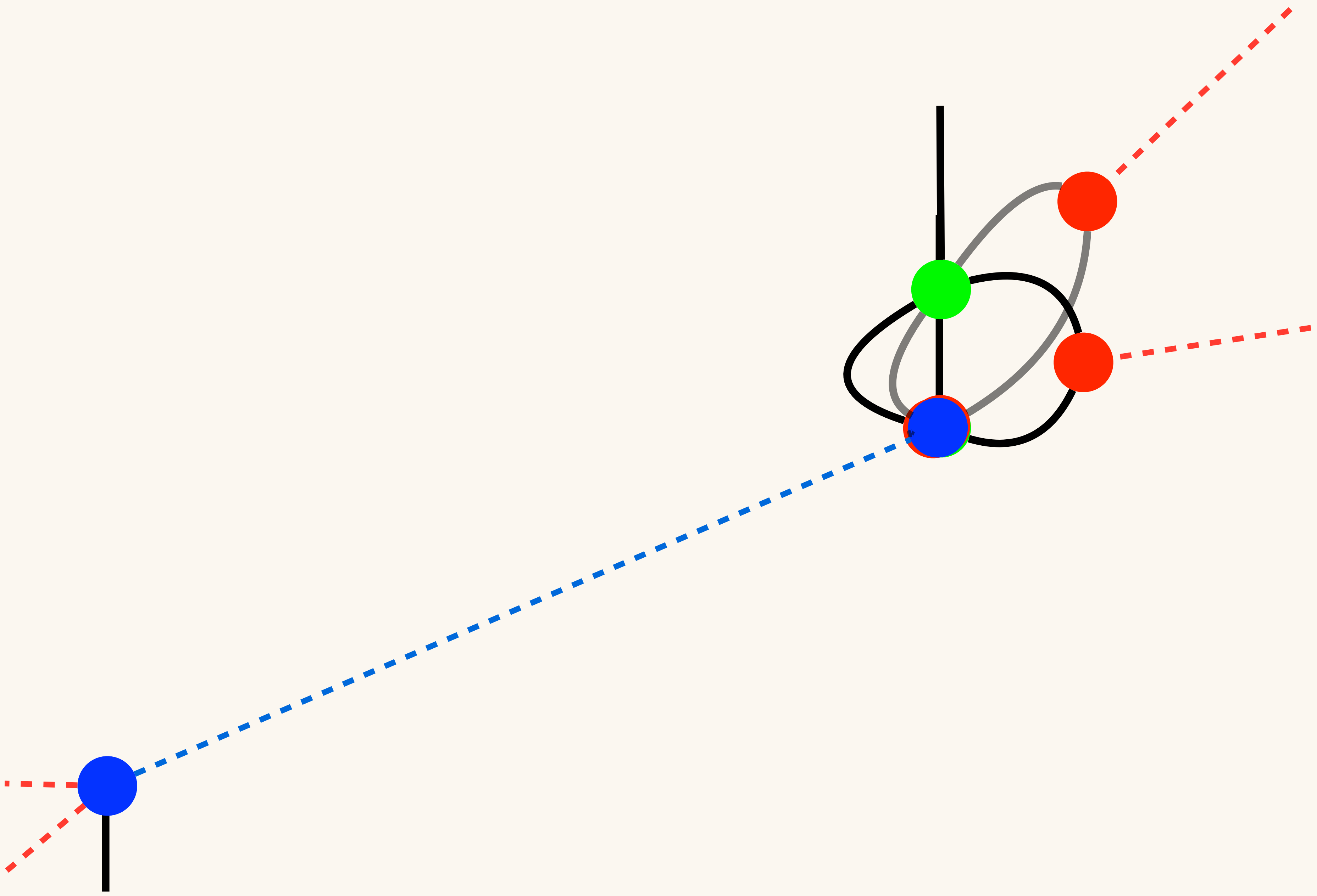


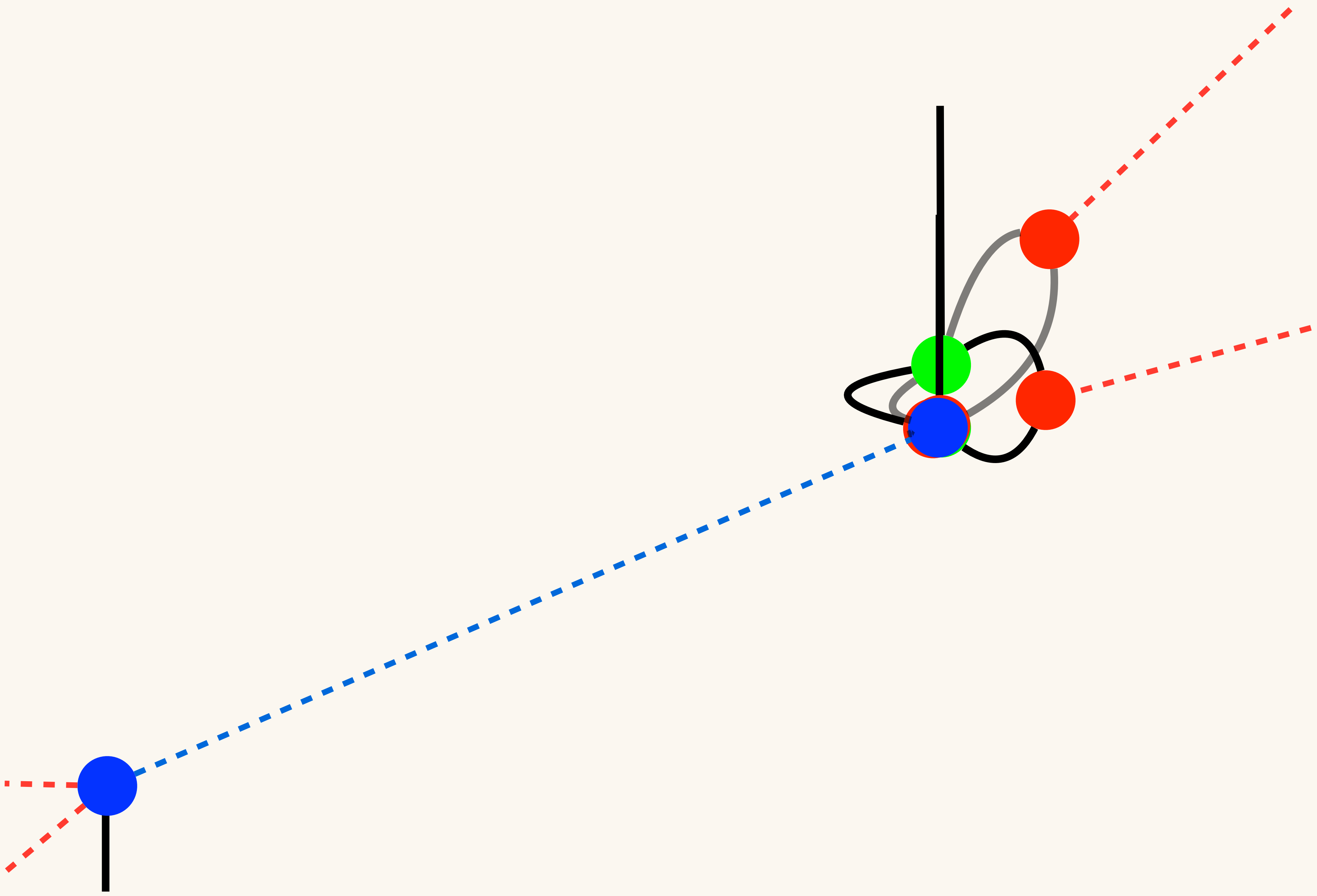


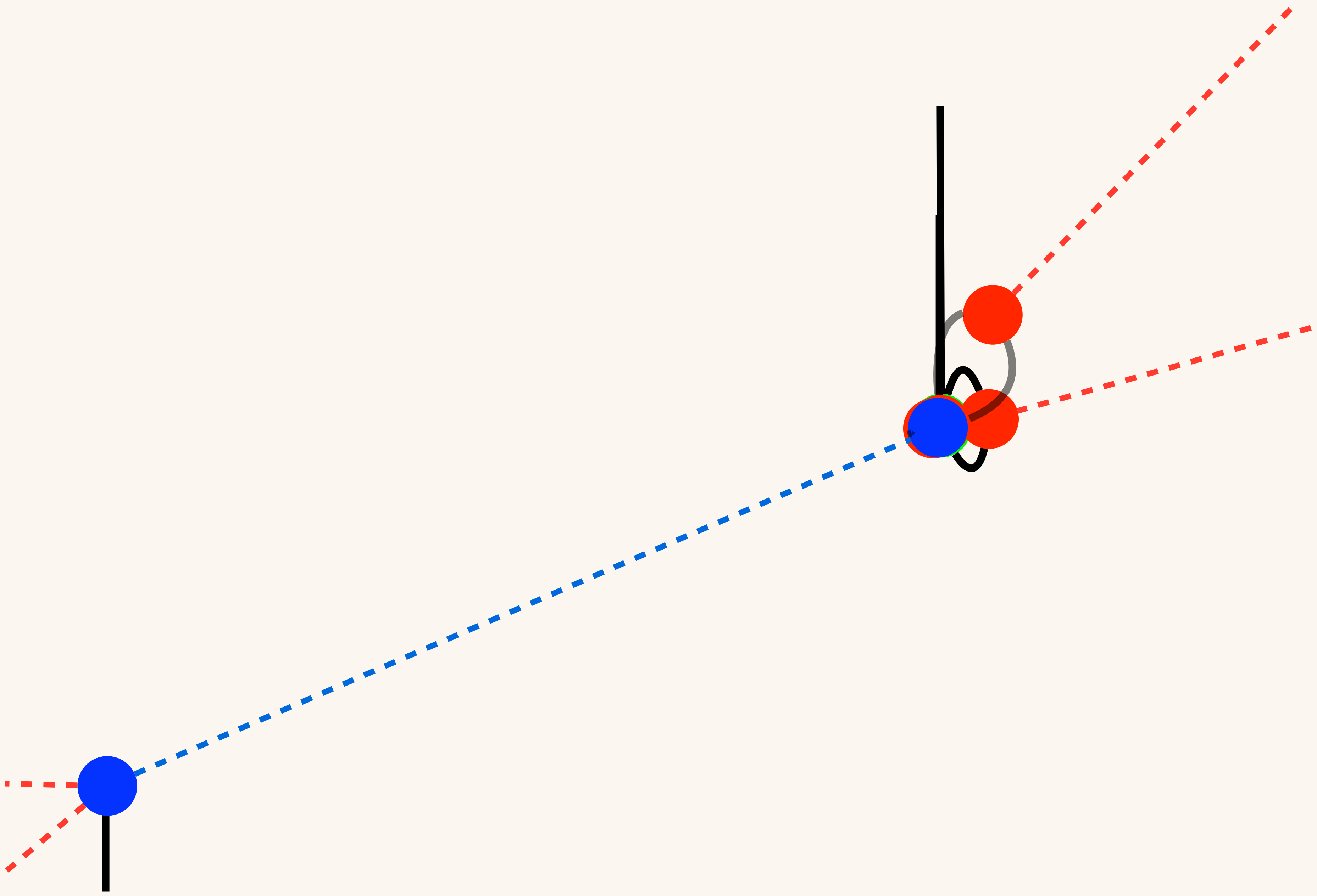


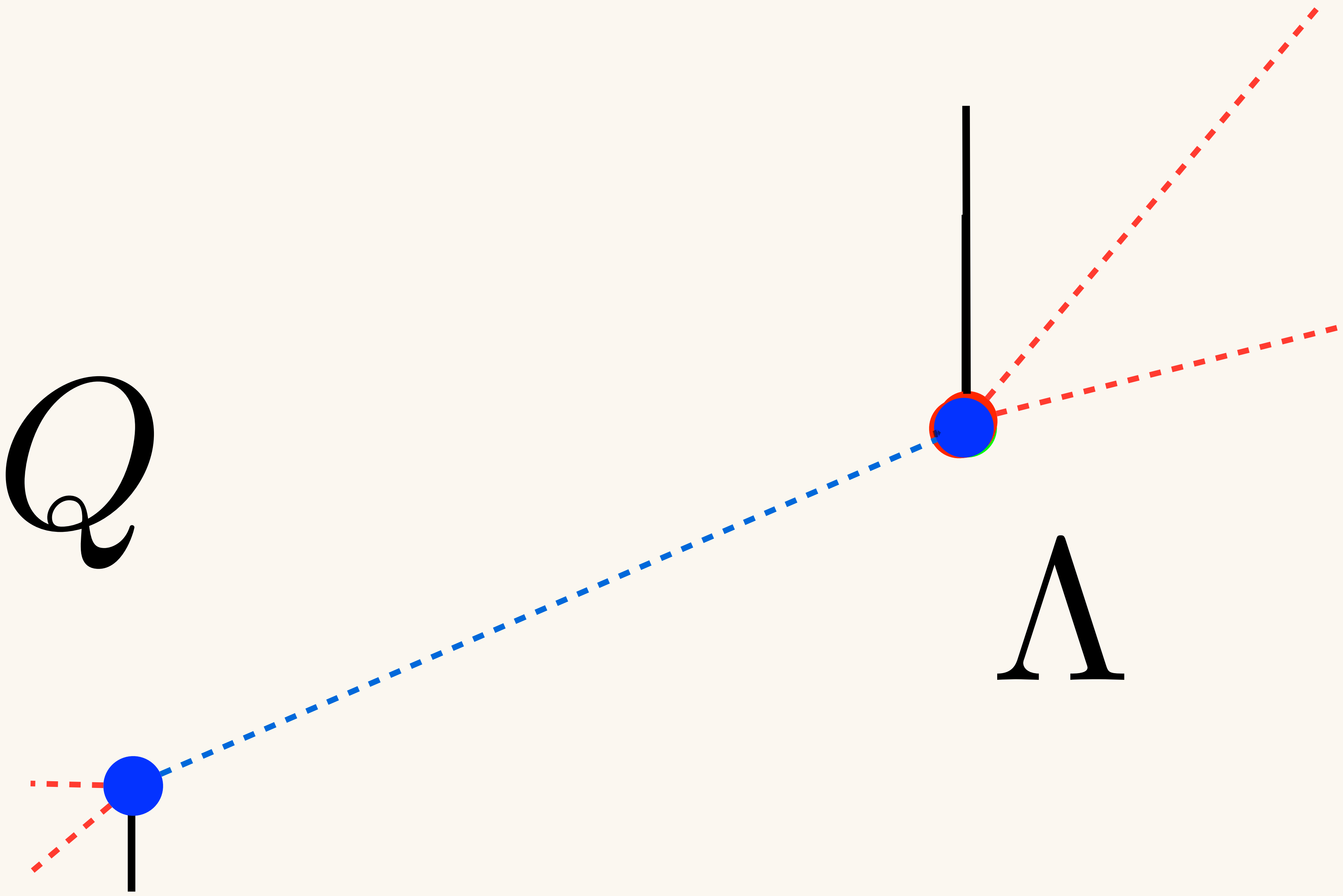










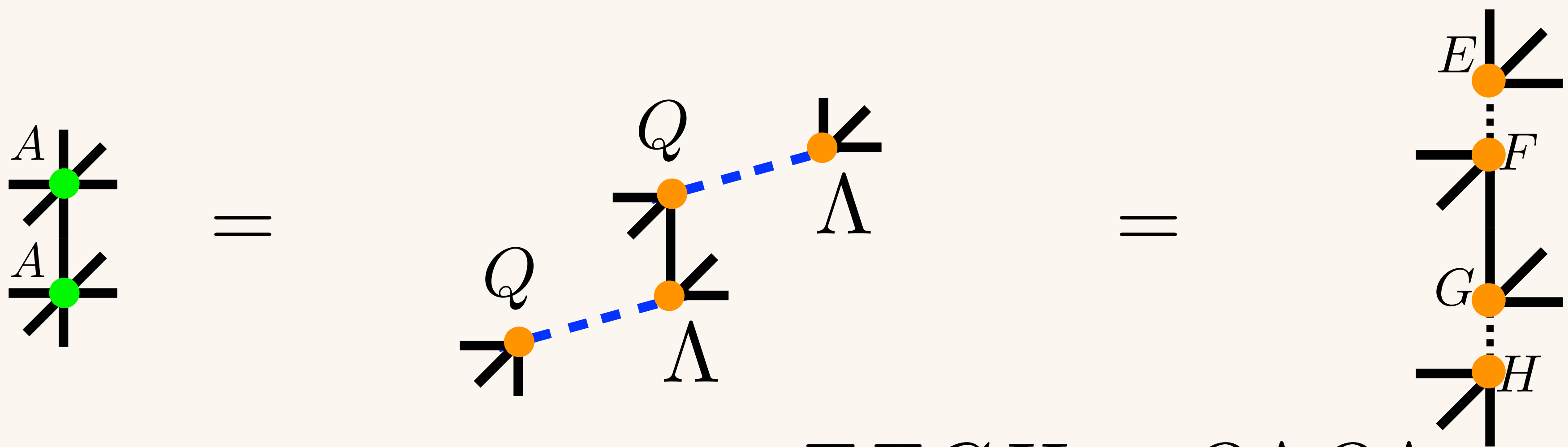

Q
Λ

Cost and systematic error reduction

◉ Minimally-decomposed TRG(MDTRG)

[K.N. arXiv:2307.14191]

→ We already have tensor of order d+1 rep. of Q and Λ.

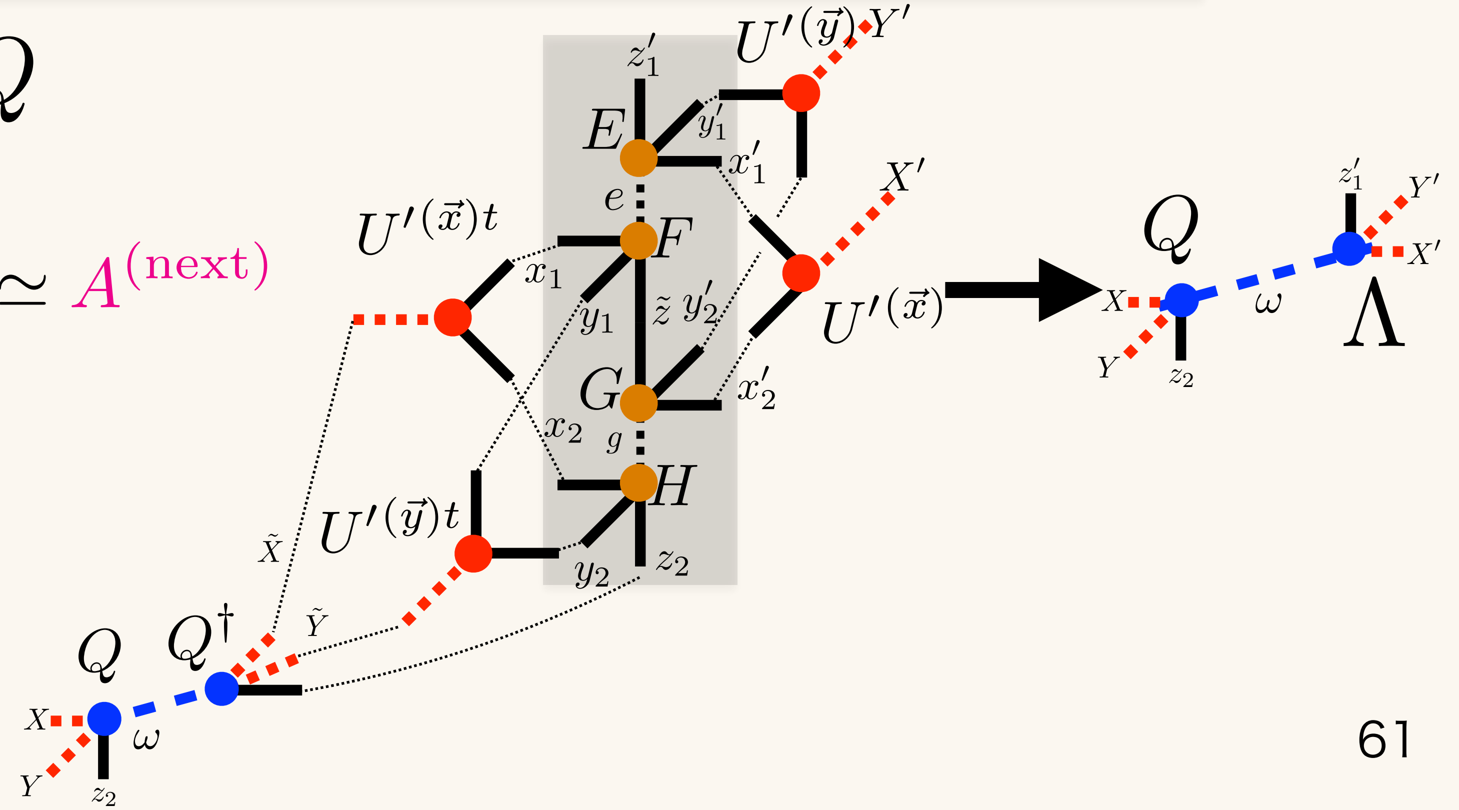


$$EFGH = Q\Lambda Q\Lambda$$

A:Order 2d E,F,G,H:Order d+1 Q, Λ:Order d+1

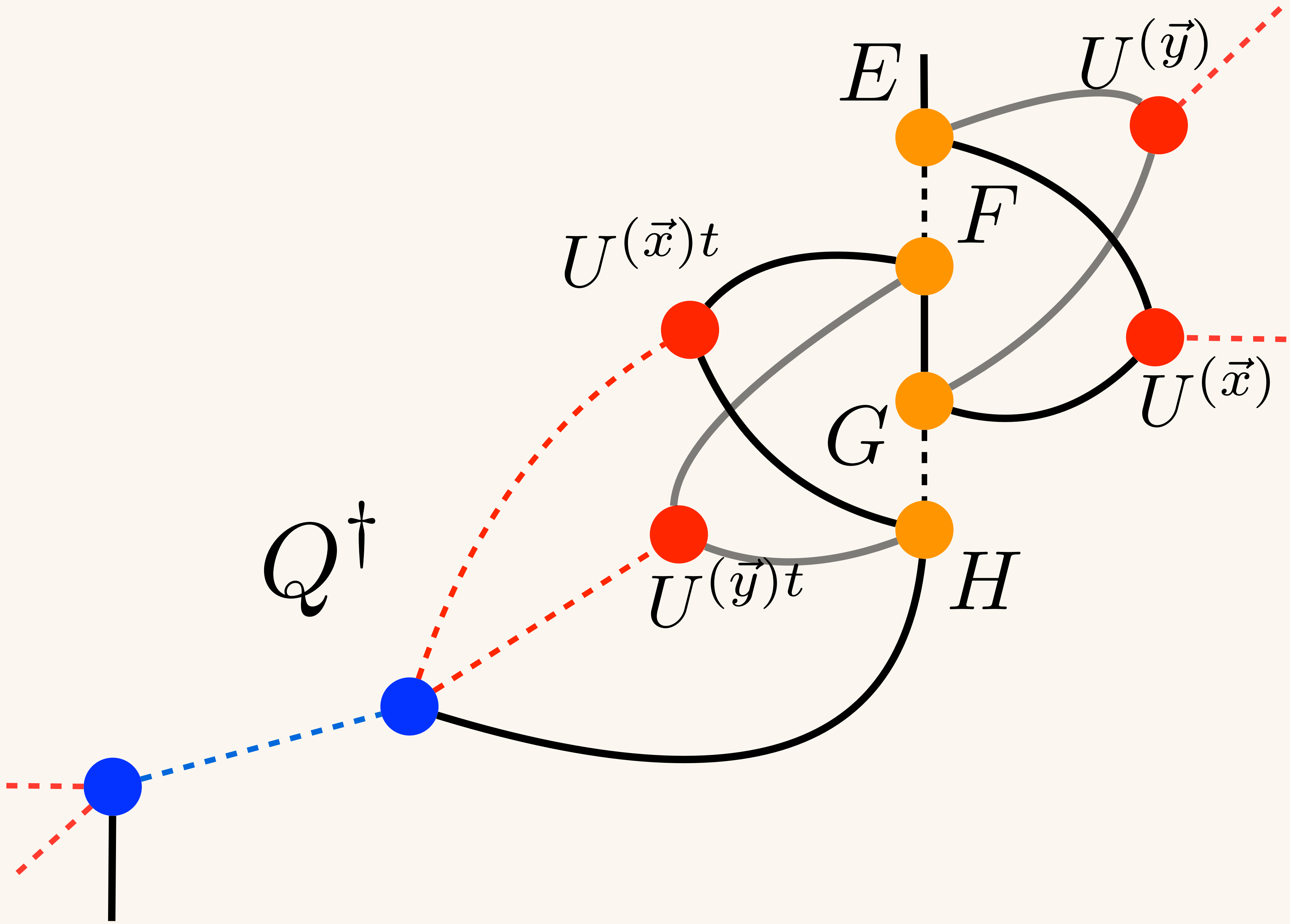
◇ Contraction with EFGH, Q

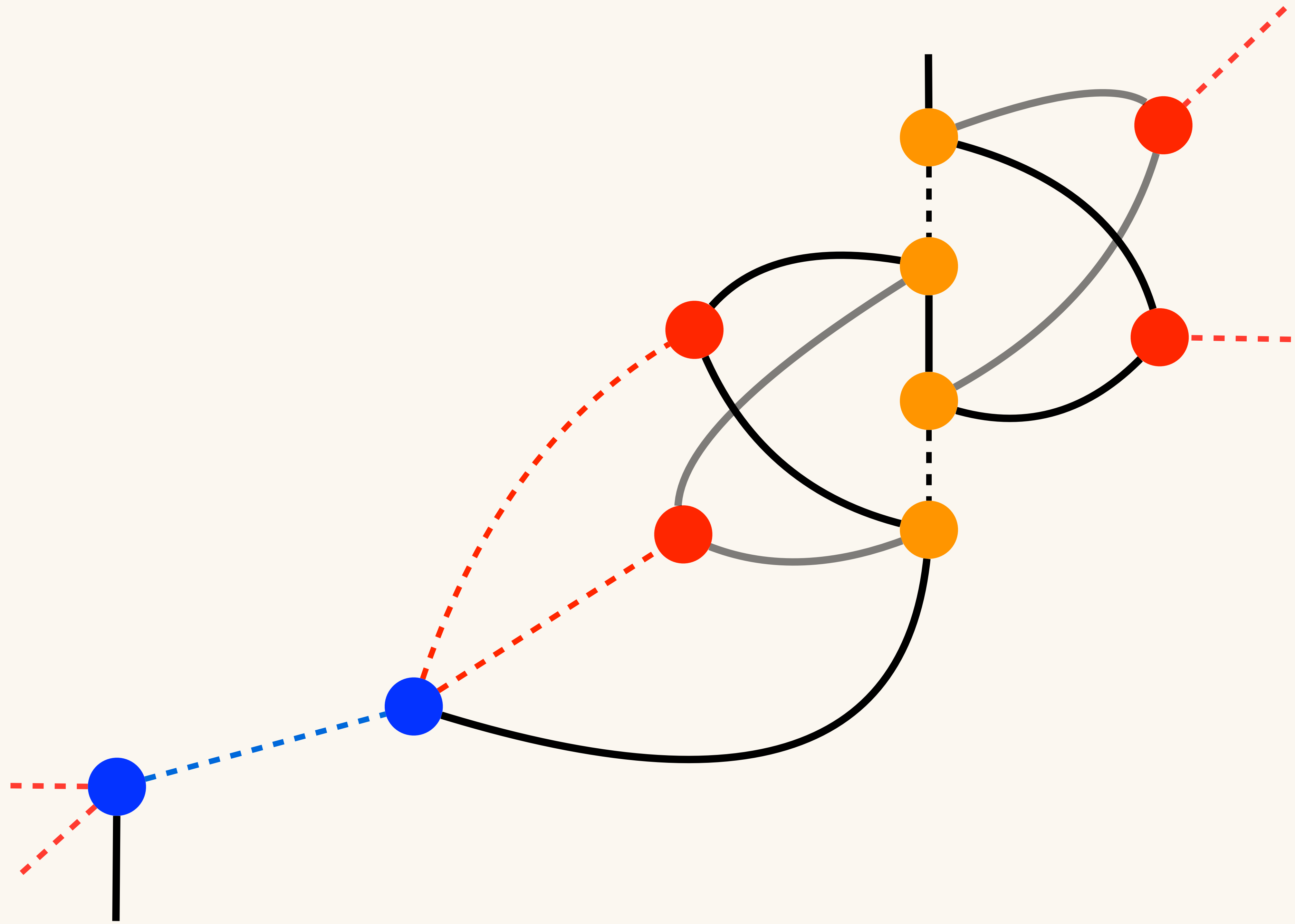
$$QQ^{\dagger}U'^{(\vec{y})t}U'^{(\vec{x})t}EFGHU'^{(\vec{x})}U'^{(\vec{y})} = Q\Lambda \simeq A^{(\mathrm{next})}$$

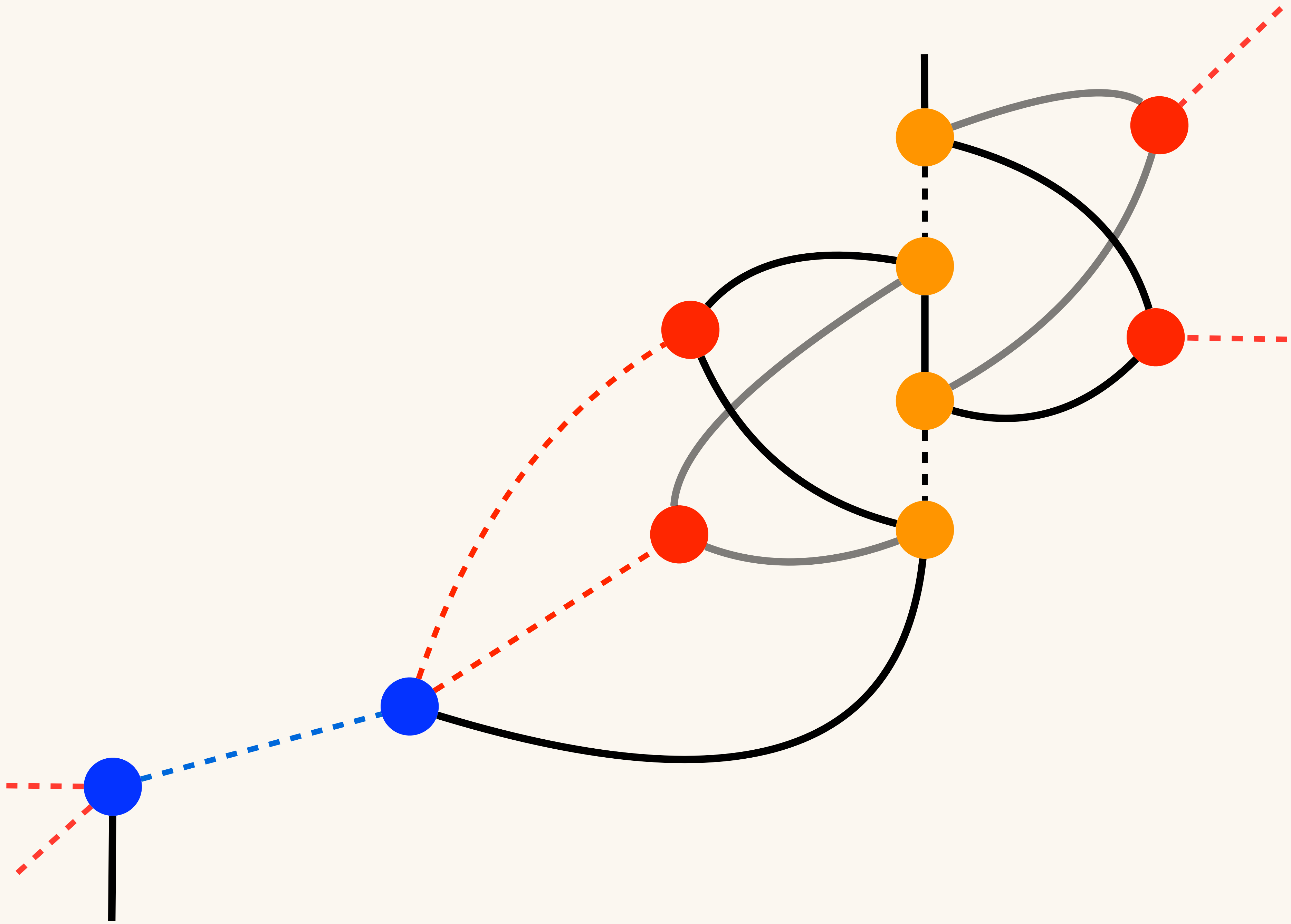


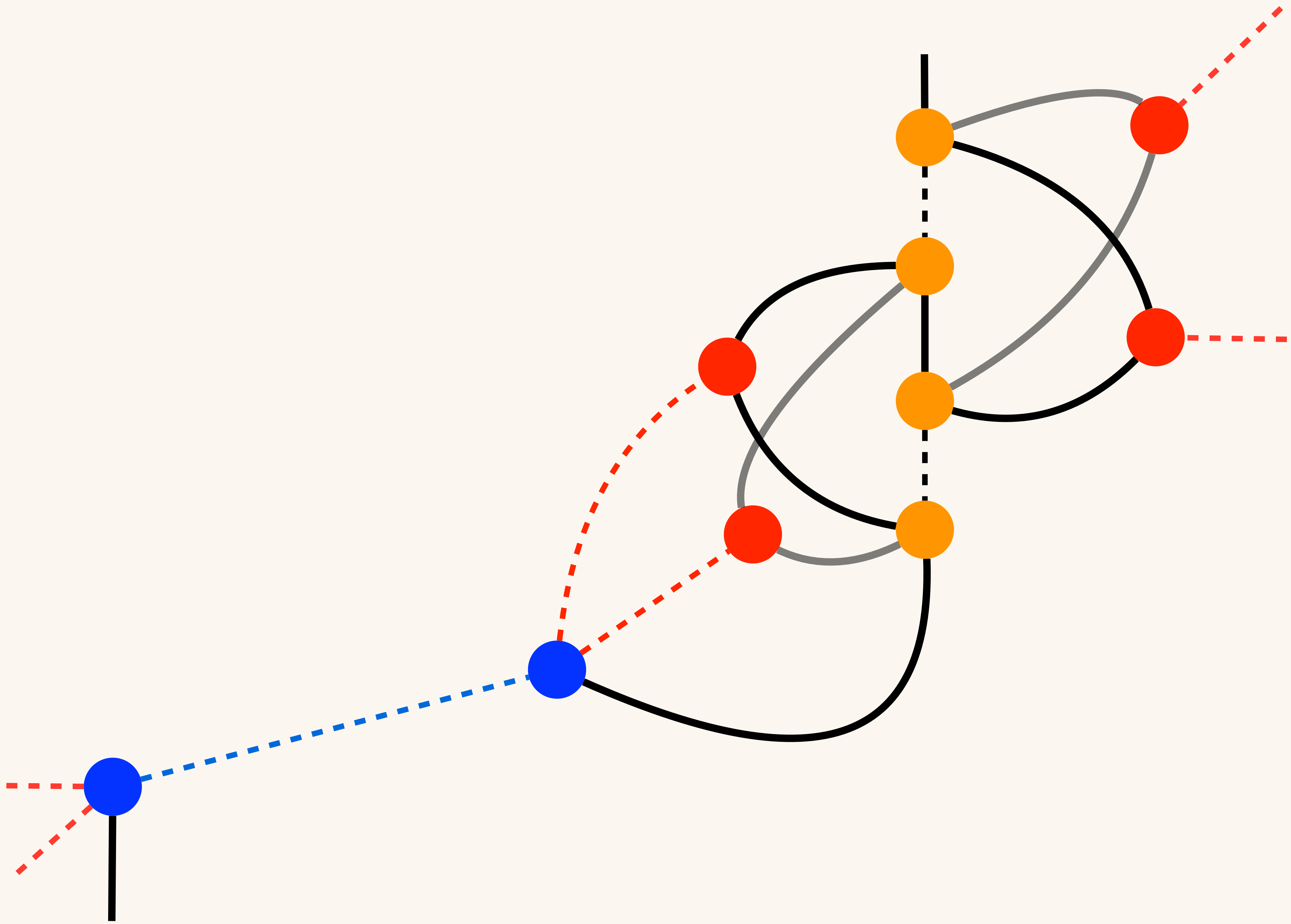
◇ Cost reduction

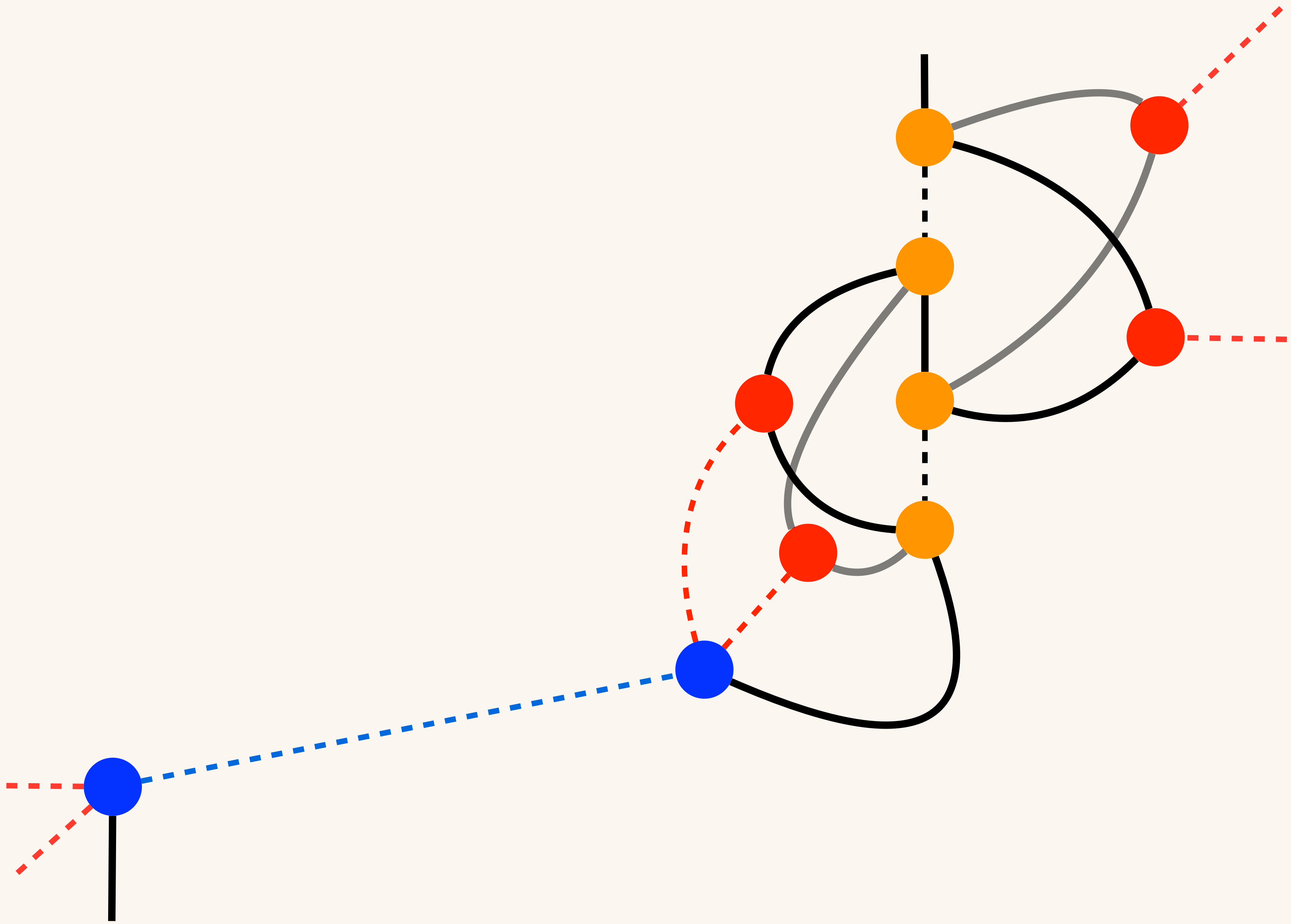
$$O(D^{3d}) \to O(D^{2d+1})$$

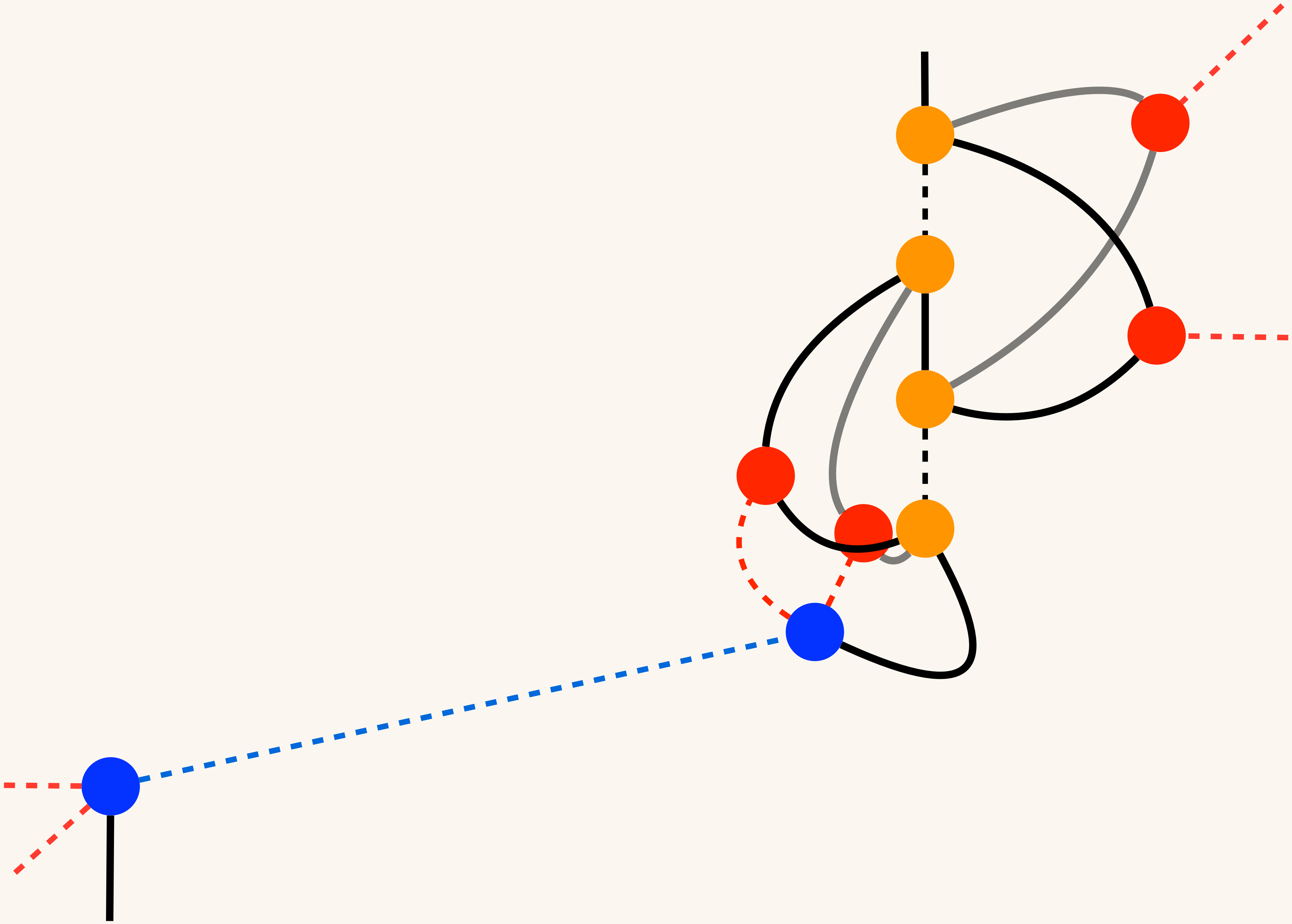

$U^{(\vec{y})}$
E
F
$U^{(\vec{x})t}$
$U^{(\vec{x})}$
G
$Q^\dagger$
$U^{(\vec{y})t}$
H

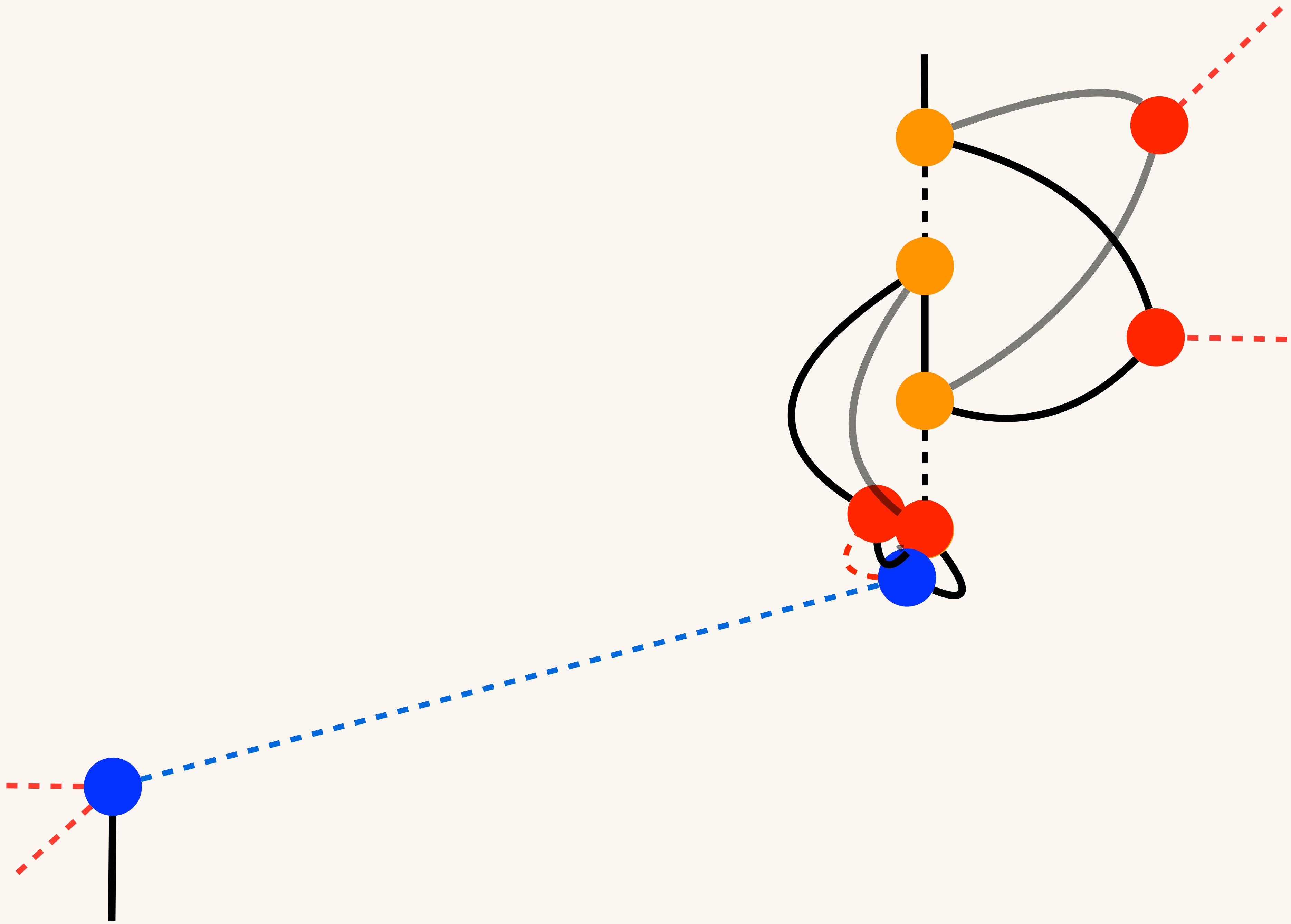


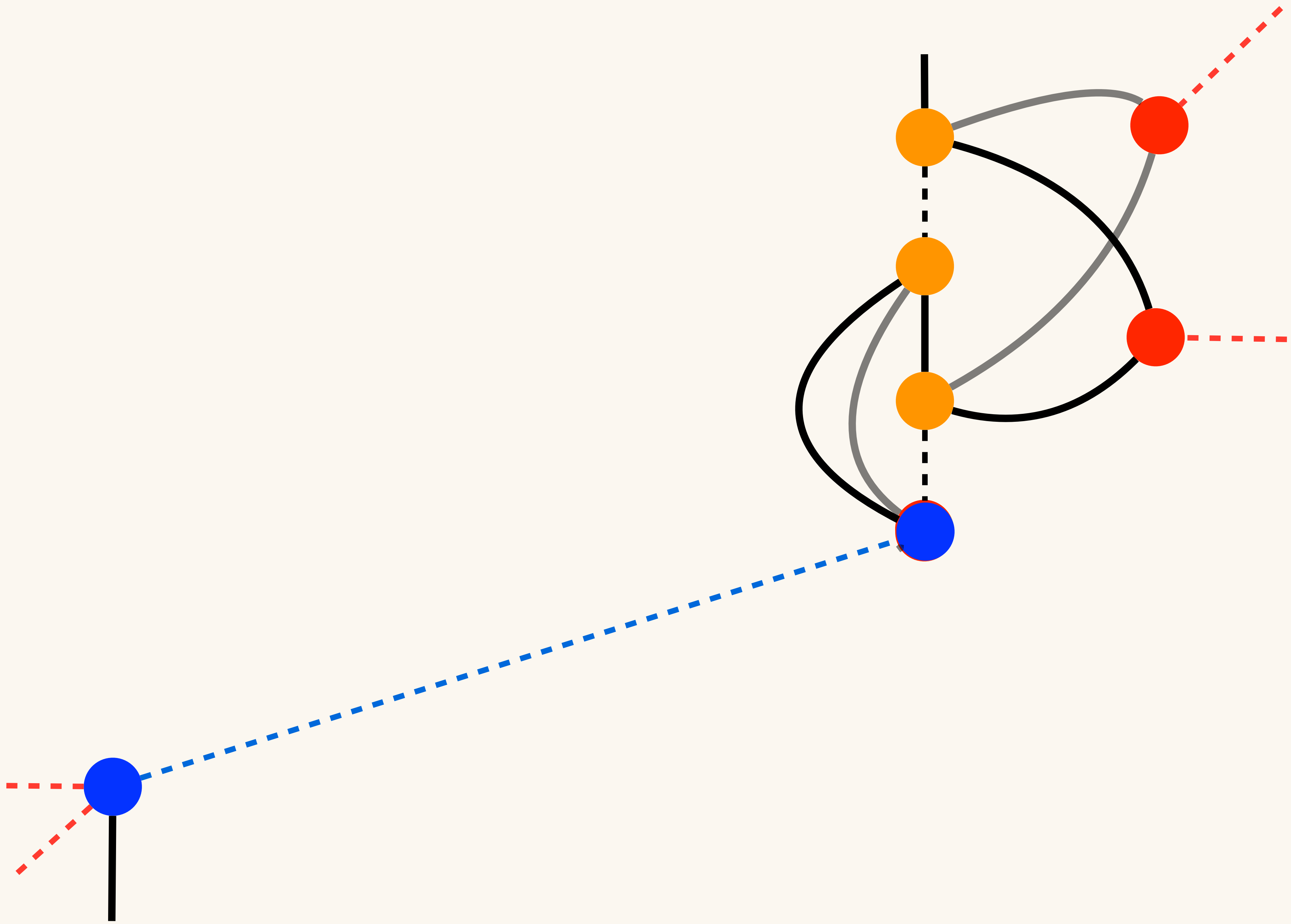


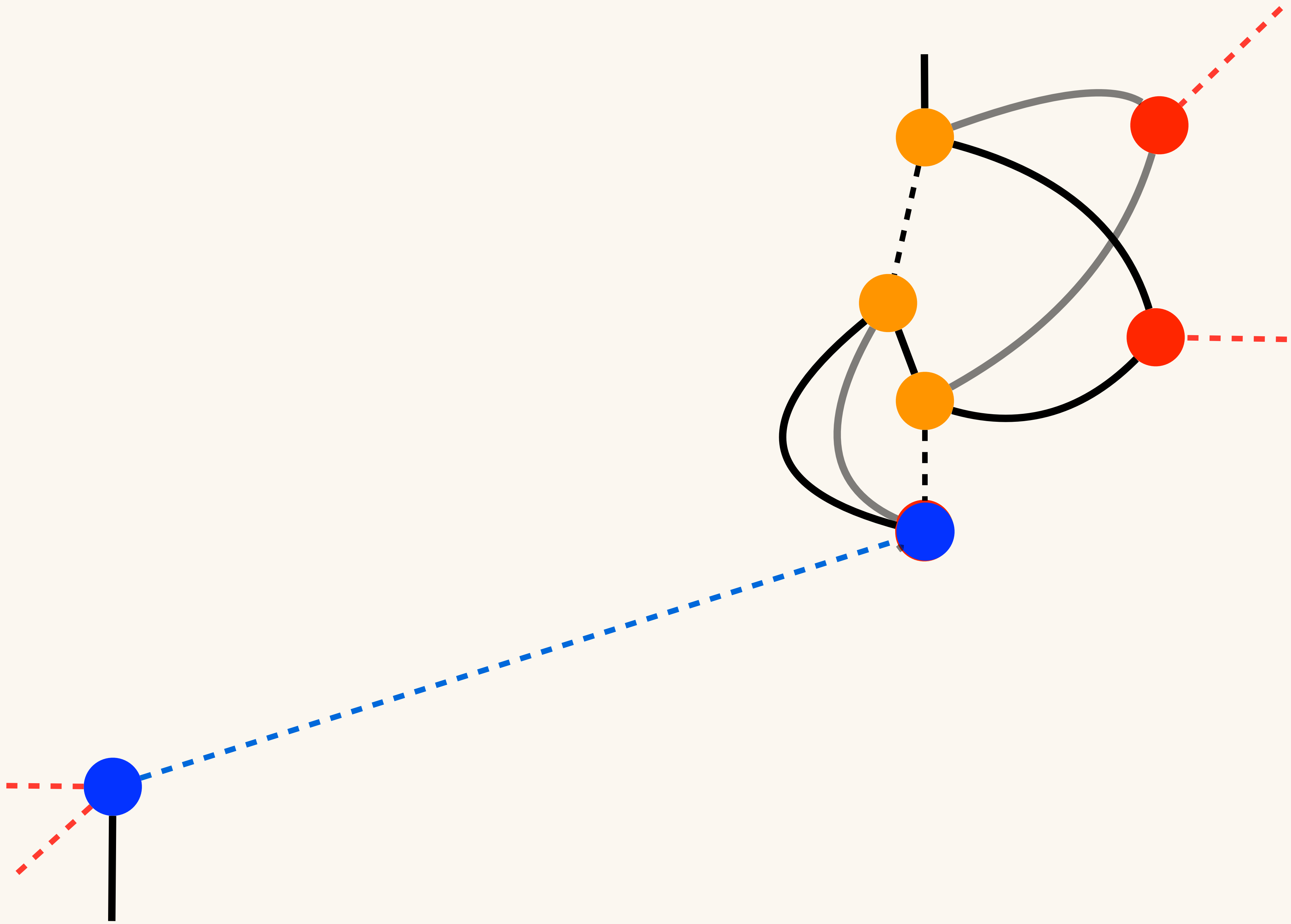


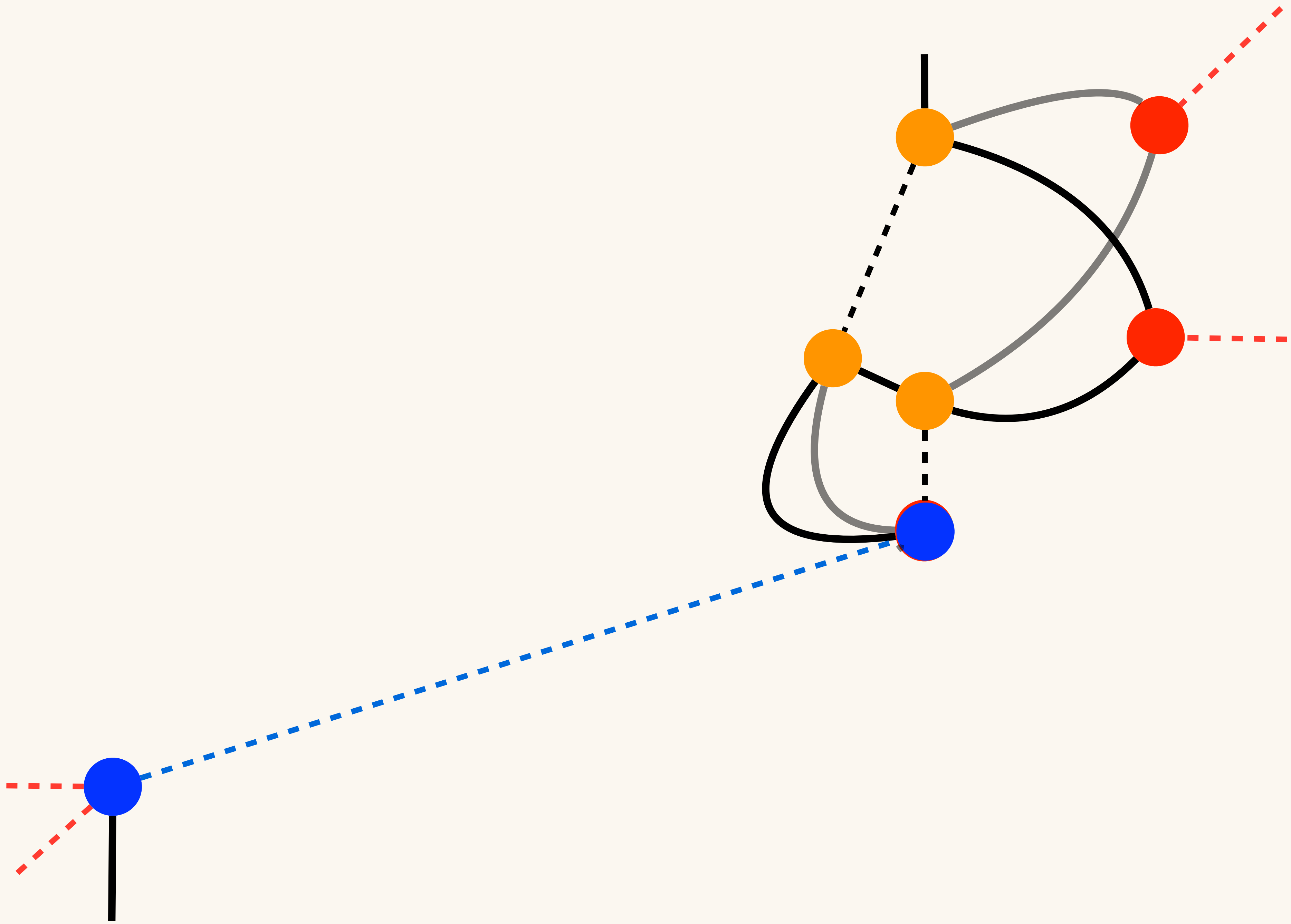


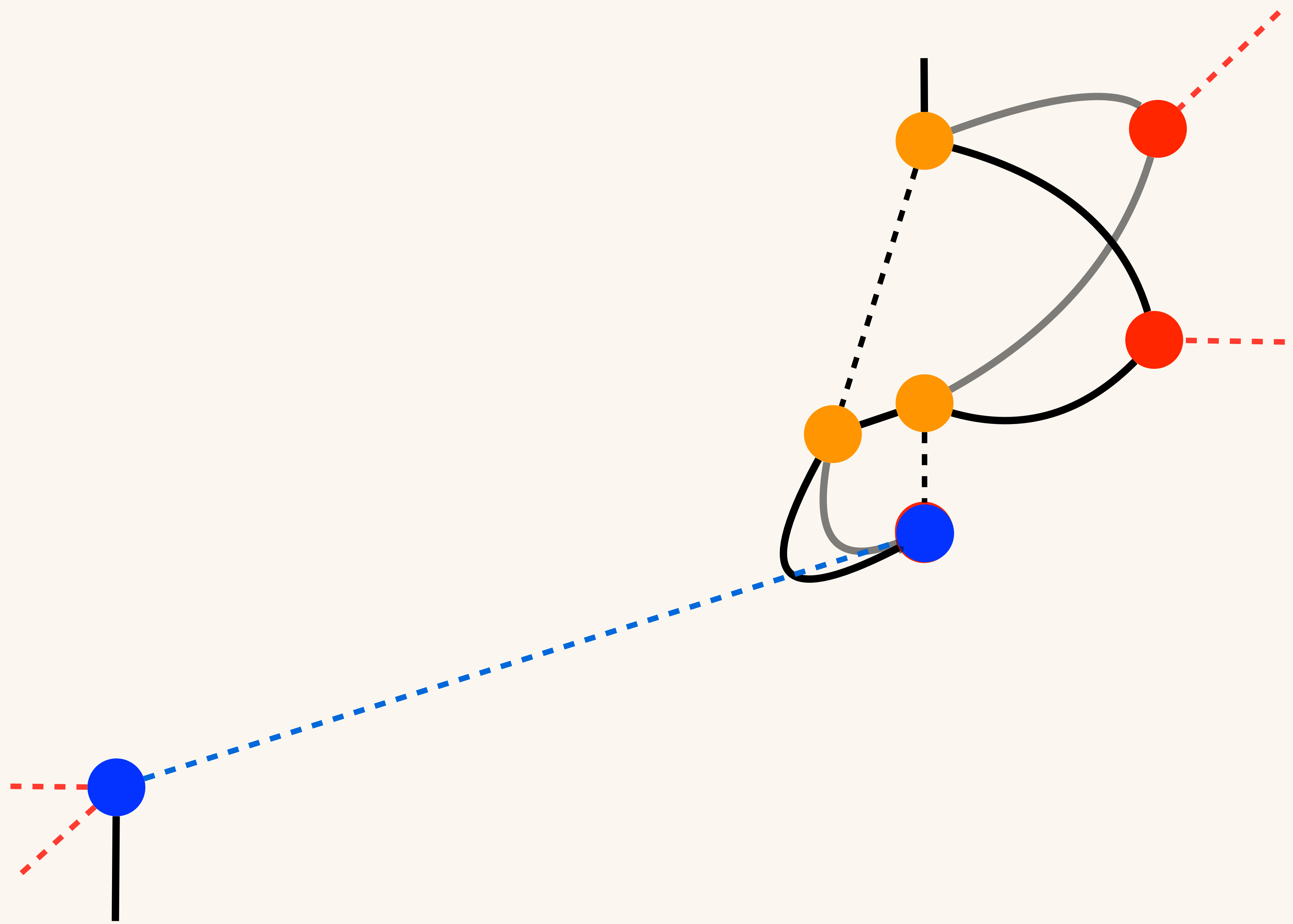


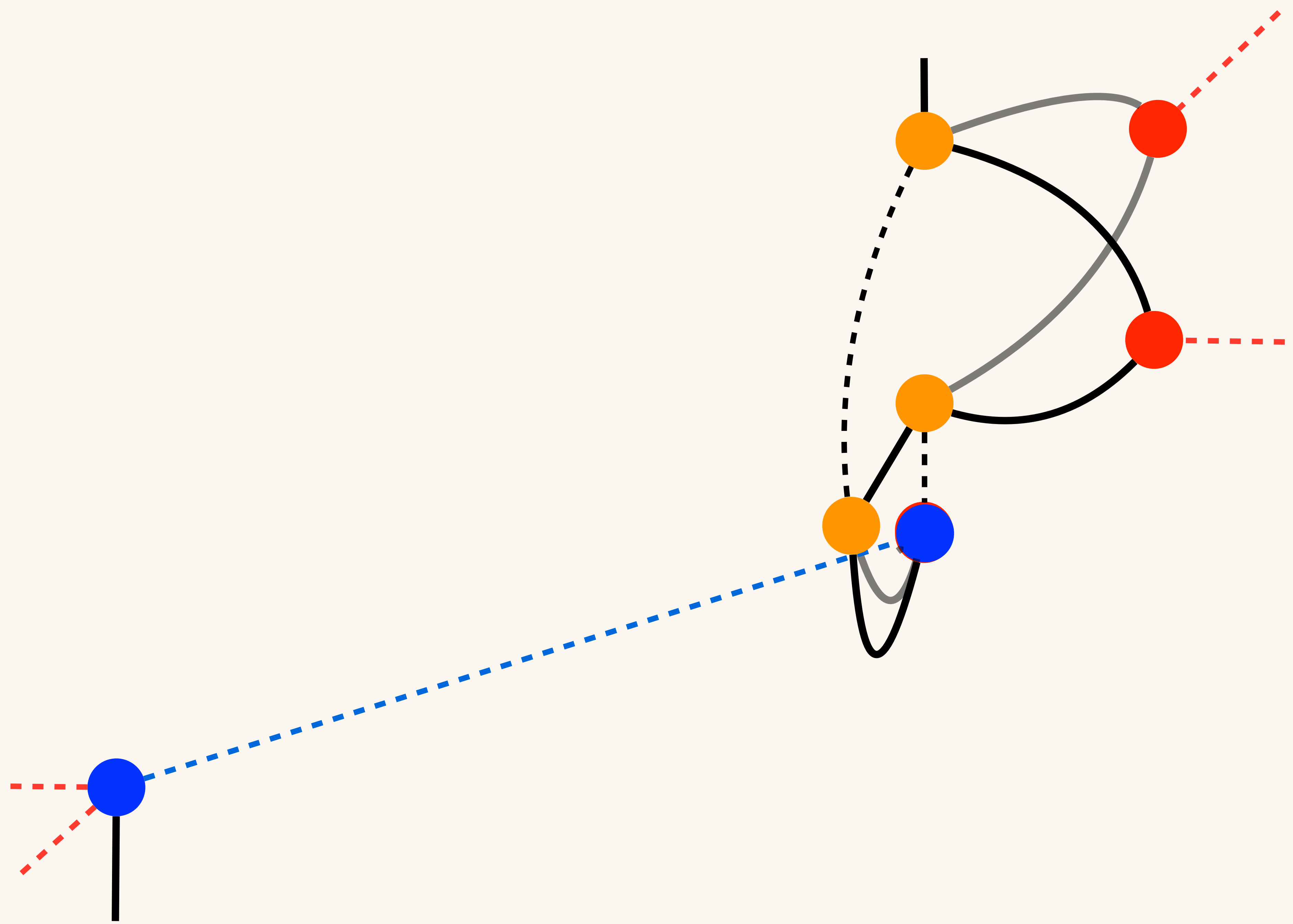


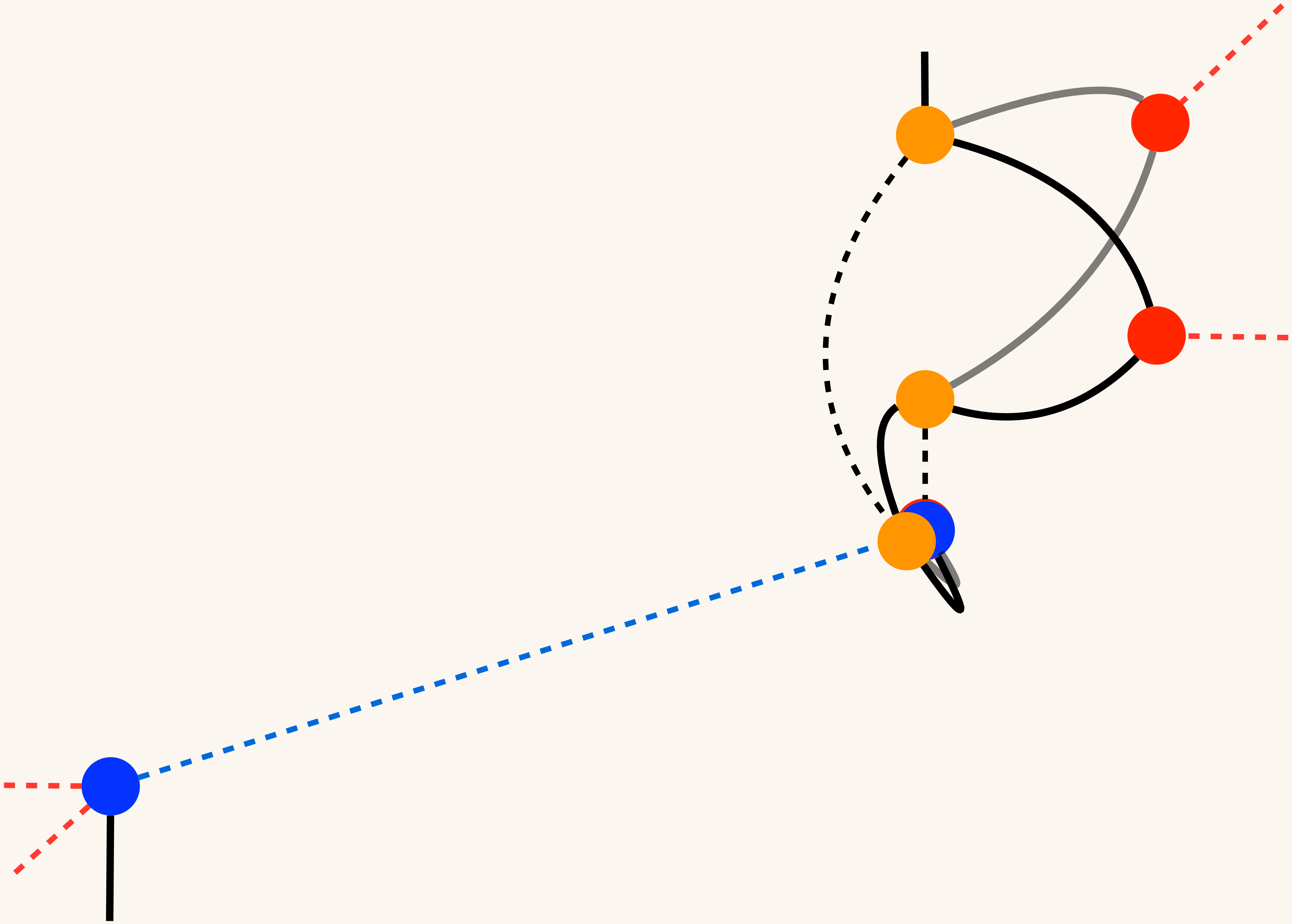


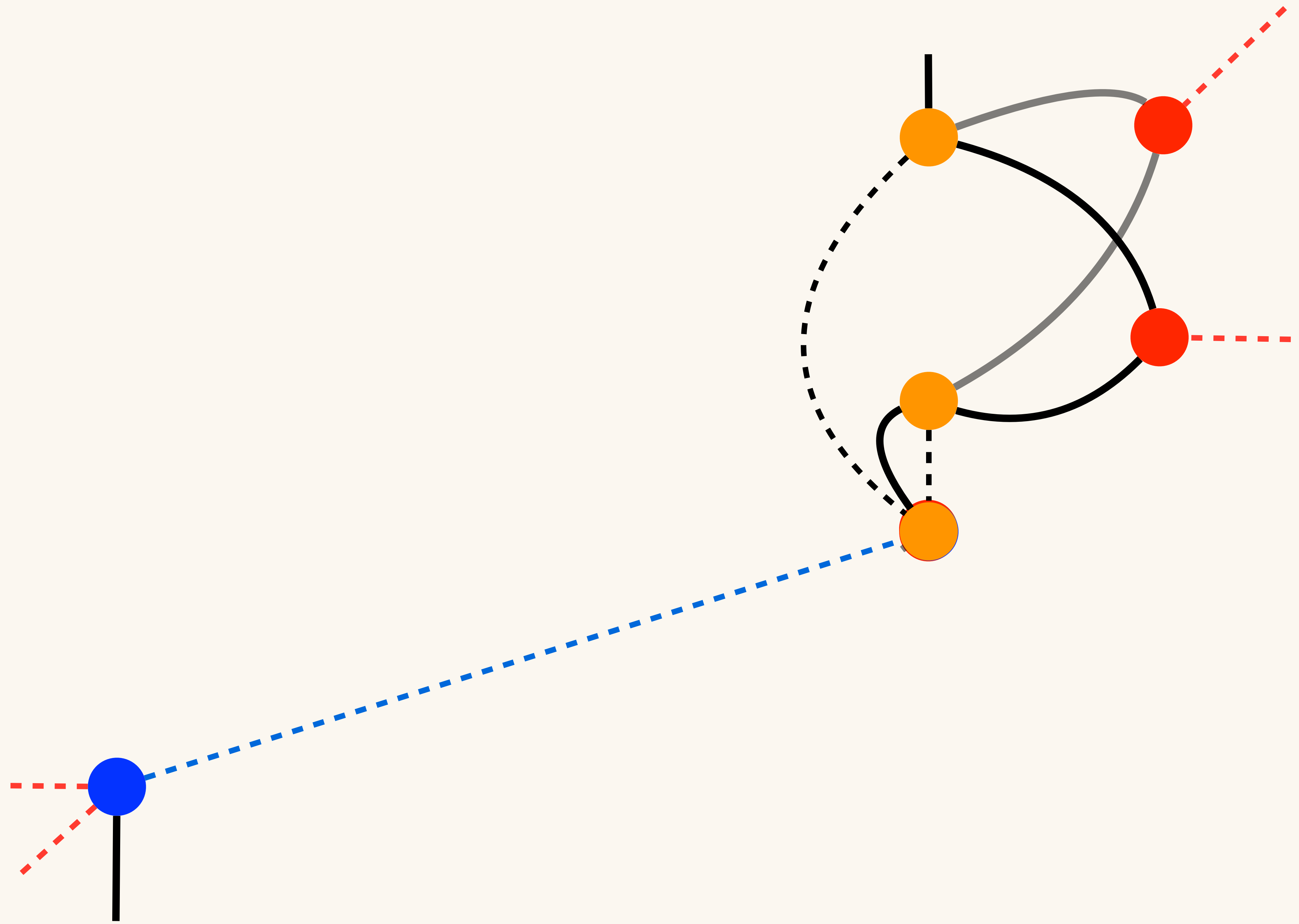


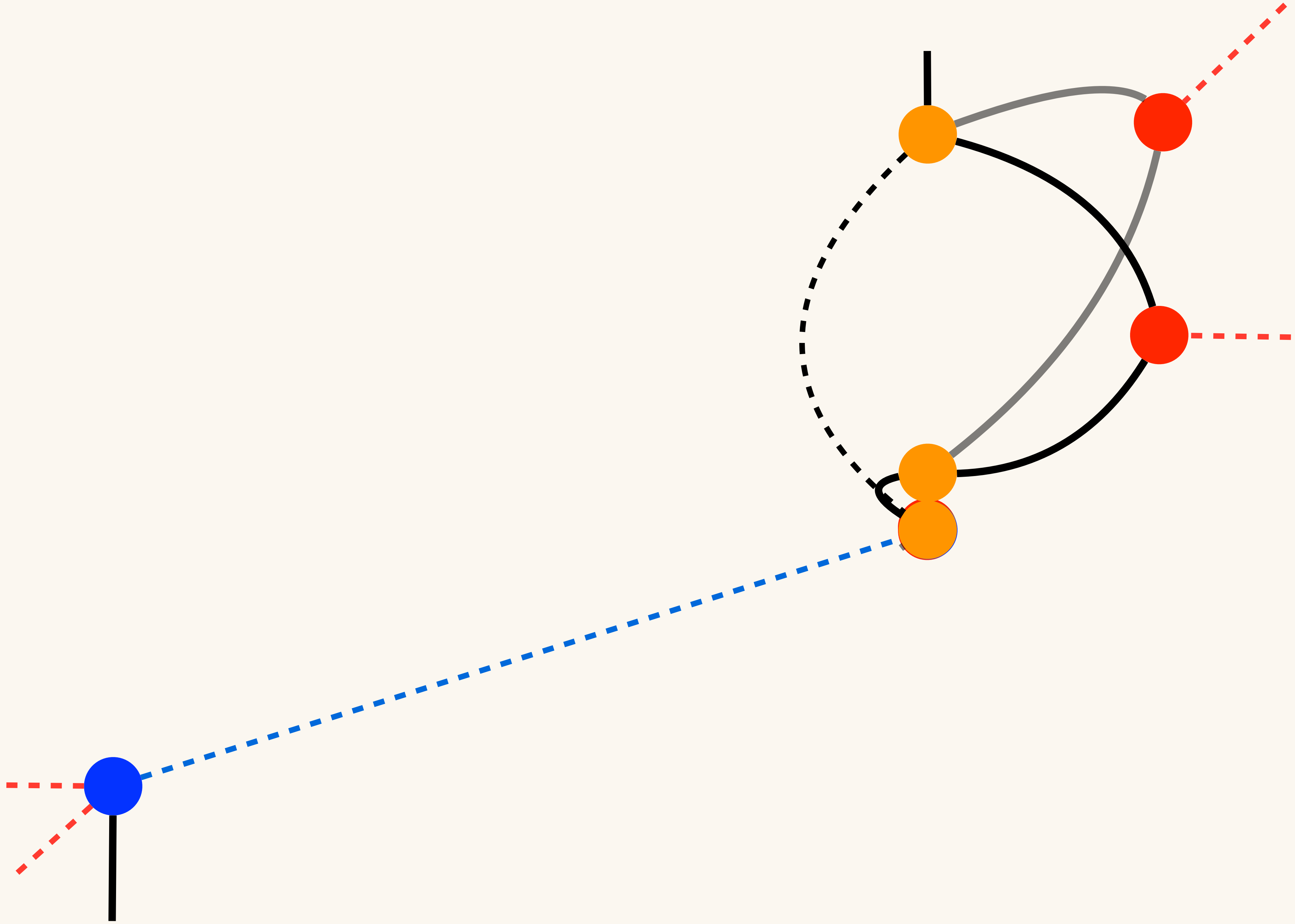


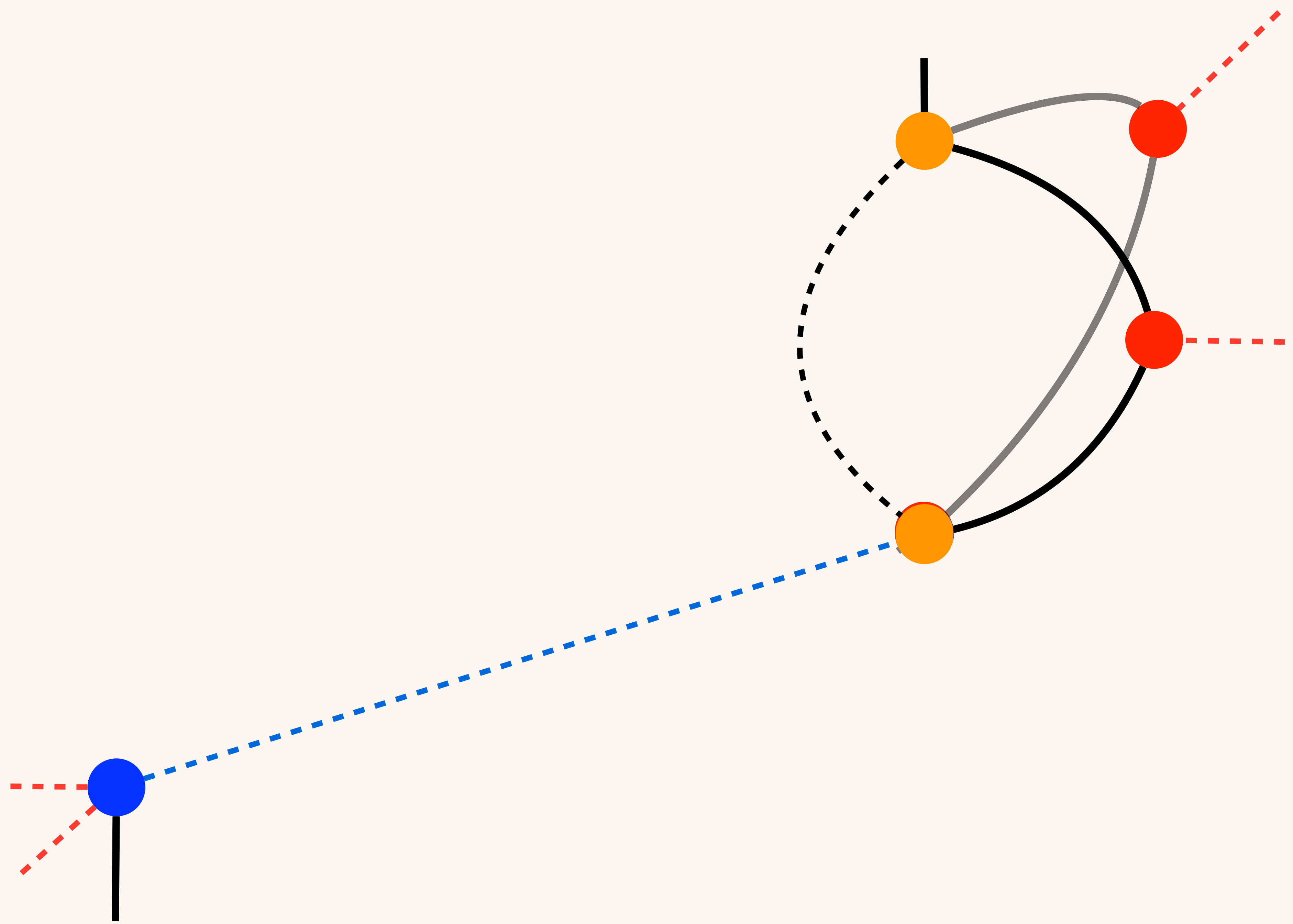


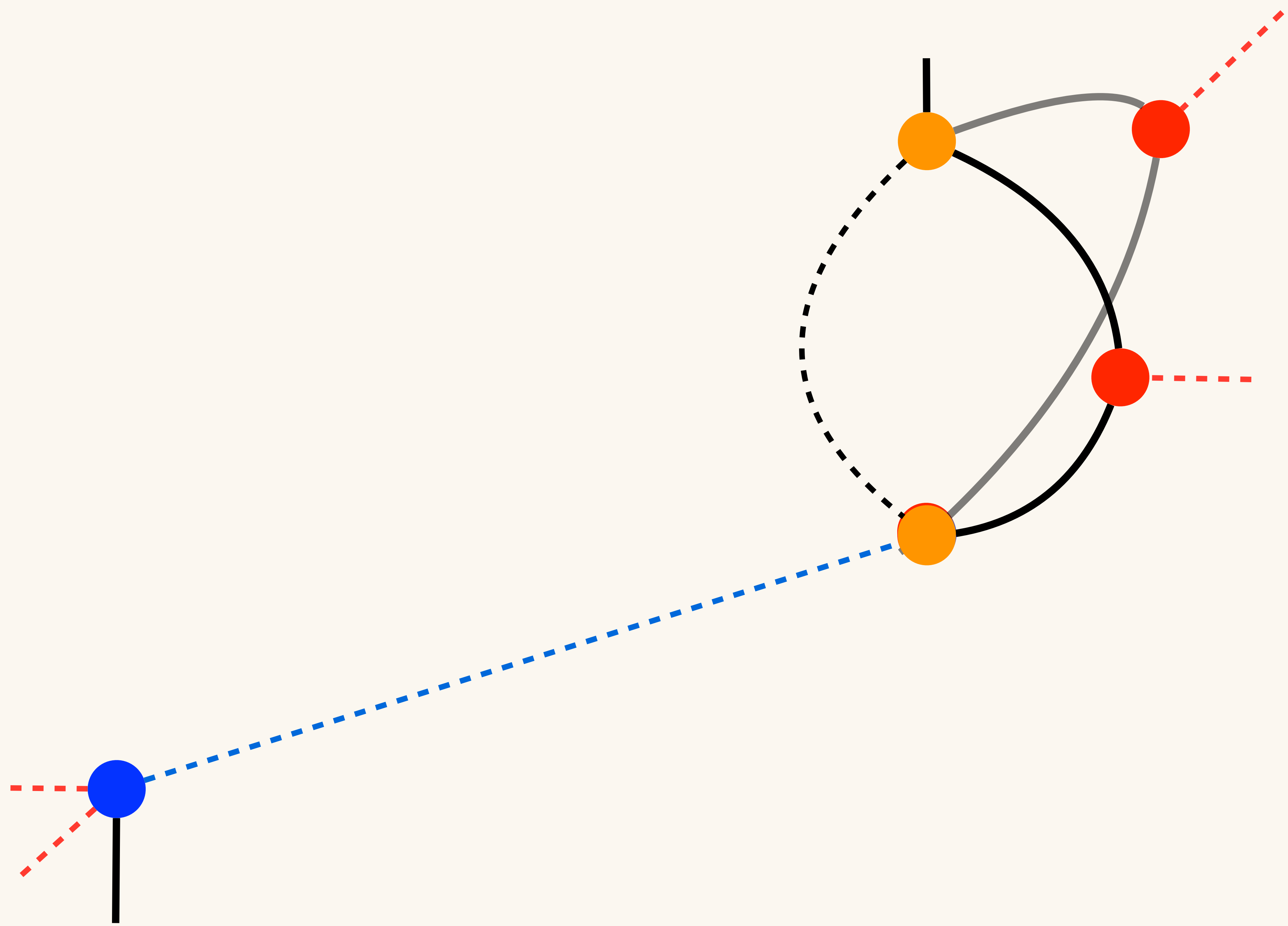


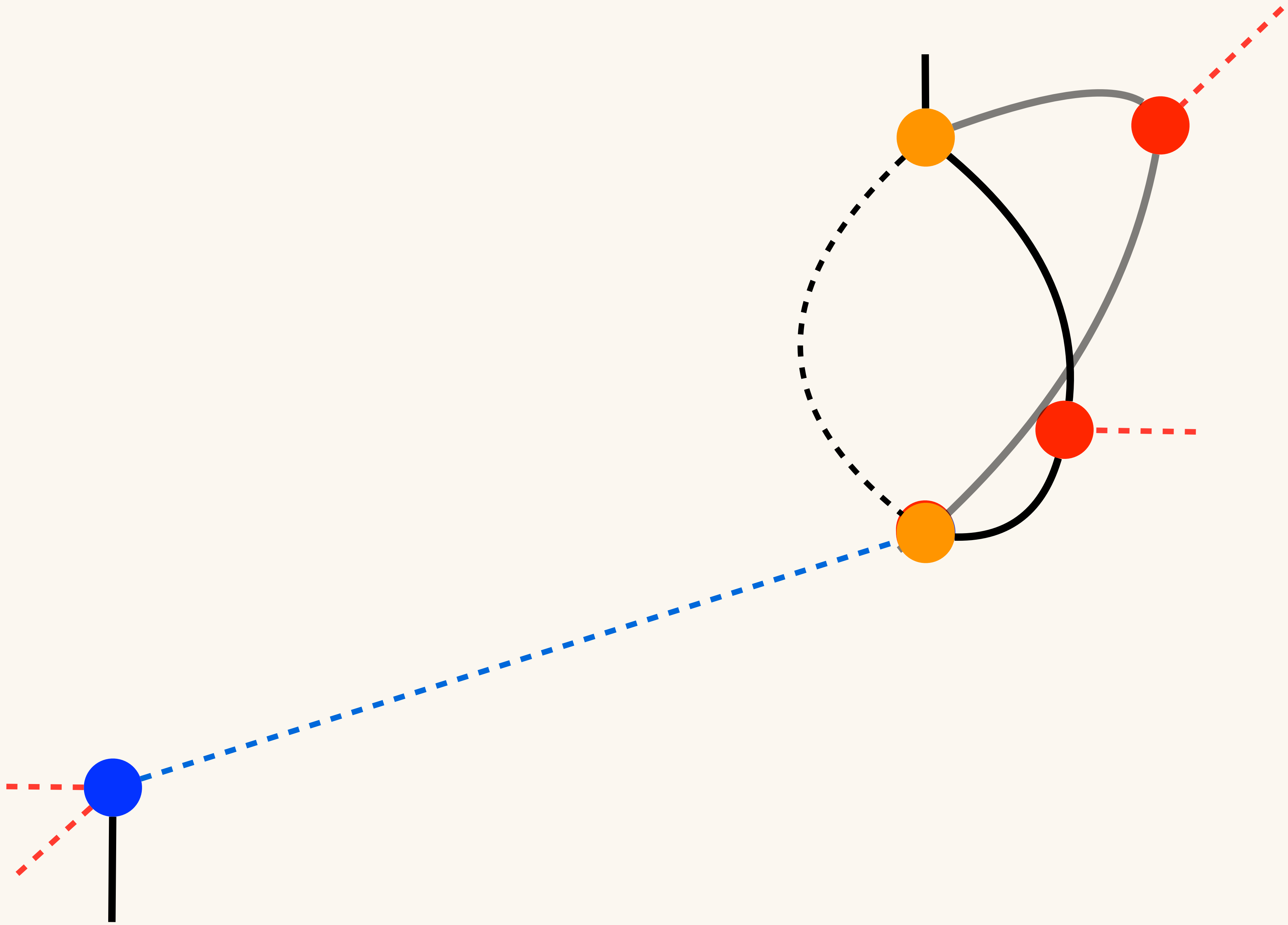


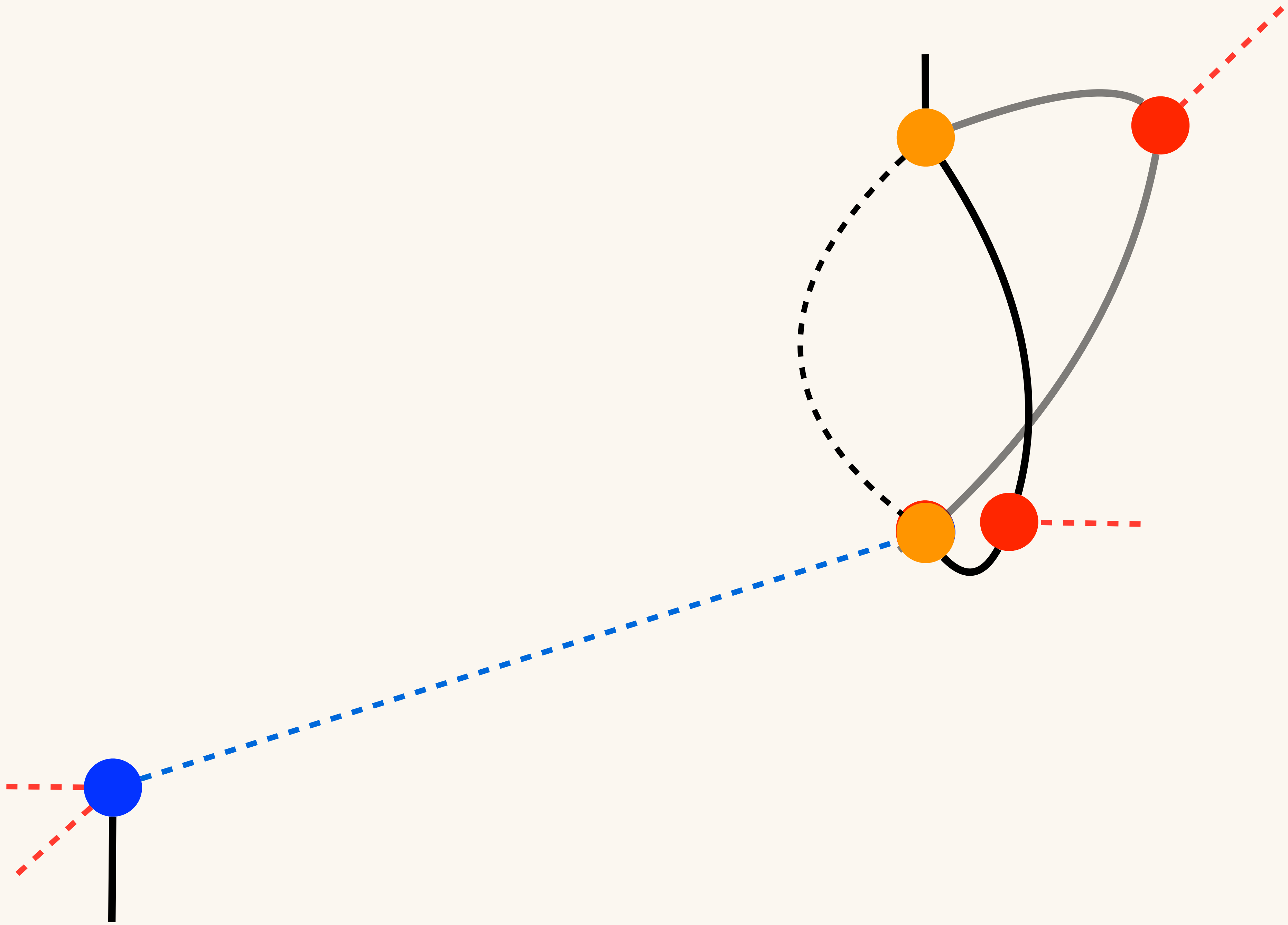


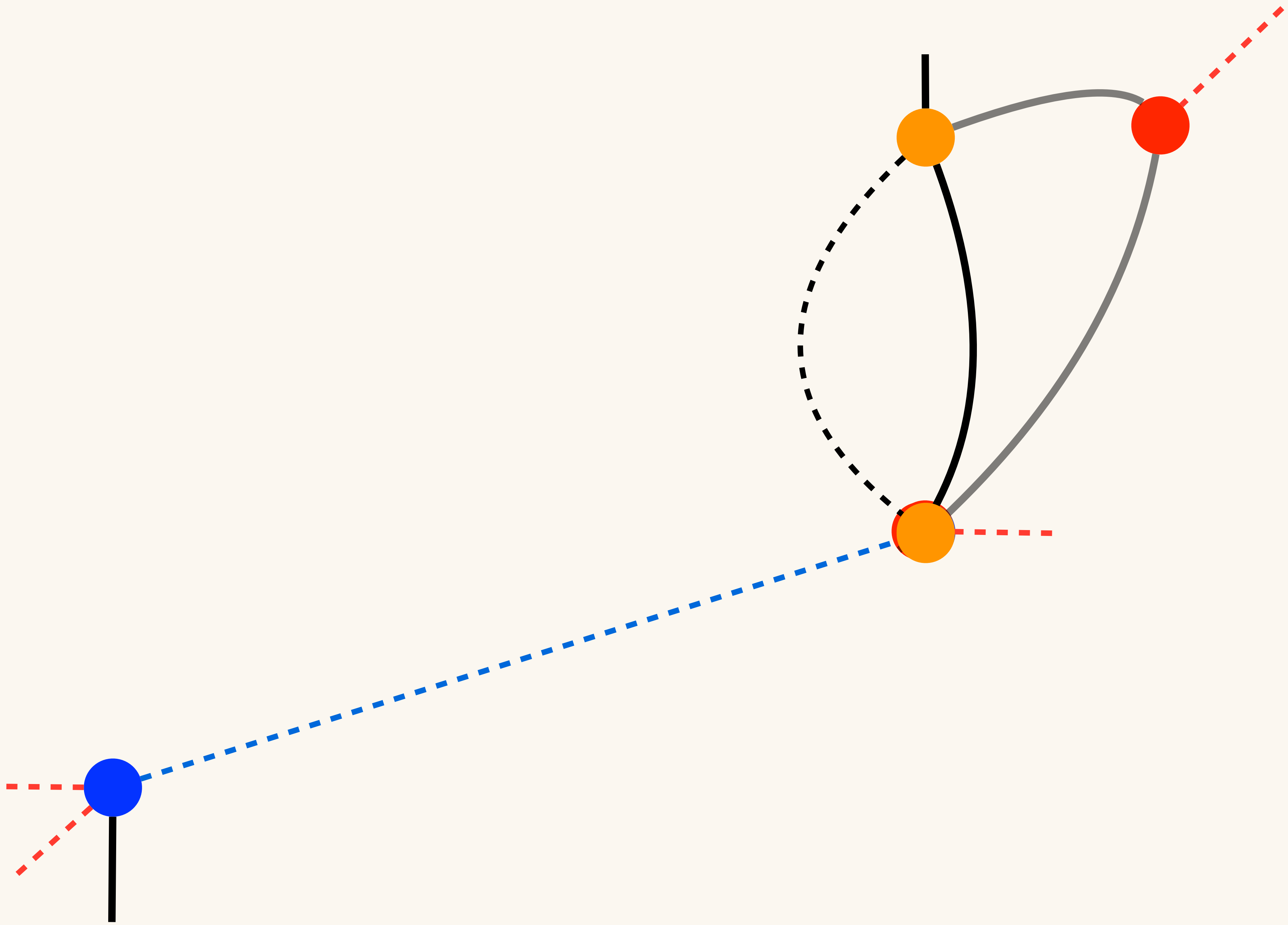


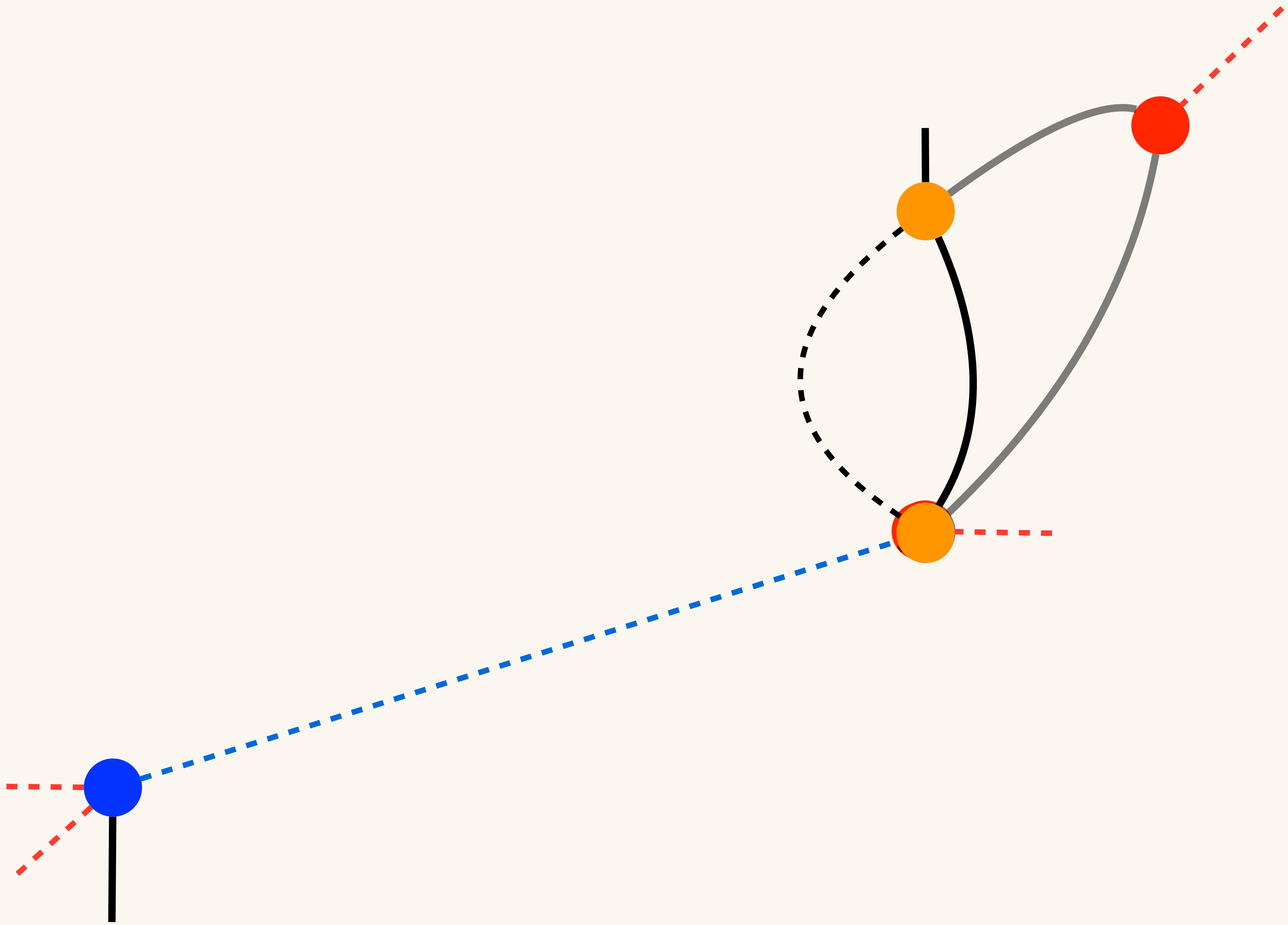


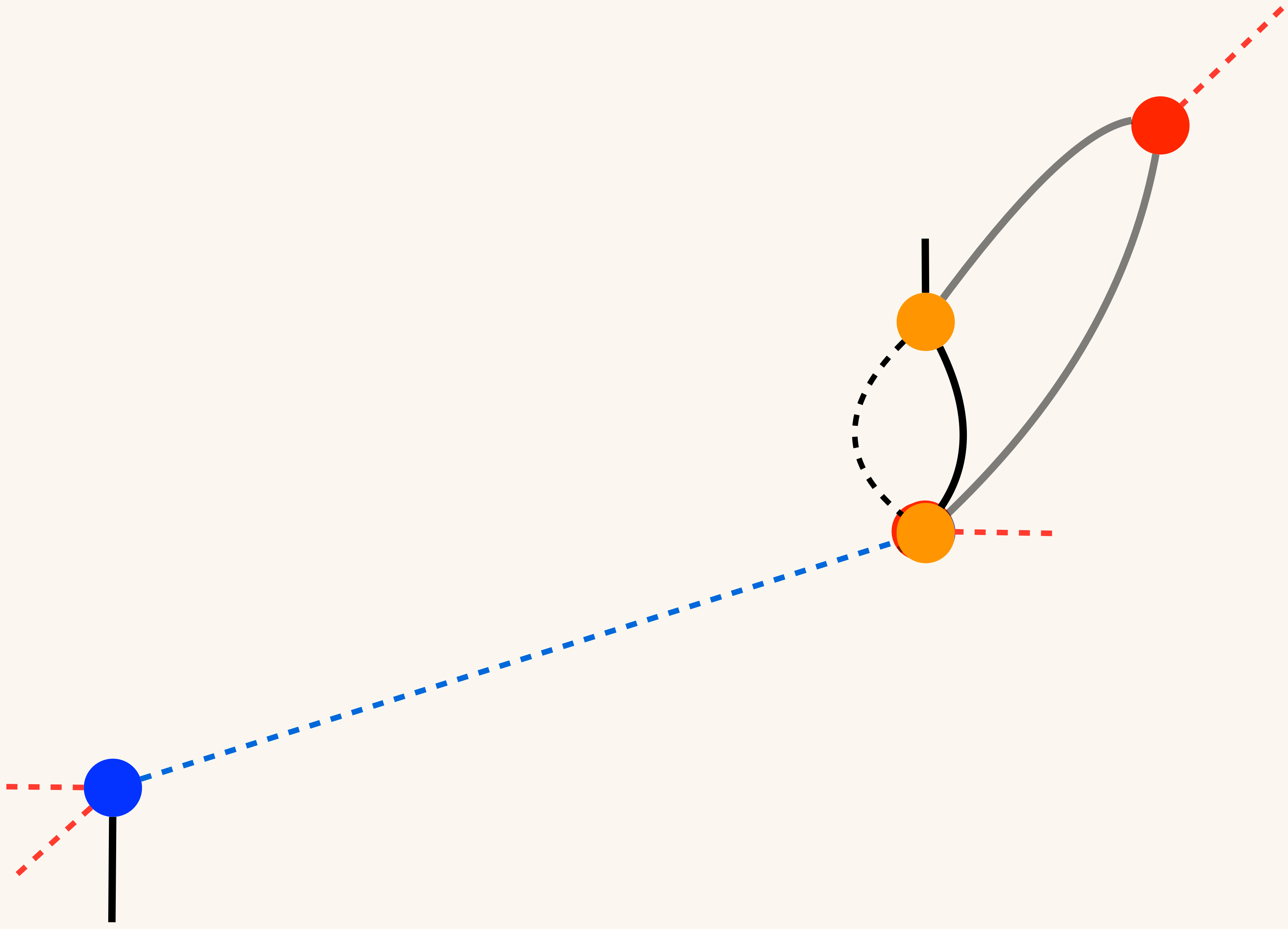


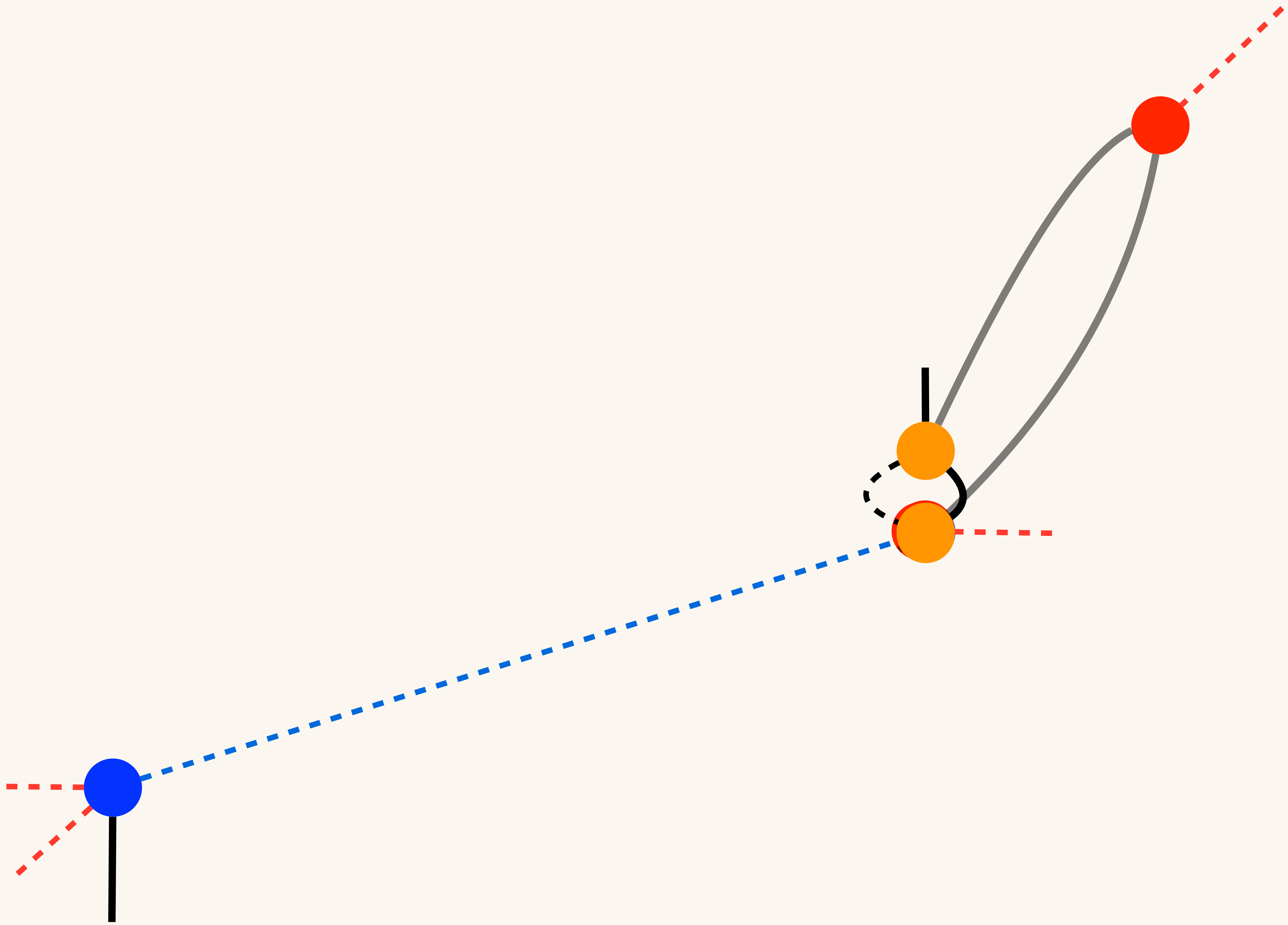


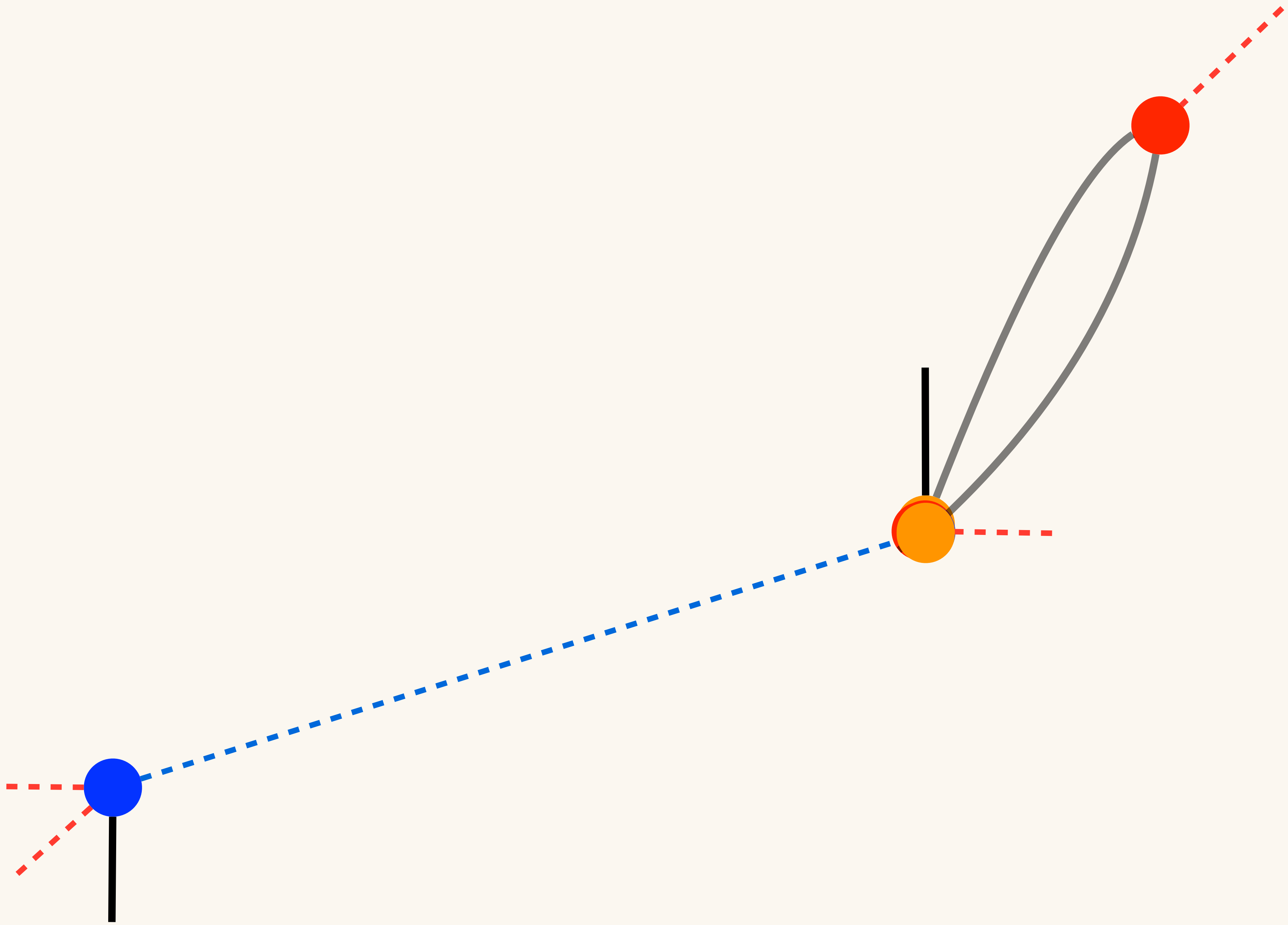


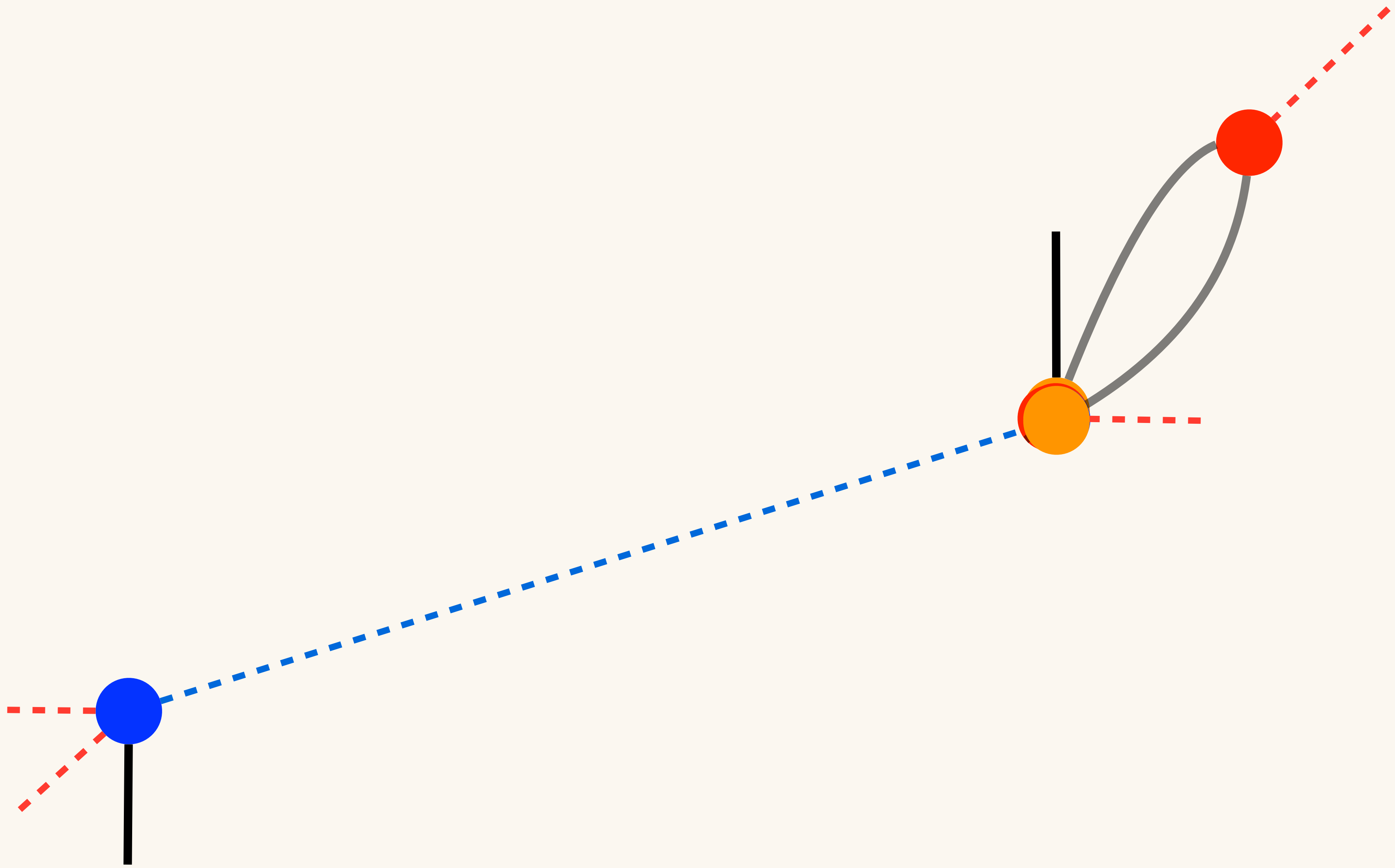


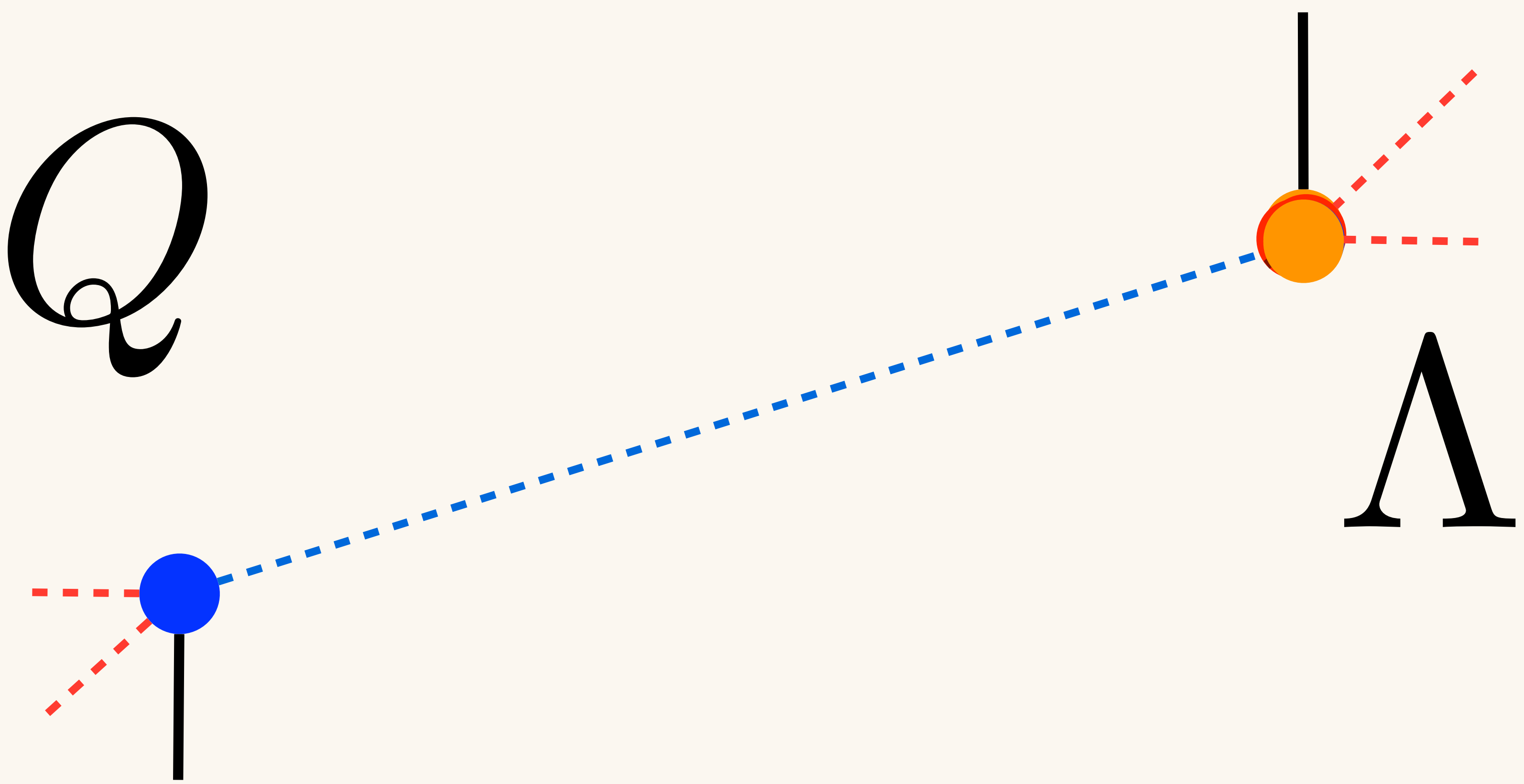






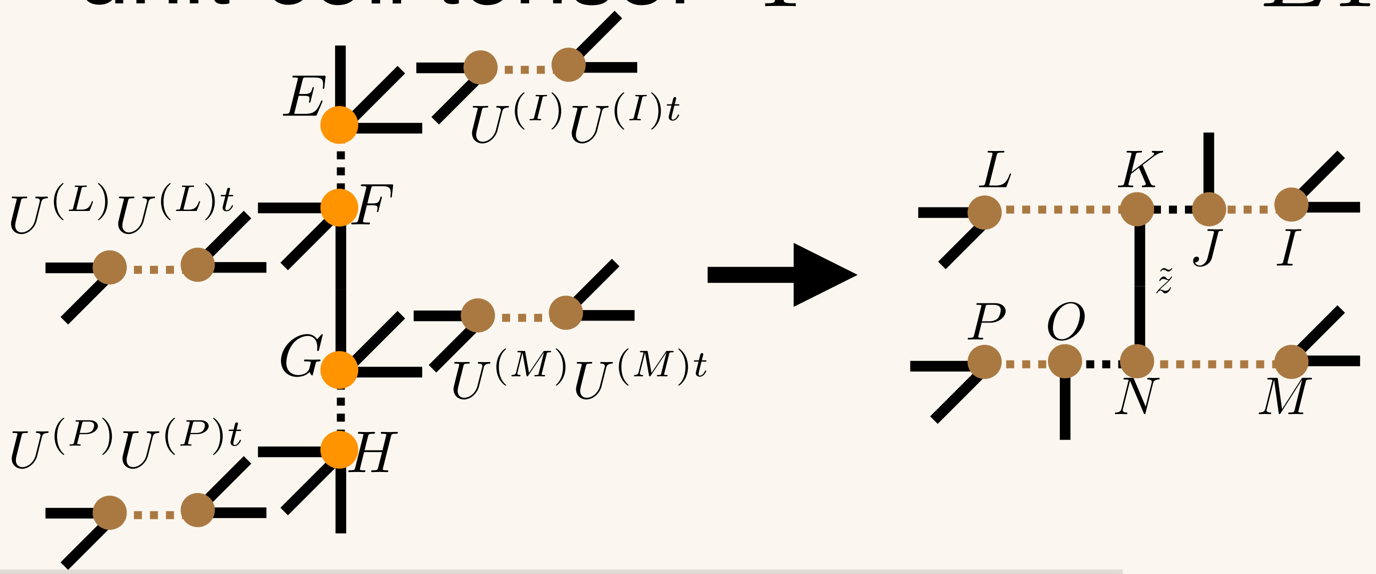


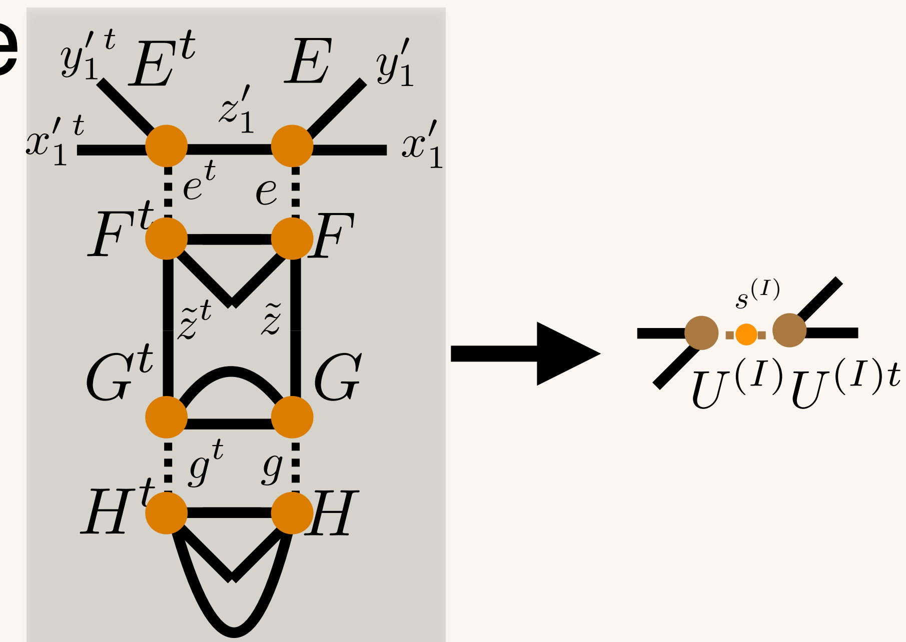



Q
Λ

◉ Minimally-decomposed TRG on triad rep.

◇ We introduce triad rep. by SVD of the unit-cell tensor $\Gamma^{(EFGH)} = EFGH$.





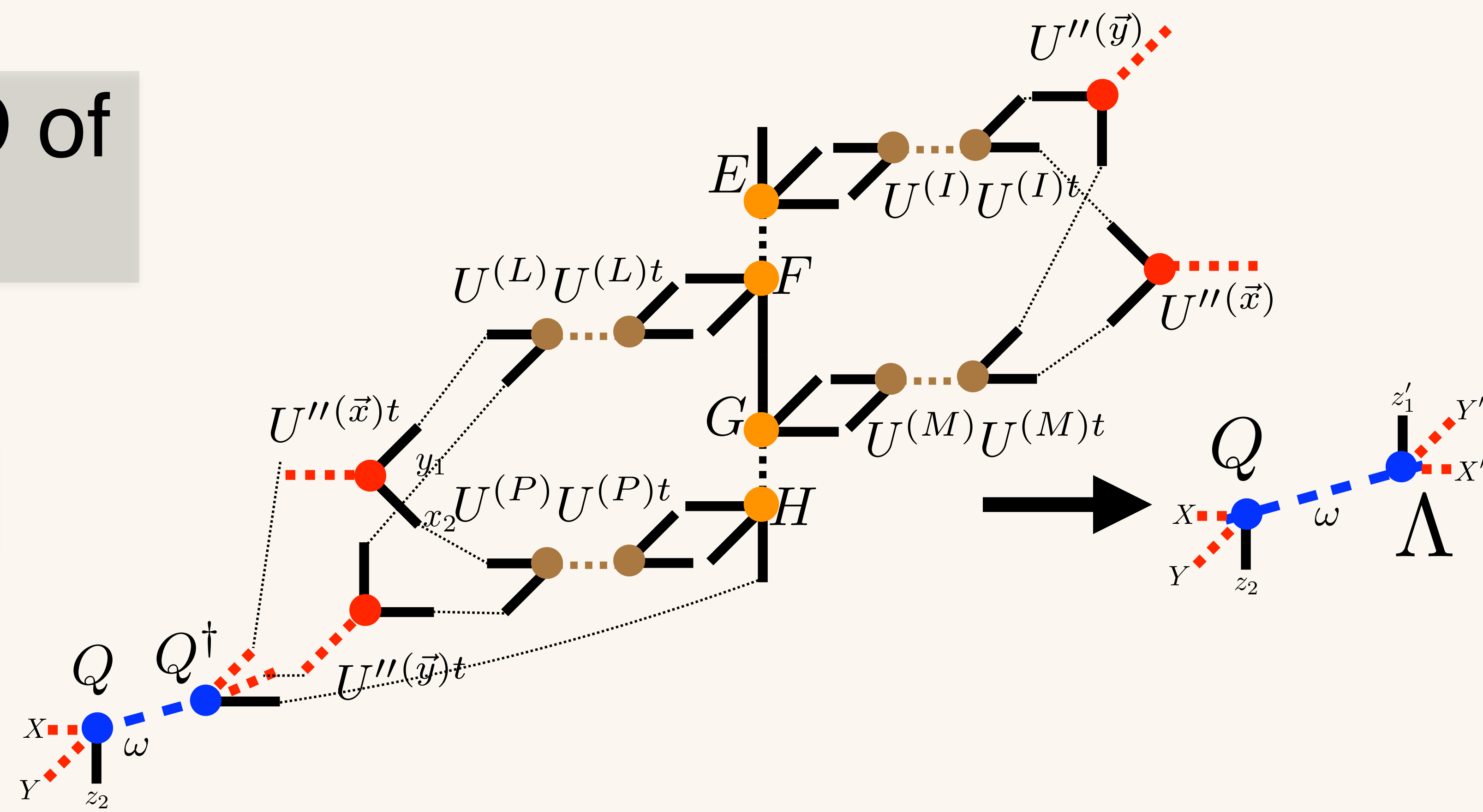
◇ We do NOT use SVD of E, F, G and H.

$$E = U^{(0)} s^{(0)} V^{(0)}$$

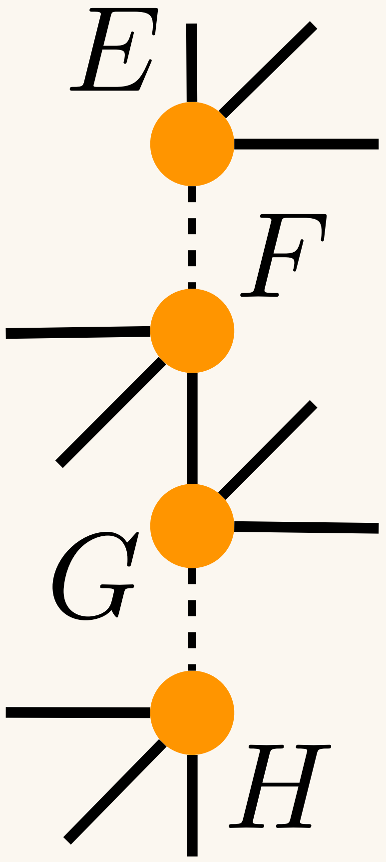
◇ We use SVD of $\Gamma\Gamma^t$.

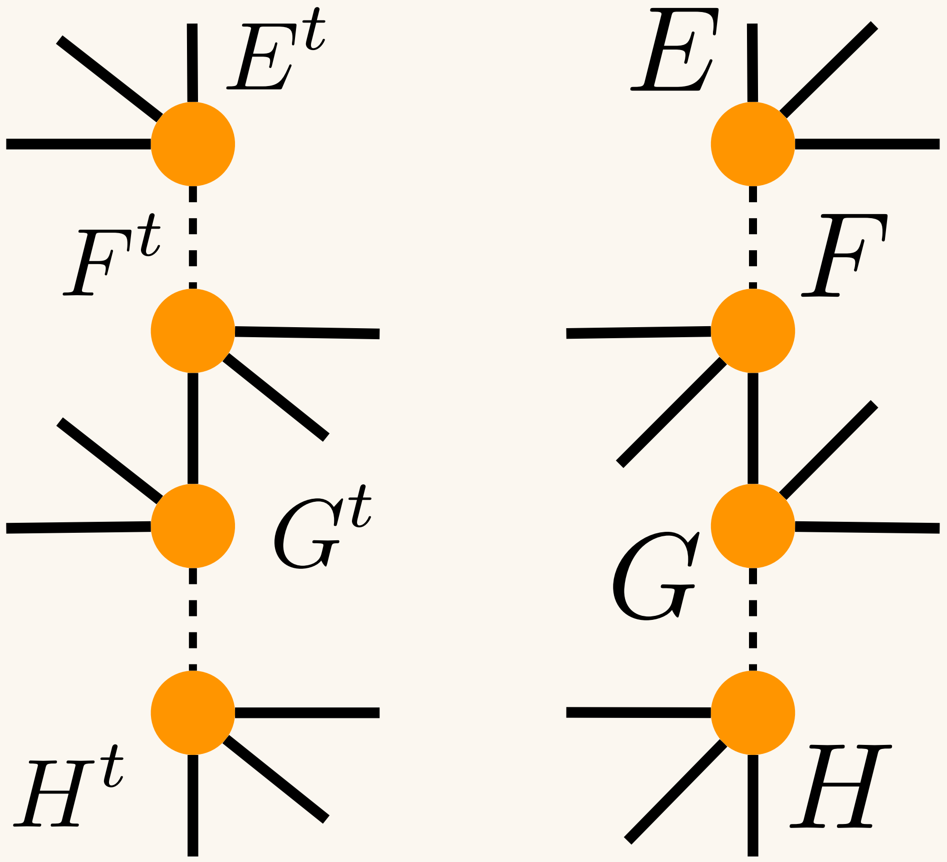
◇ Cost reduction

$$O(D^{2d+1}) \to O(D^{d+3})$$

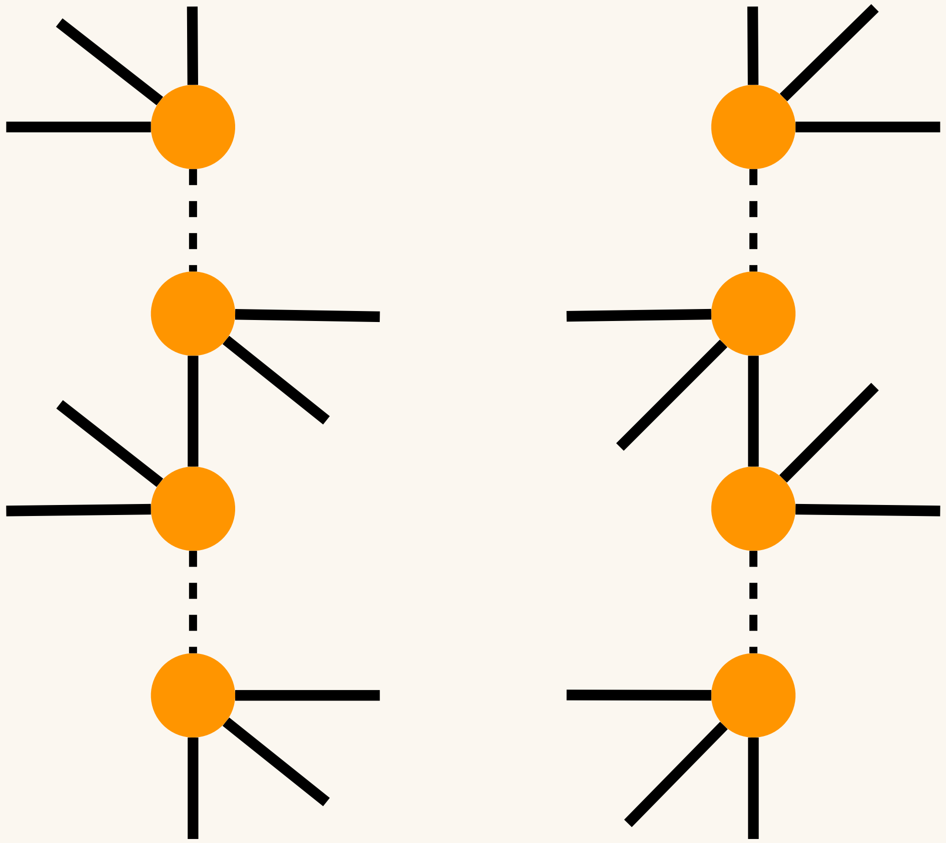


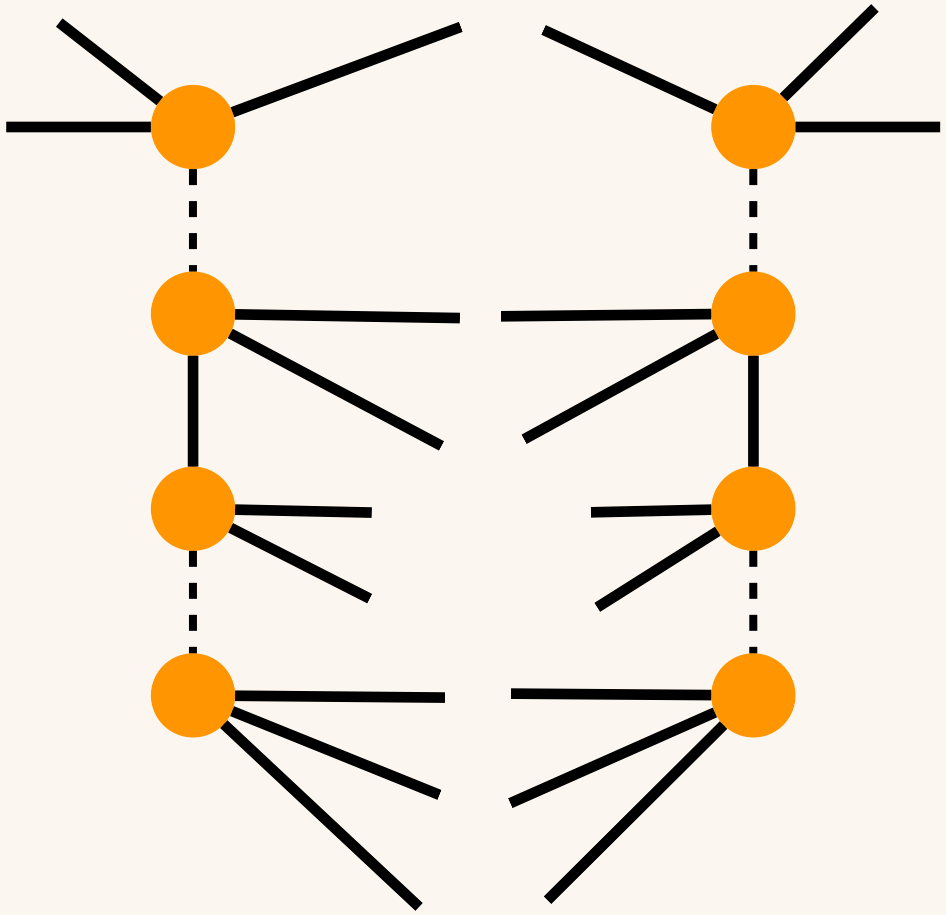
$$QQ^\dagger U''^{(\vec{y})t} U''^{(\vec{x})t} U^{(LP)} U^{(LP)t} EFGH U^{(IM)} U^{(IM)t} U''^{(\vec{x})} U''^{(\vec{y})} = Q\Lambda \simeq A^{(\text{next})}$$

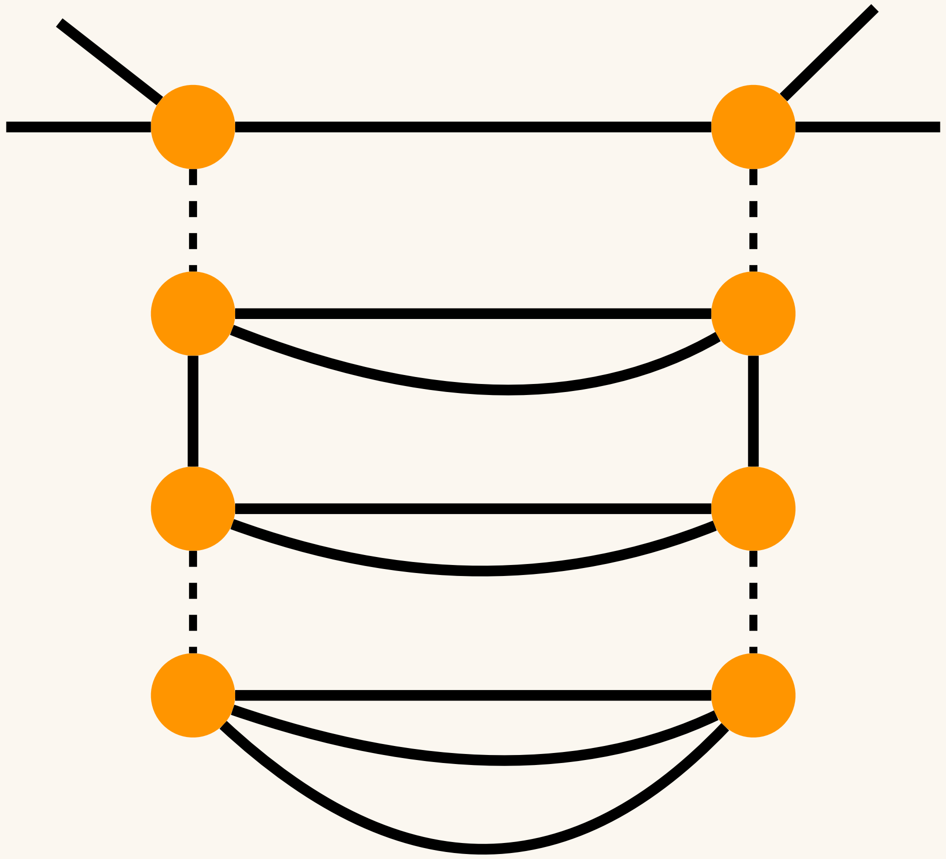

E
F
G
H

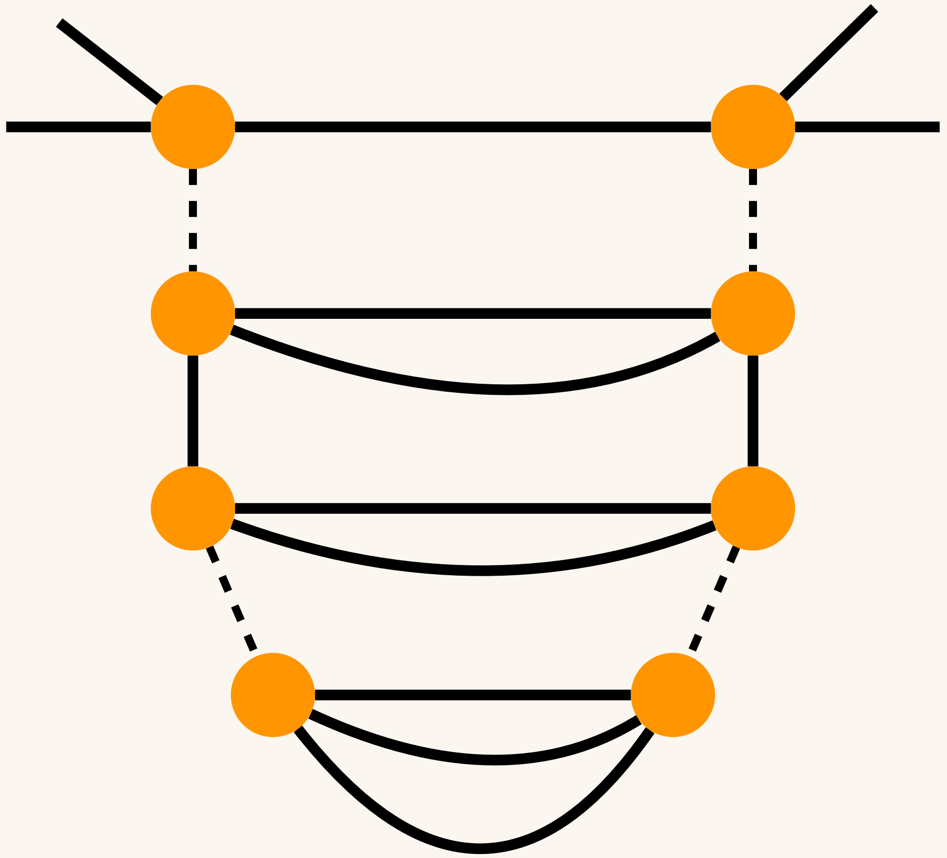


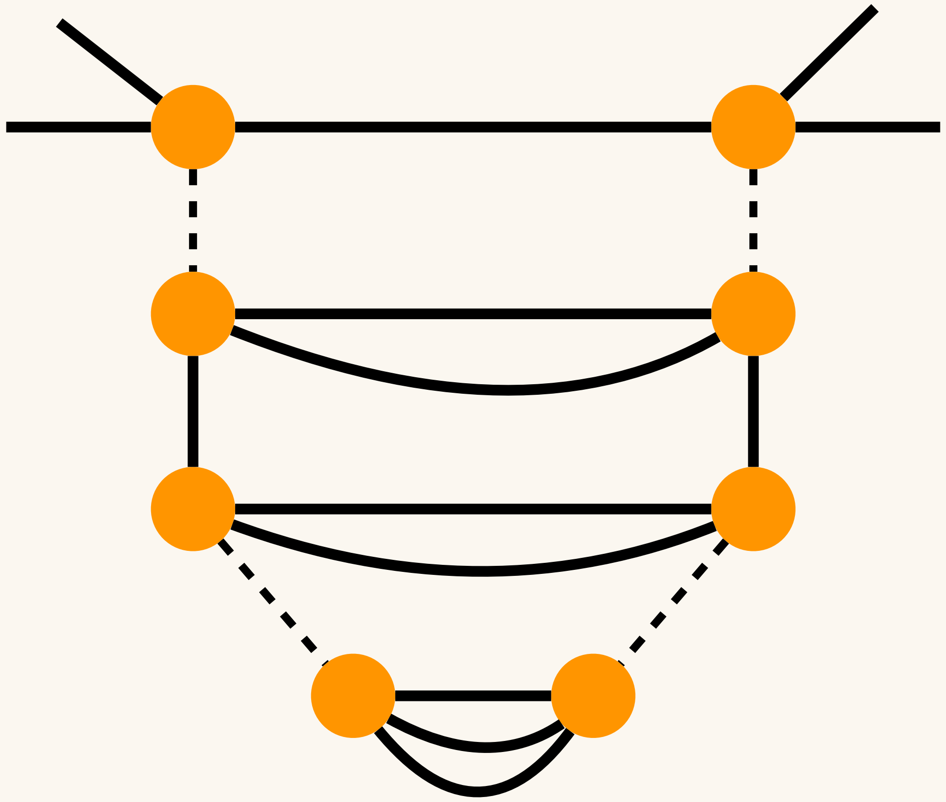
E^t
F^t
G^t
H^t
E
F
G
H

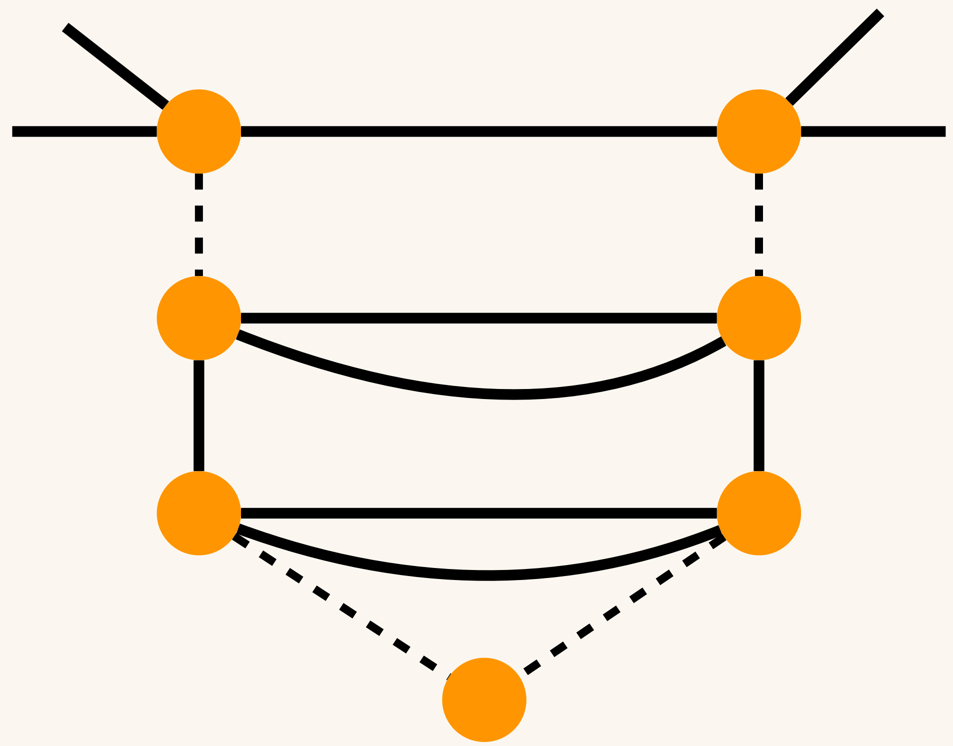


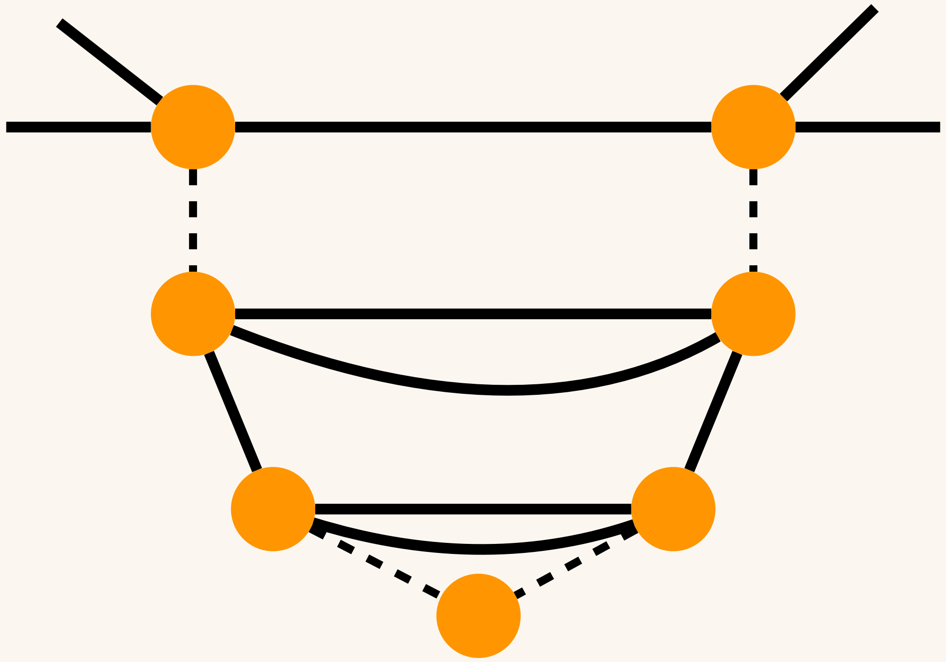


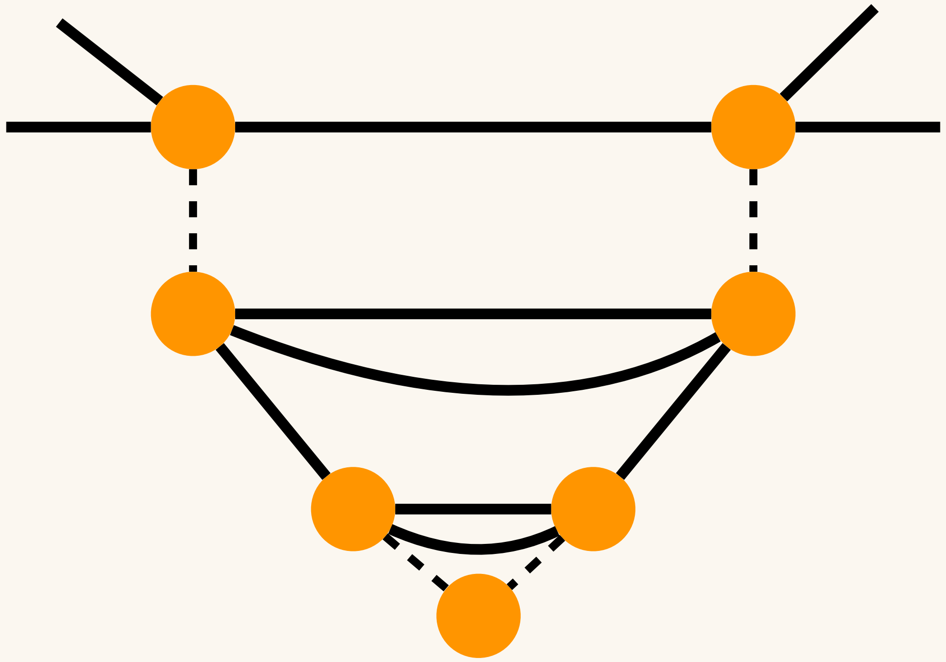


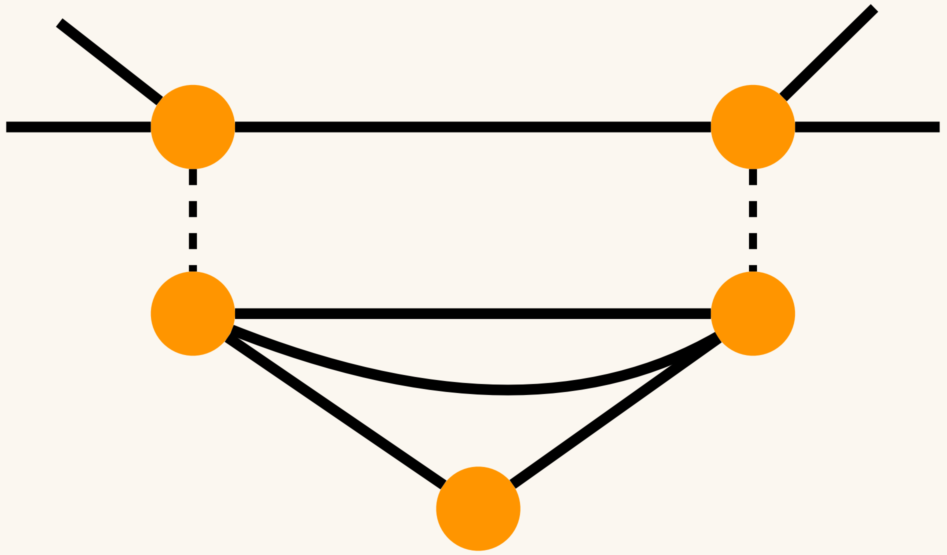


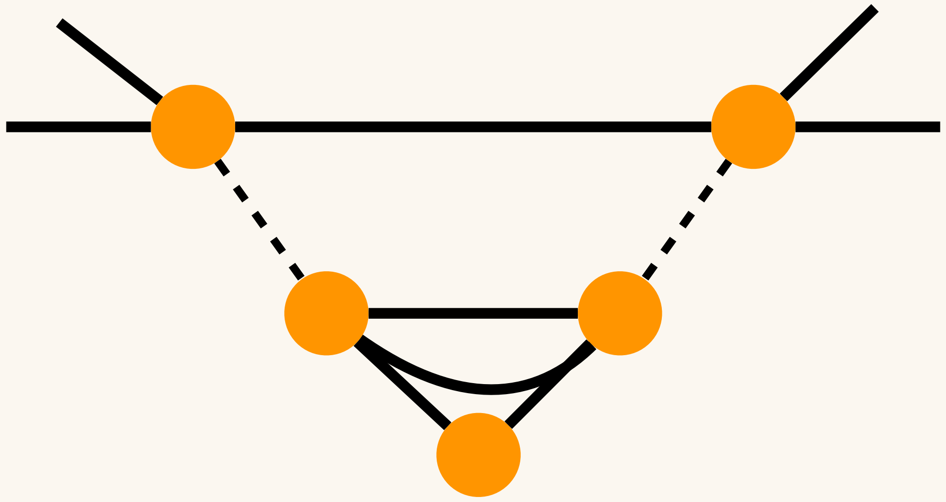


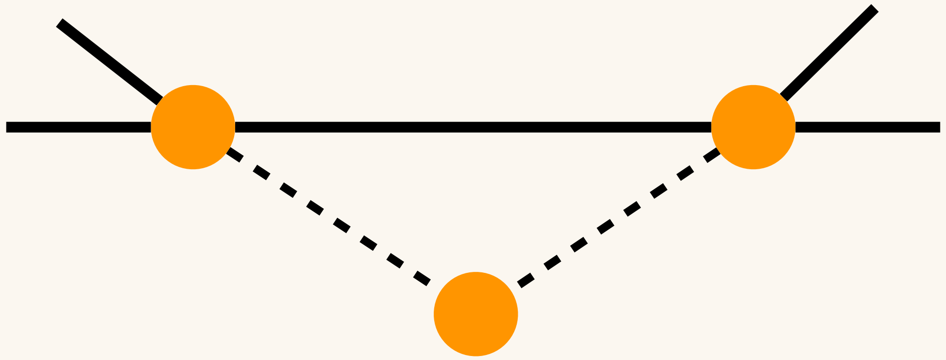


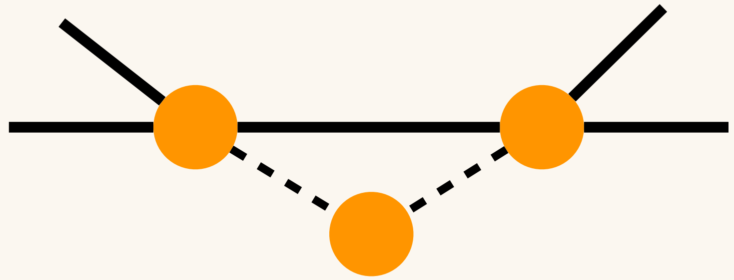


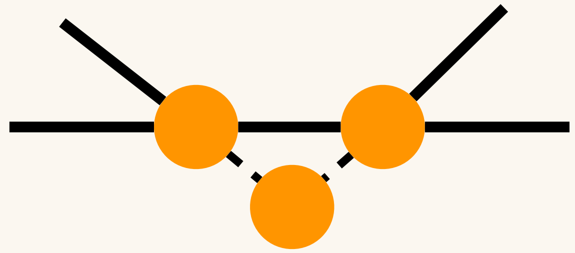


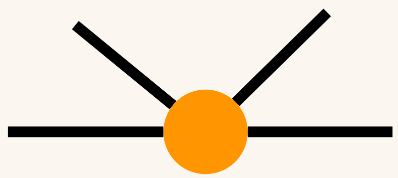


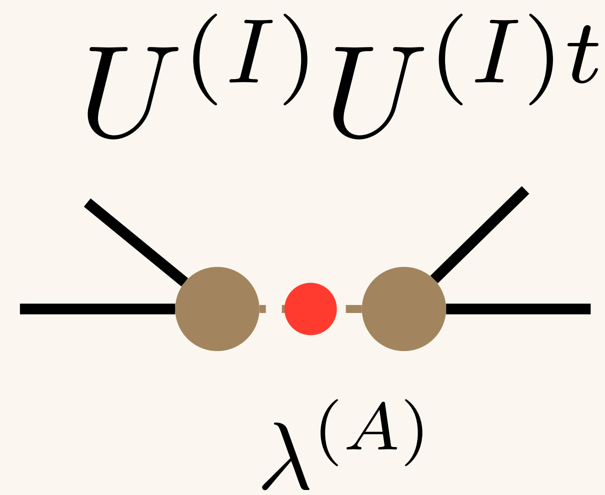


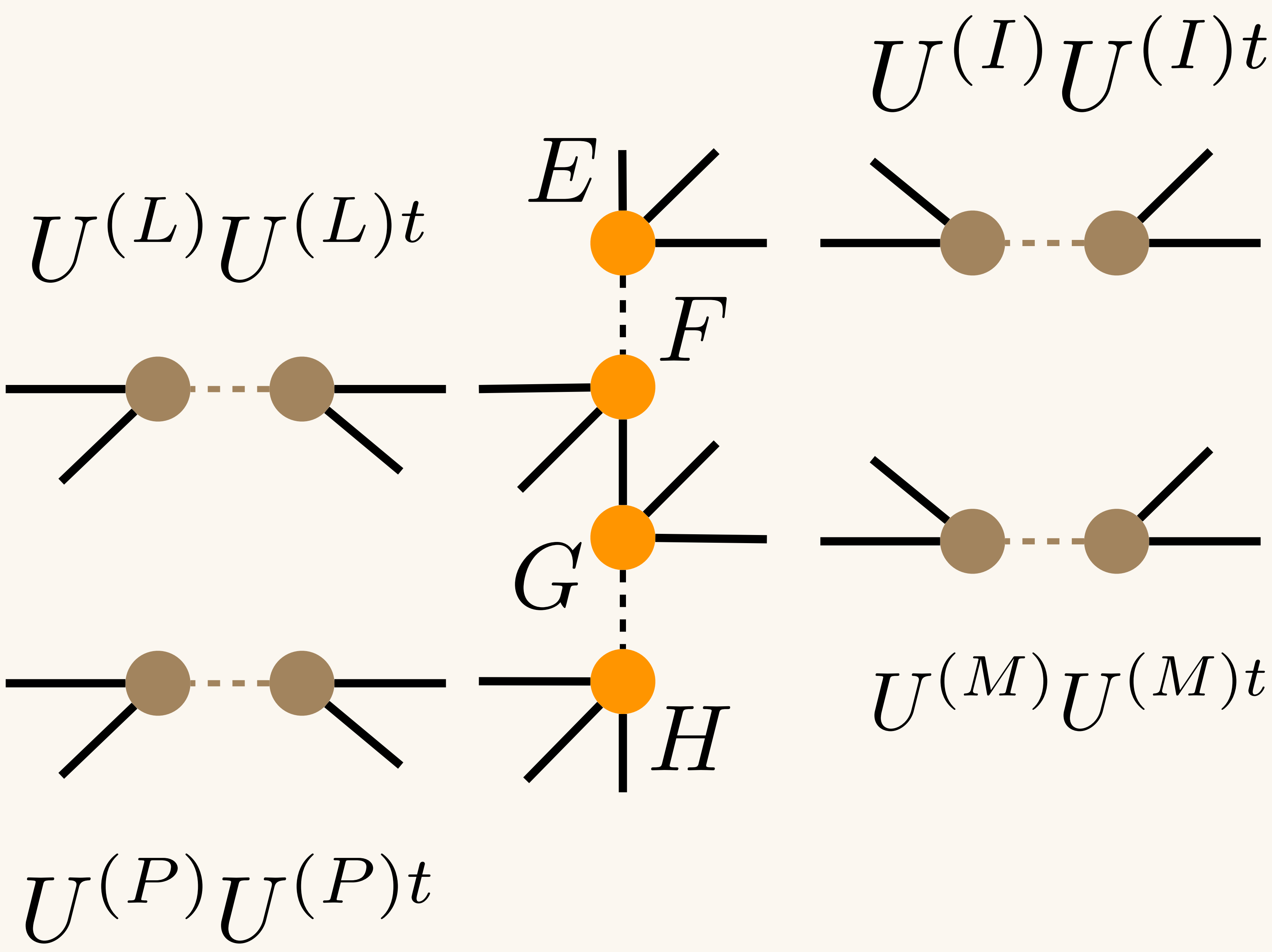




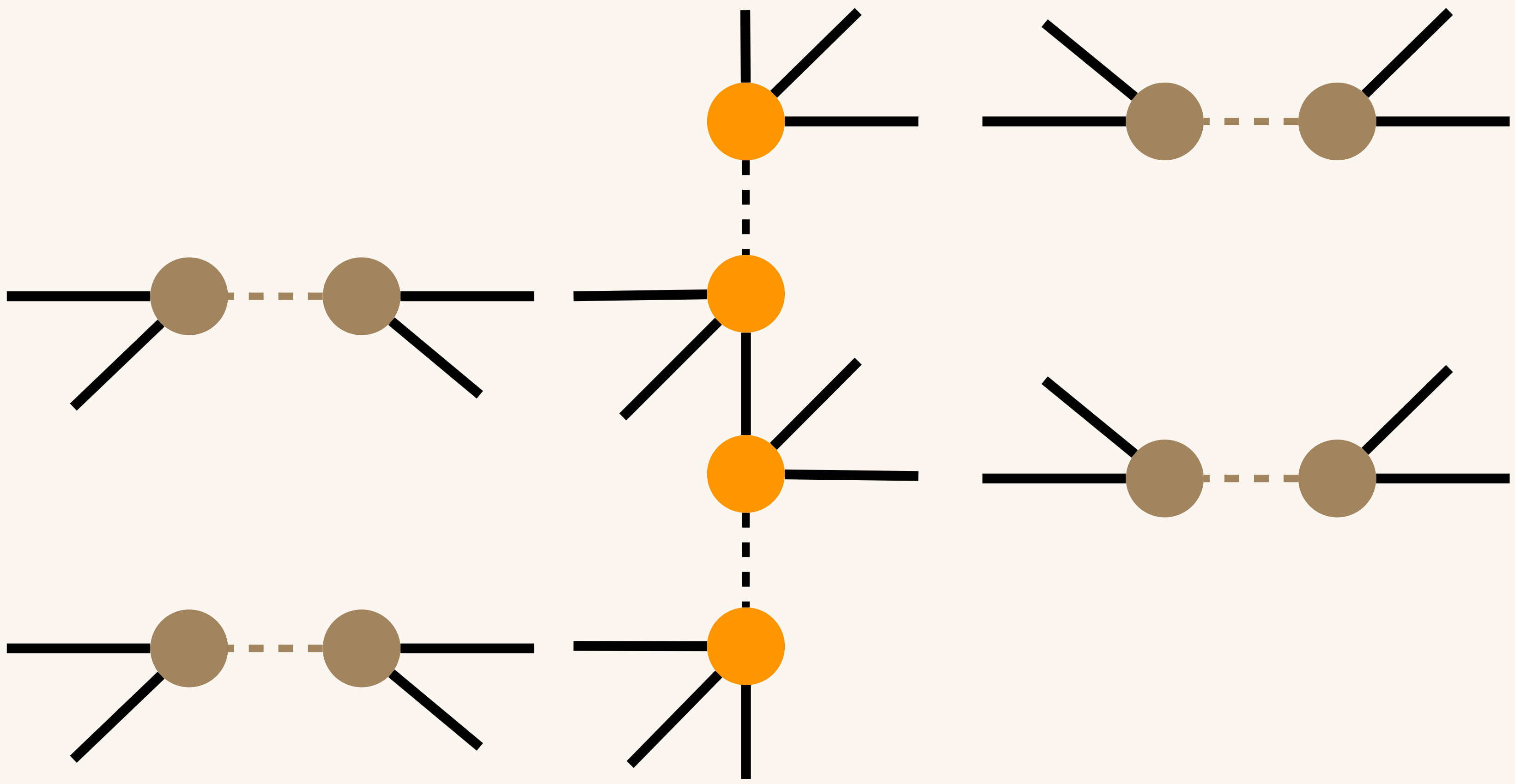


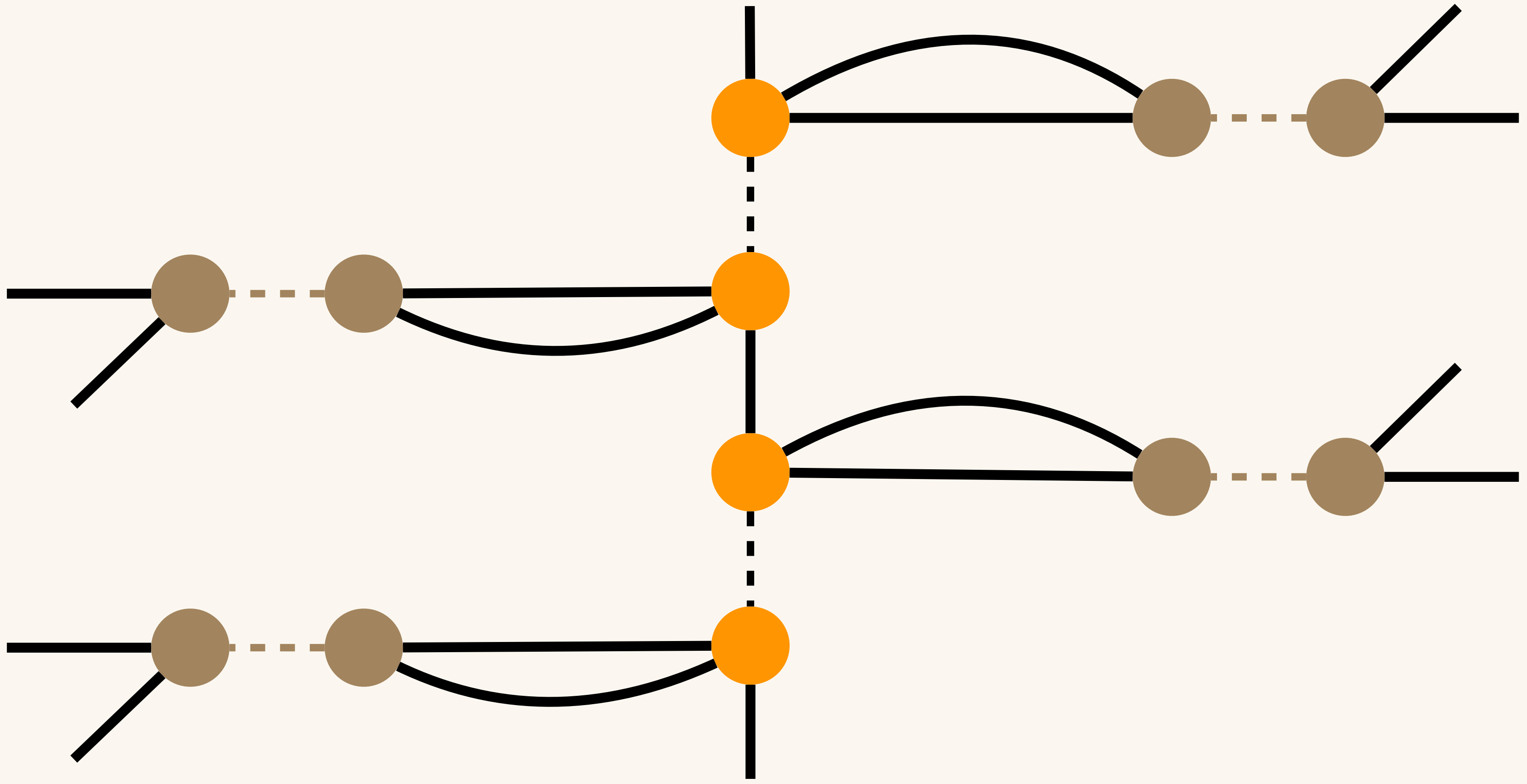


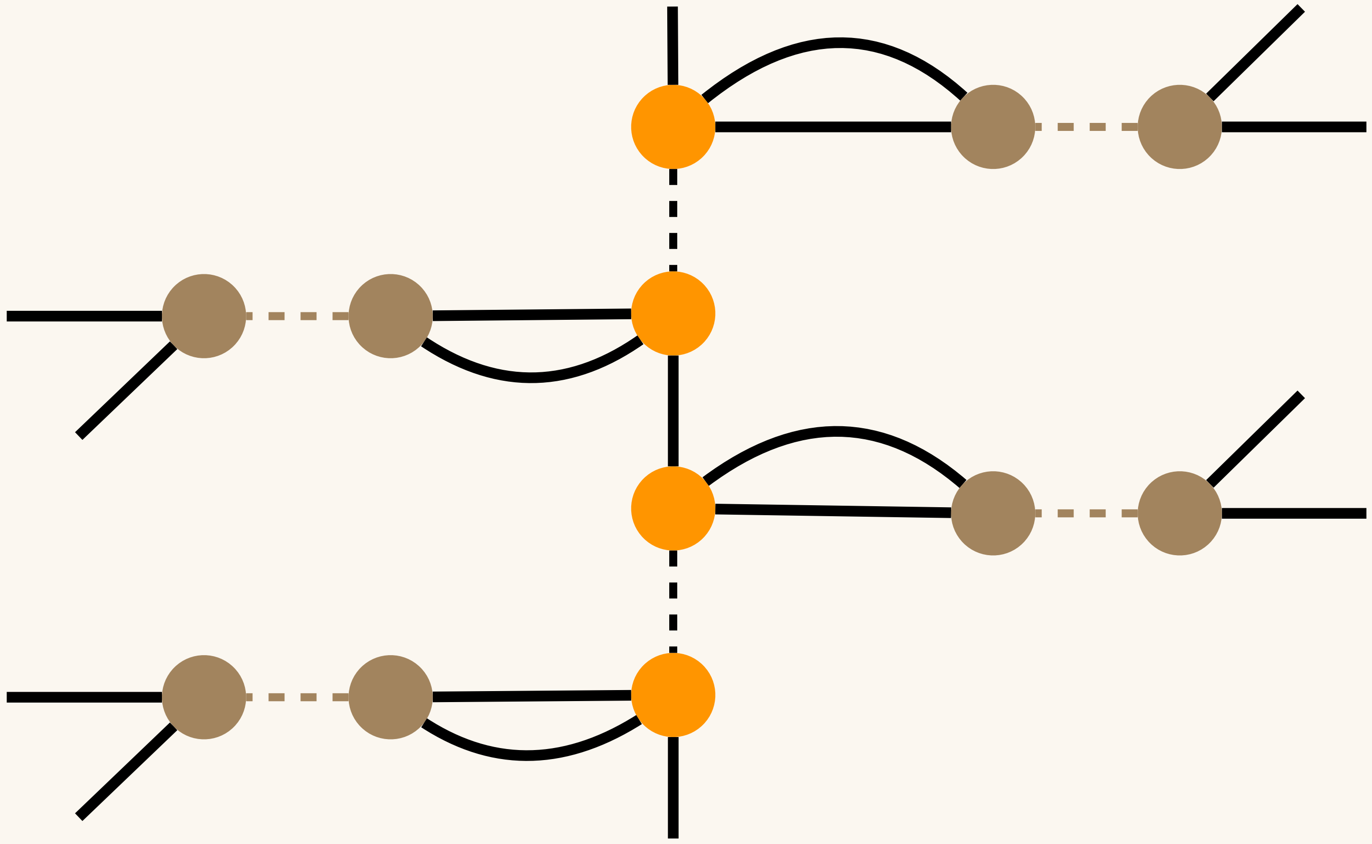

$U^{(I)} U^{(I)t}$
$\lambda^{(A)}$

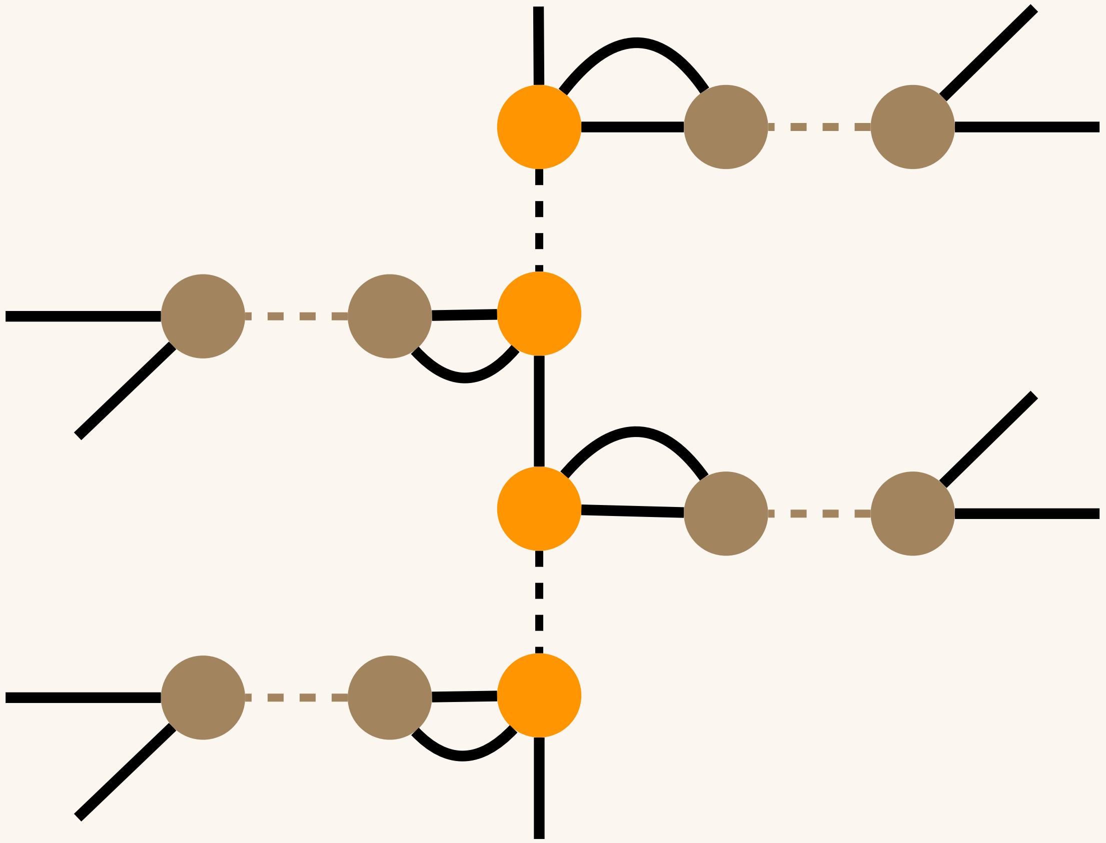


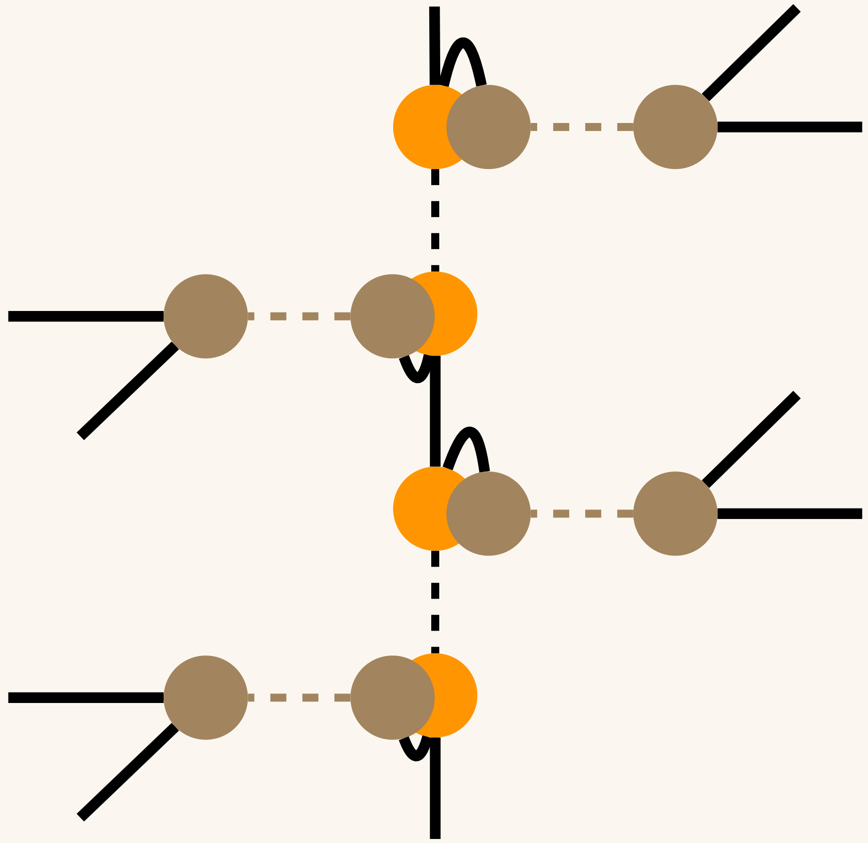
$U^{(I)}U^{(I)t}$
E
$U^{(L)}U^{(L)t}$
F
G
$U^{(M)}U^{(M)t}$
H
$U^{(P)}U^{(P)t}$

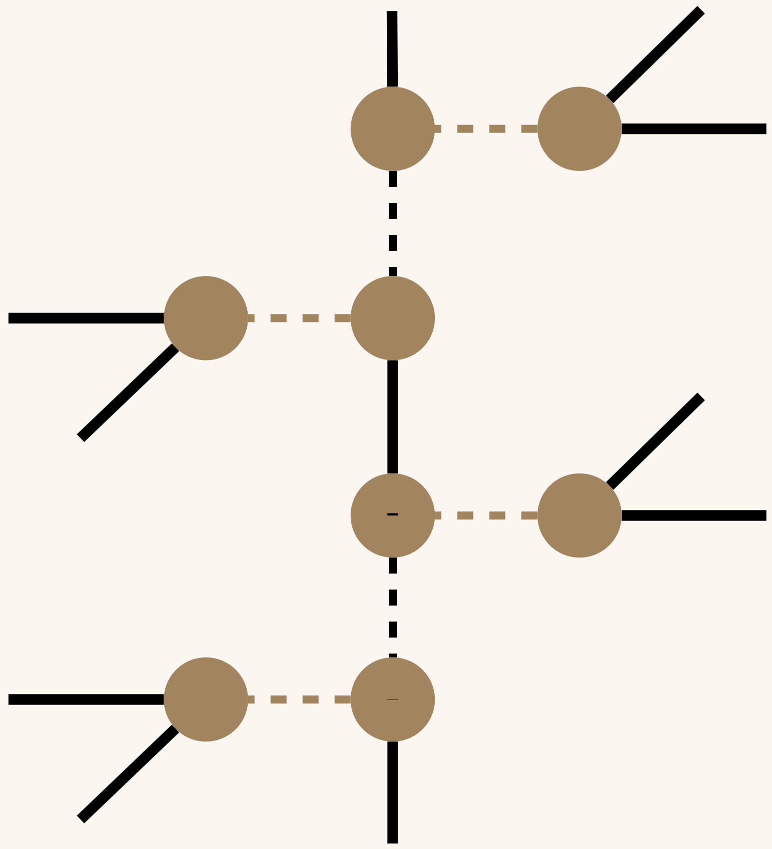


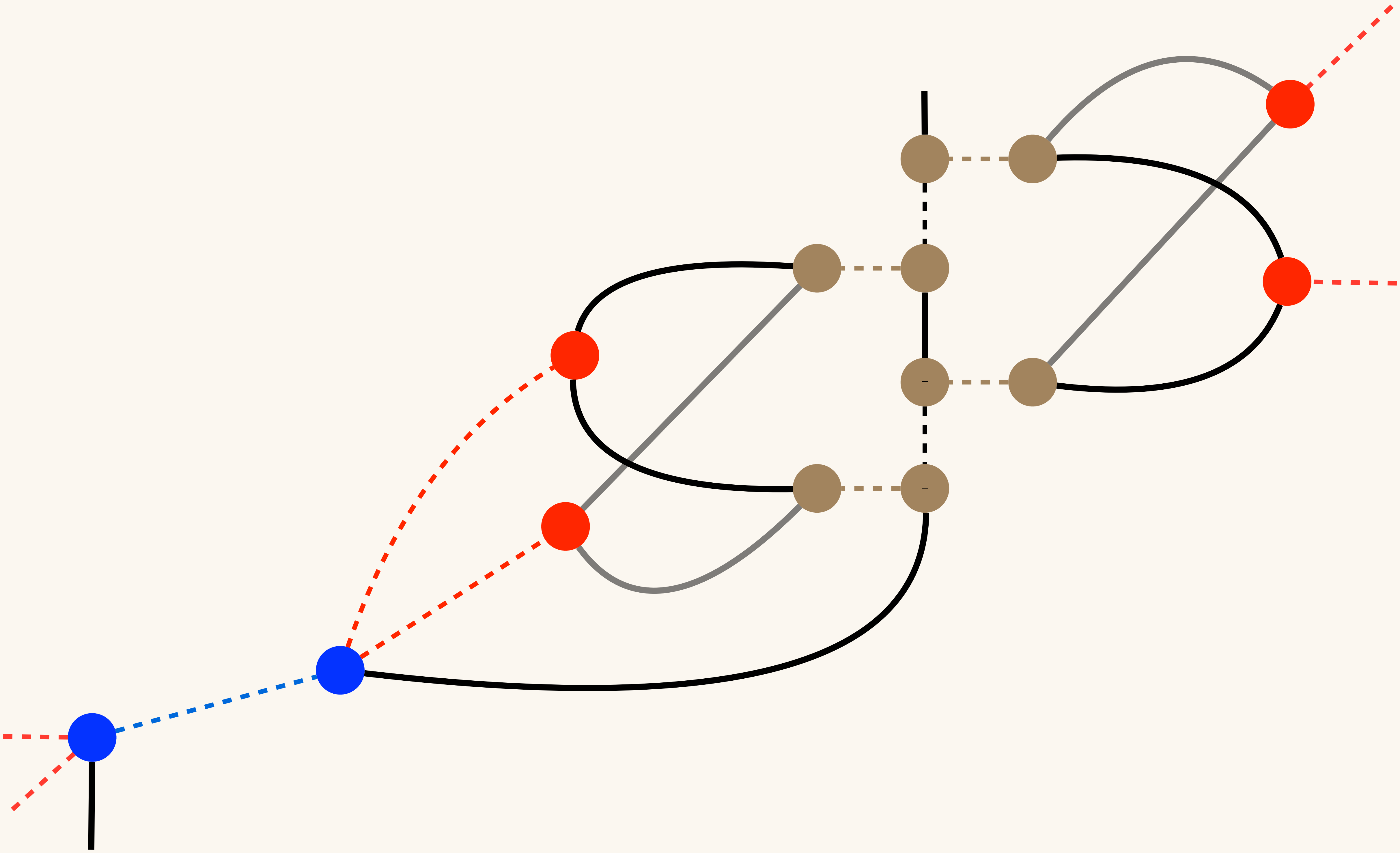


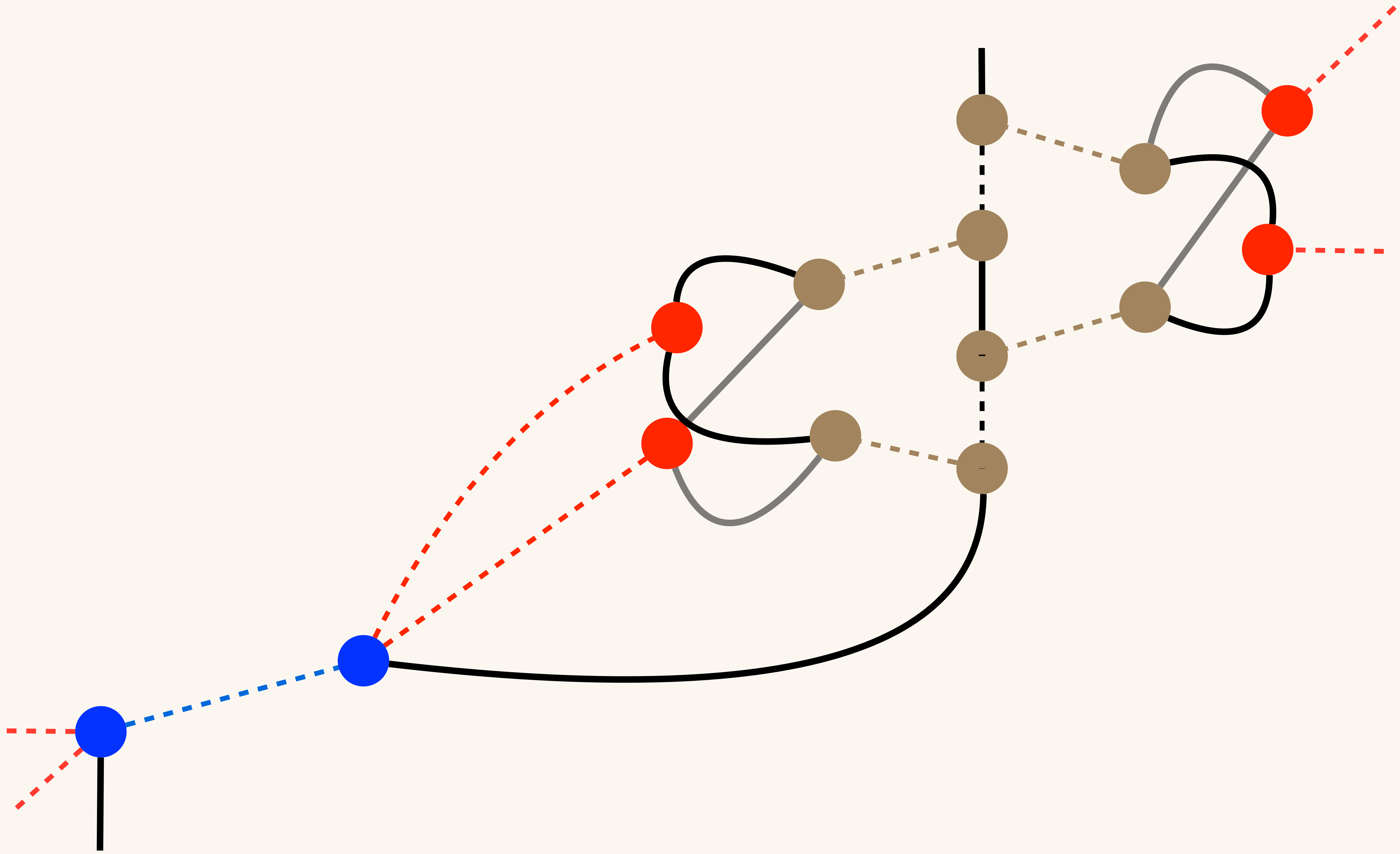


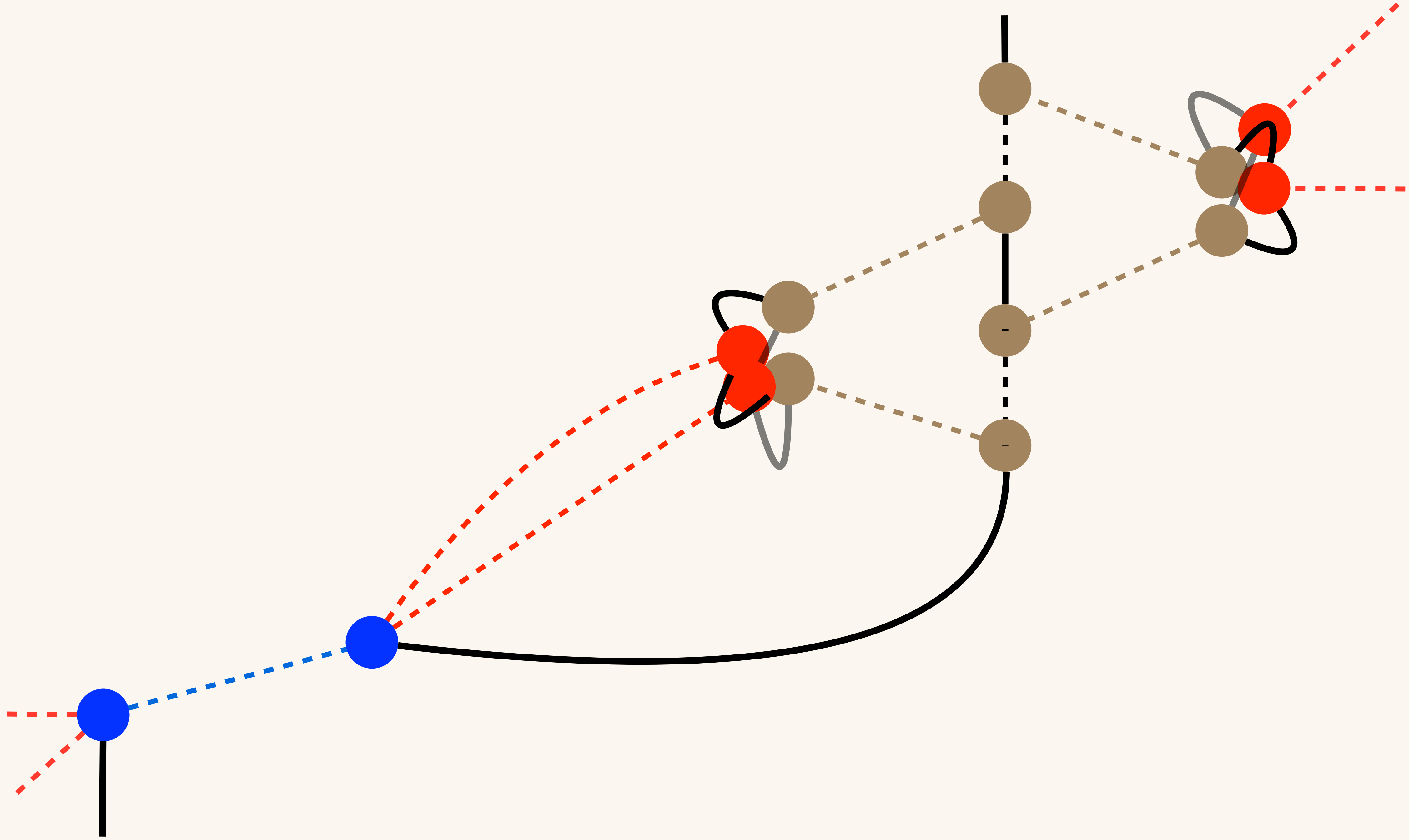


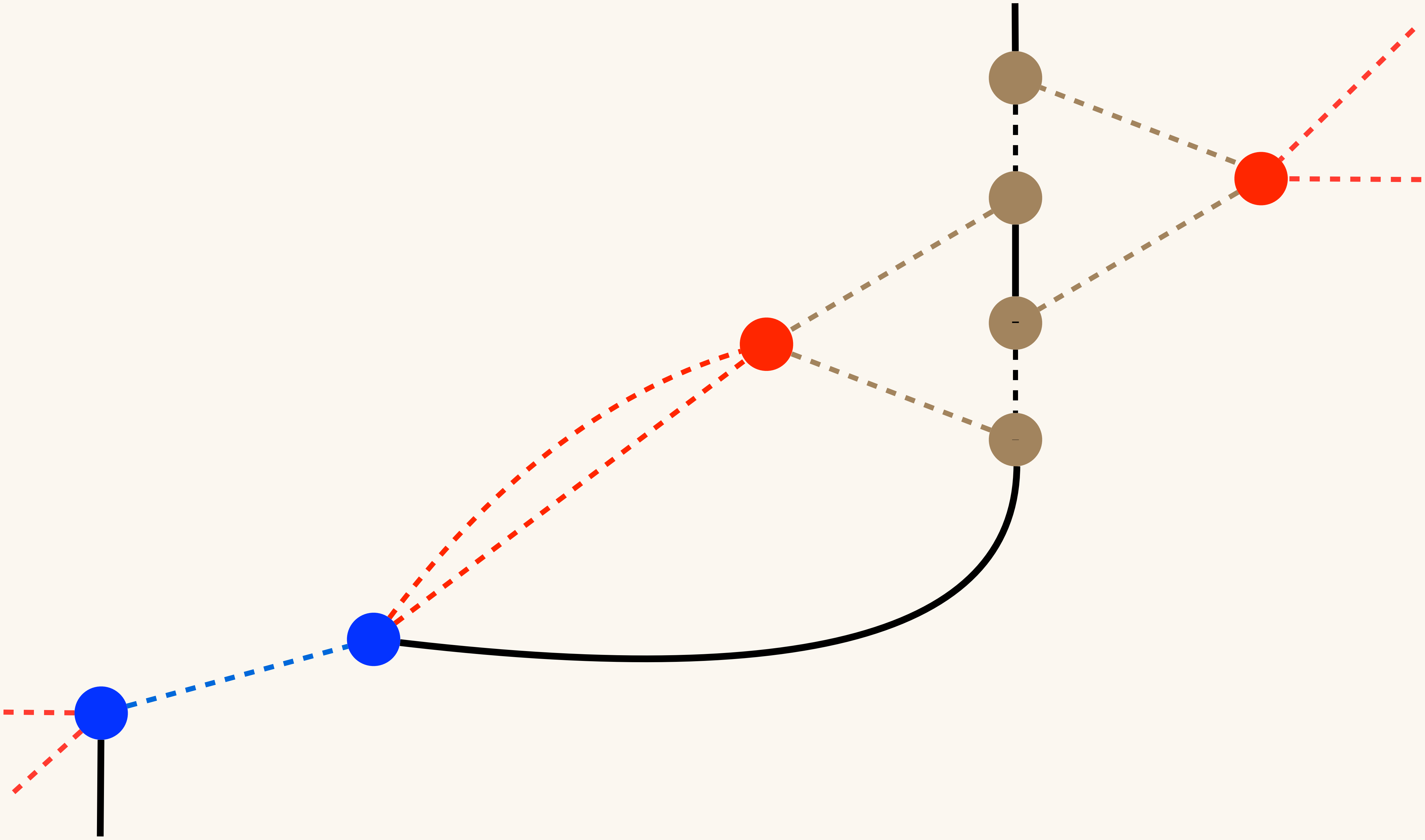


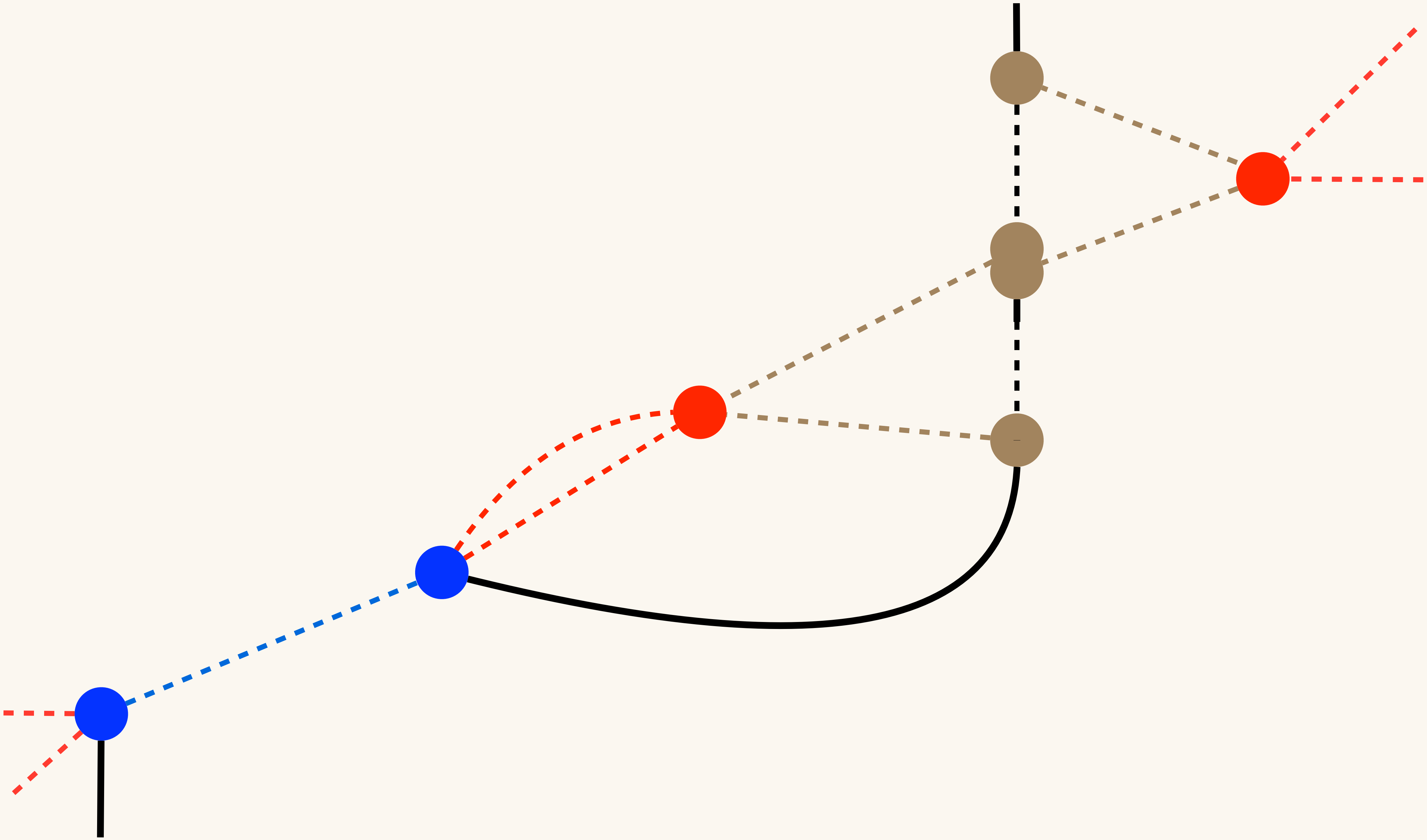


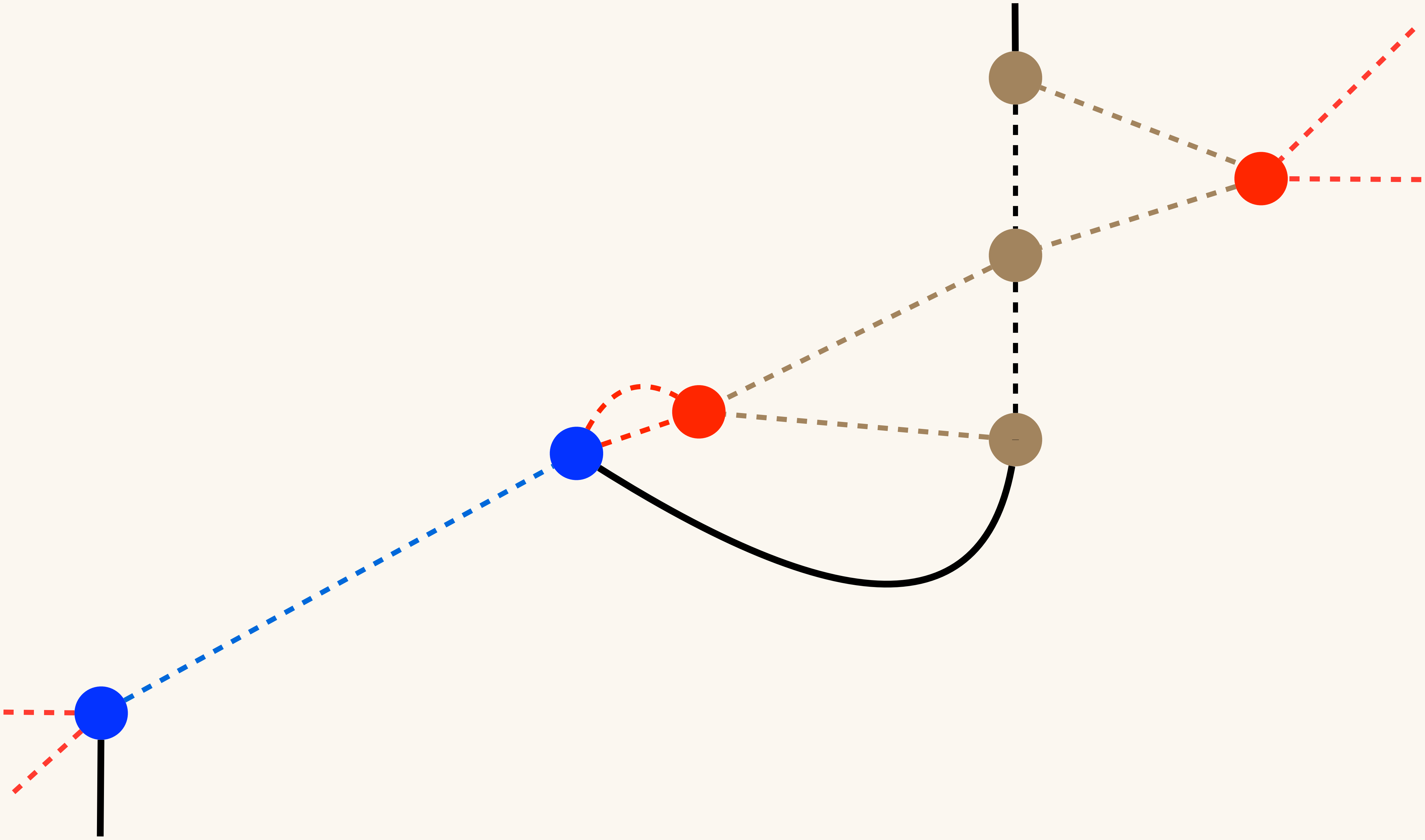


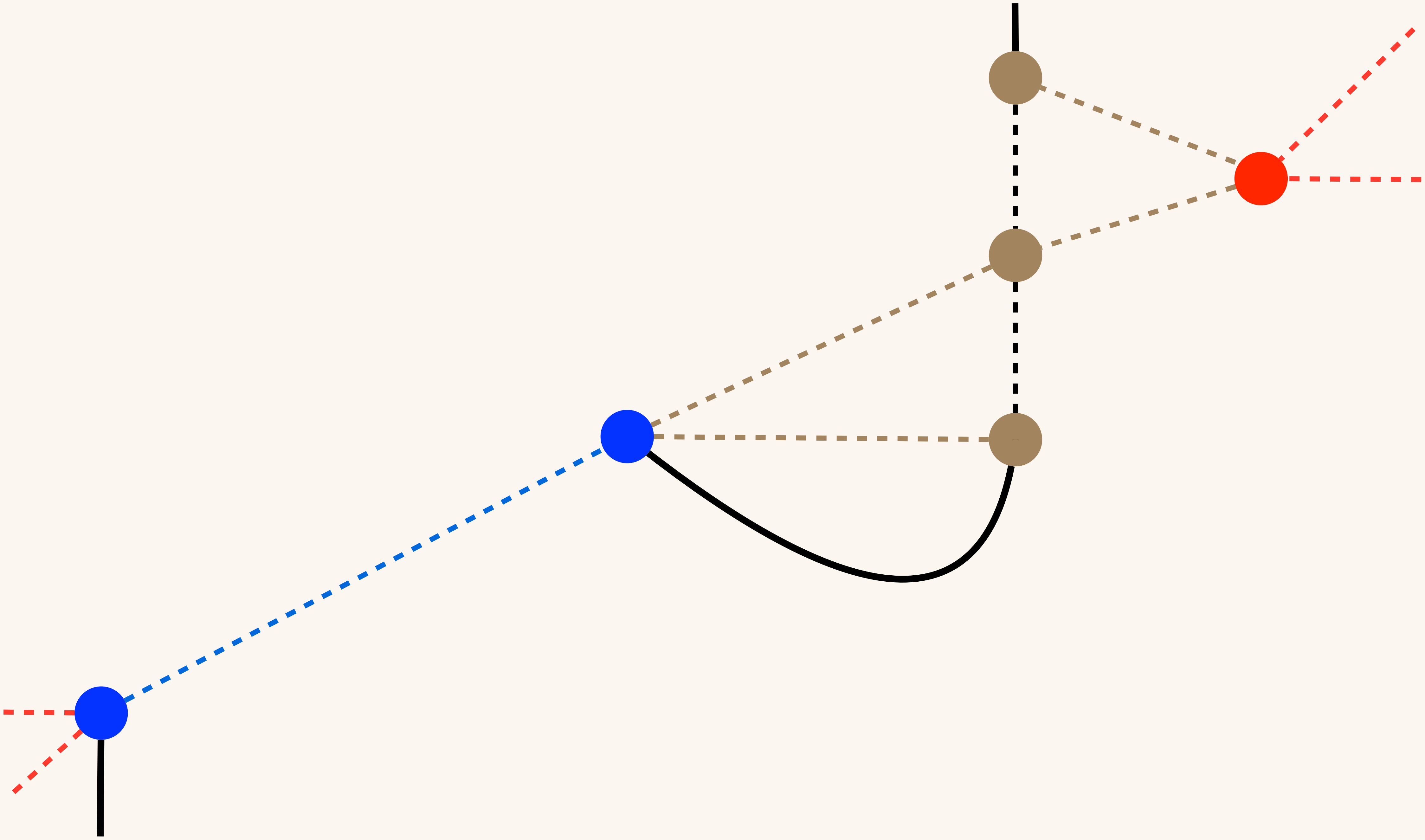


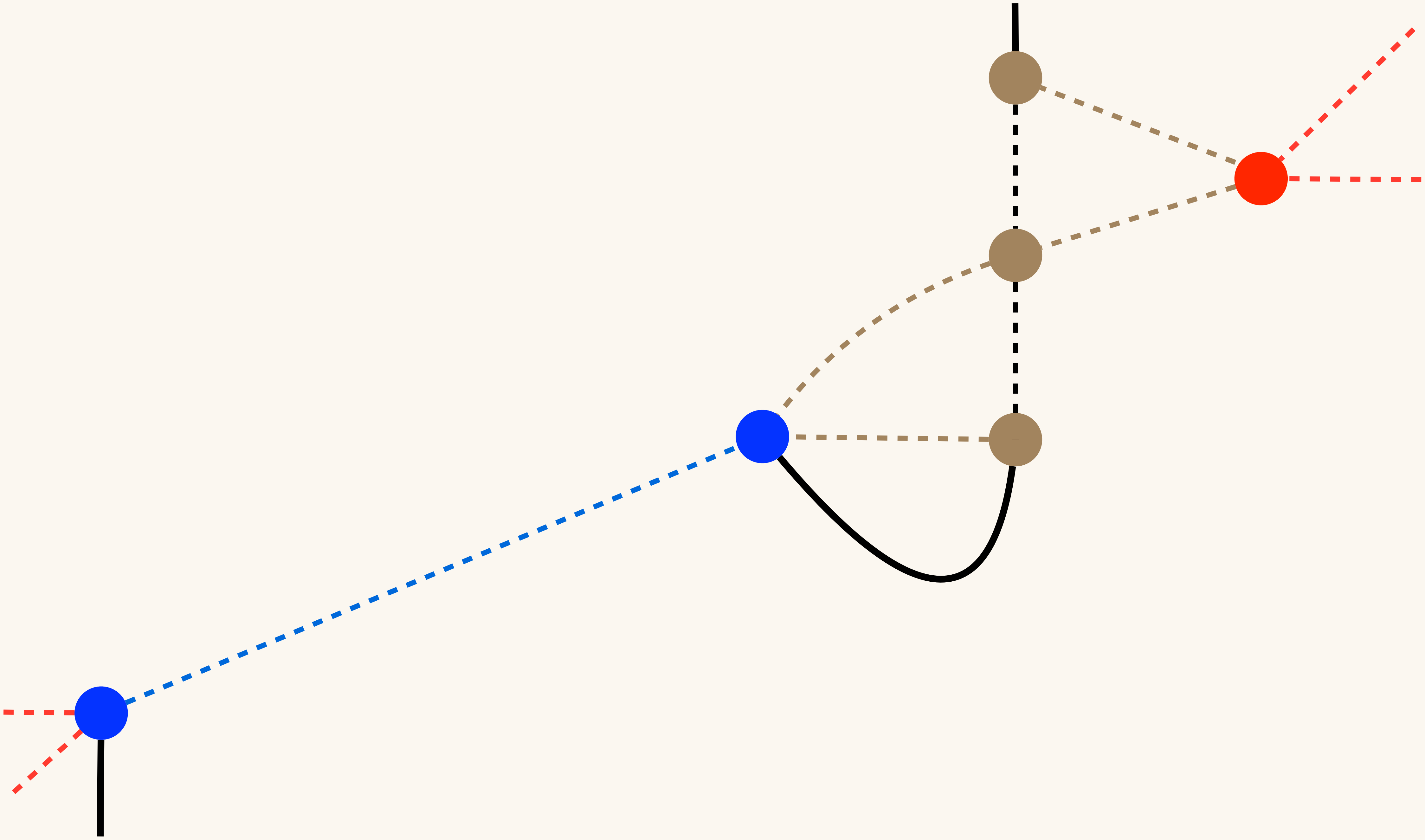


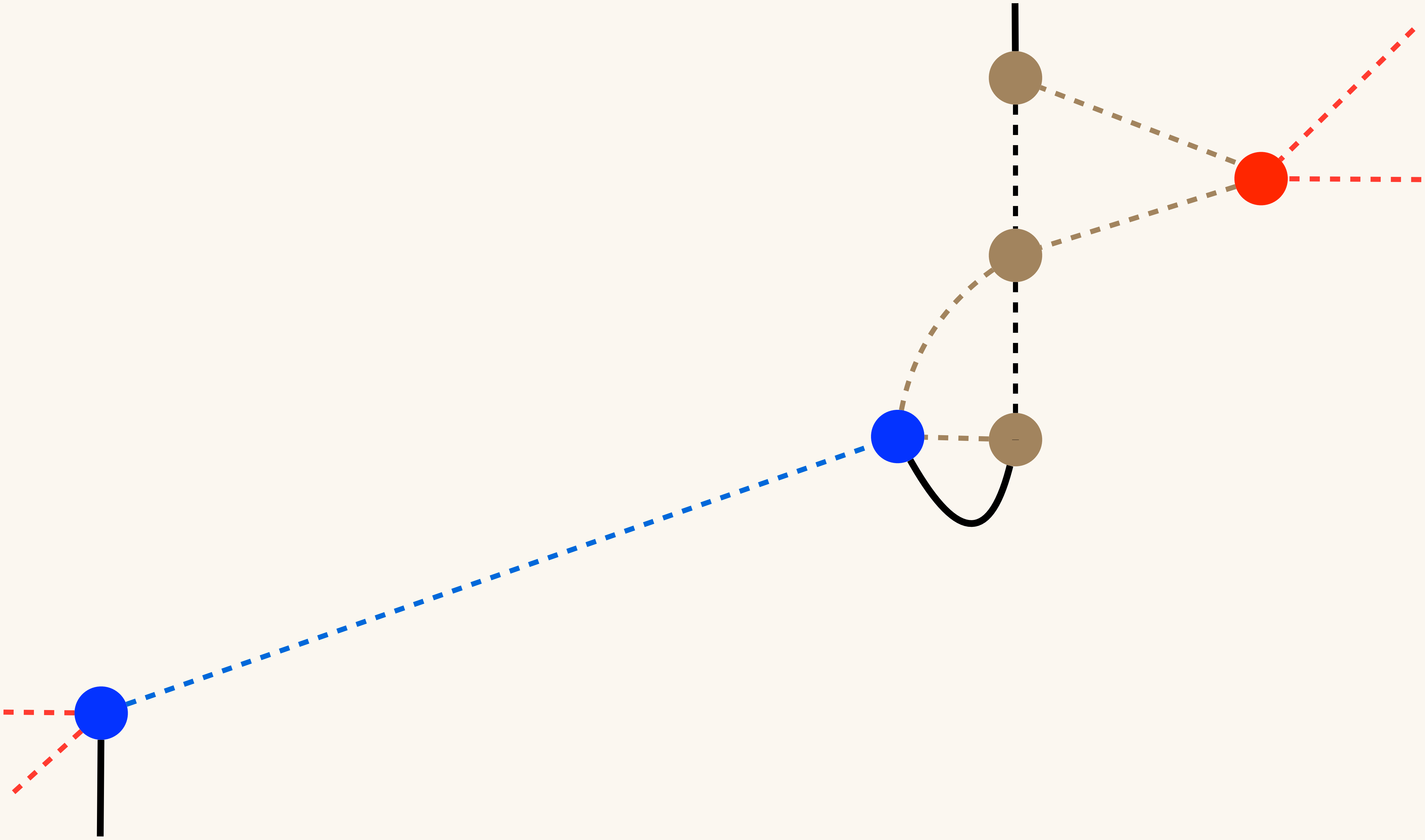


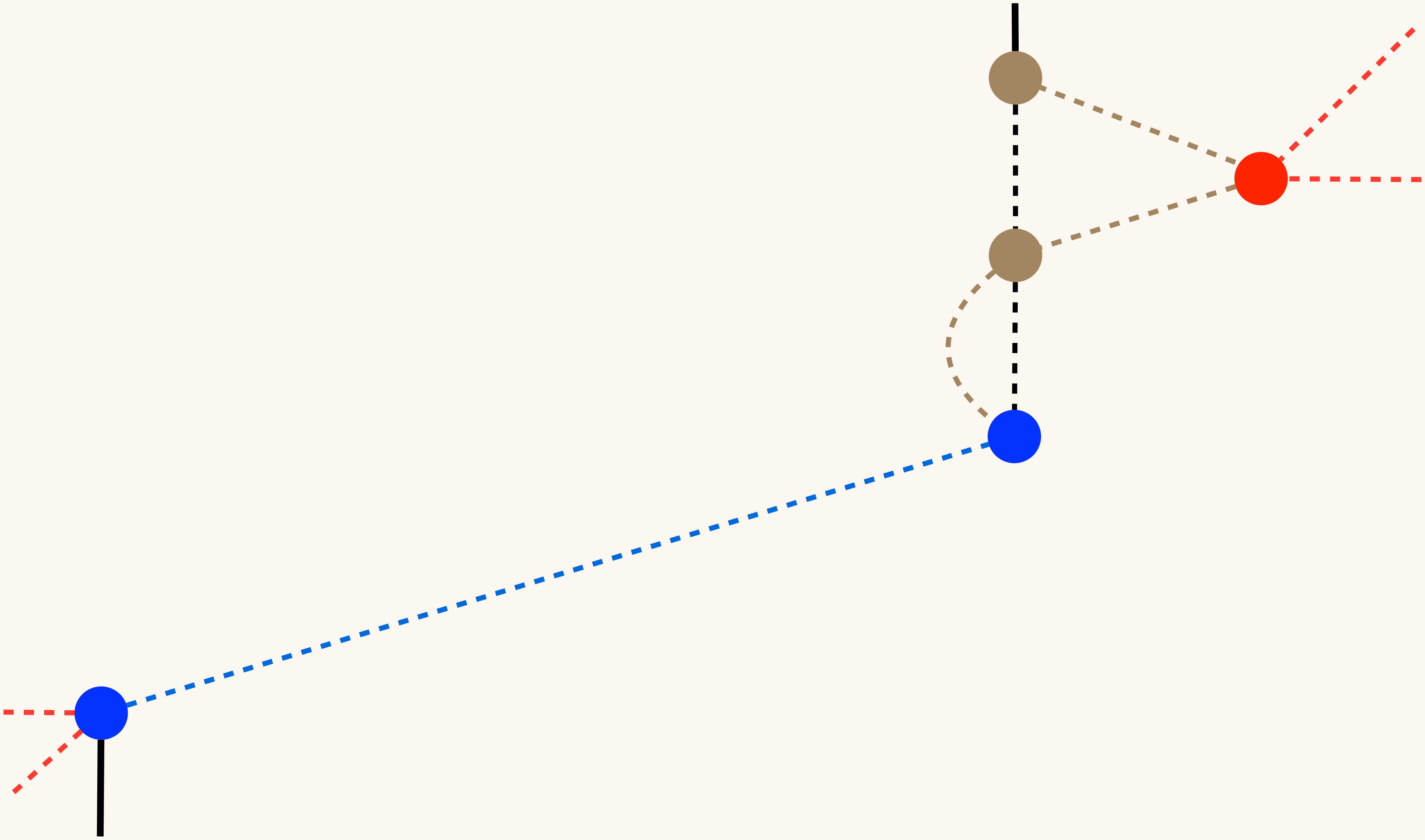


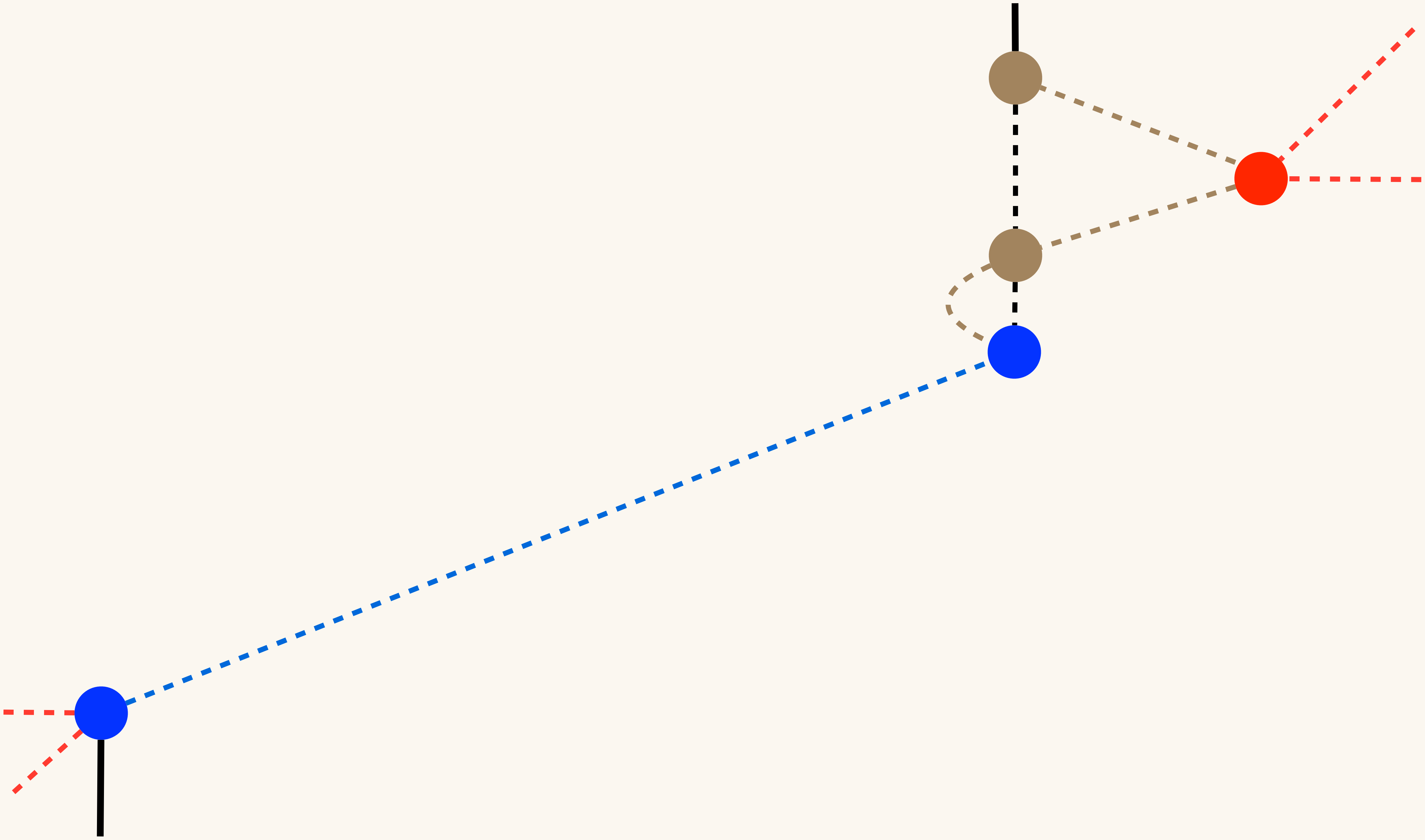


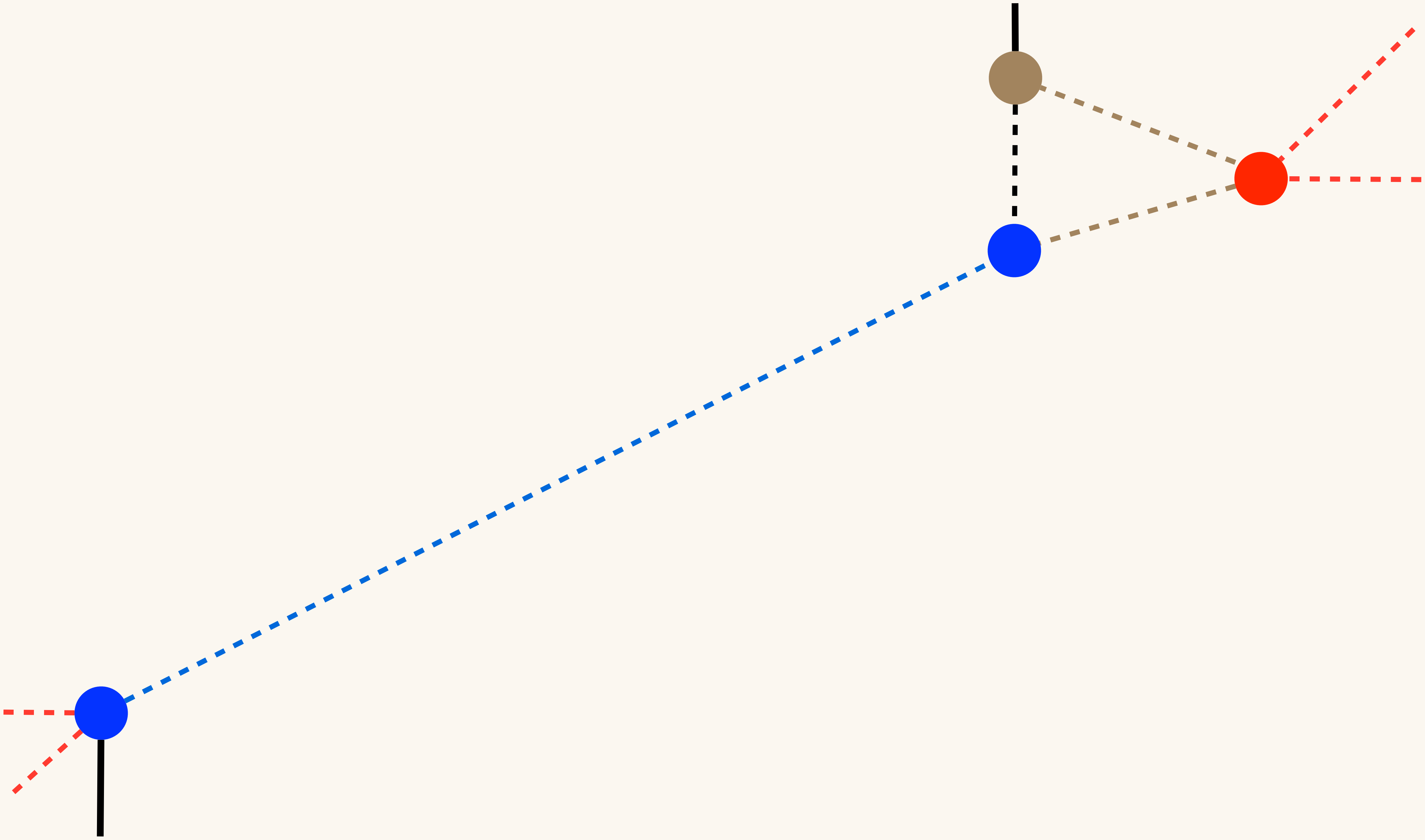


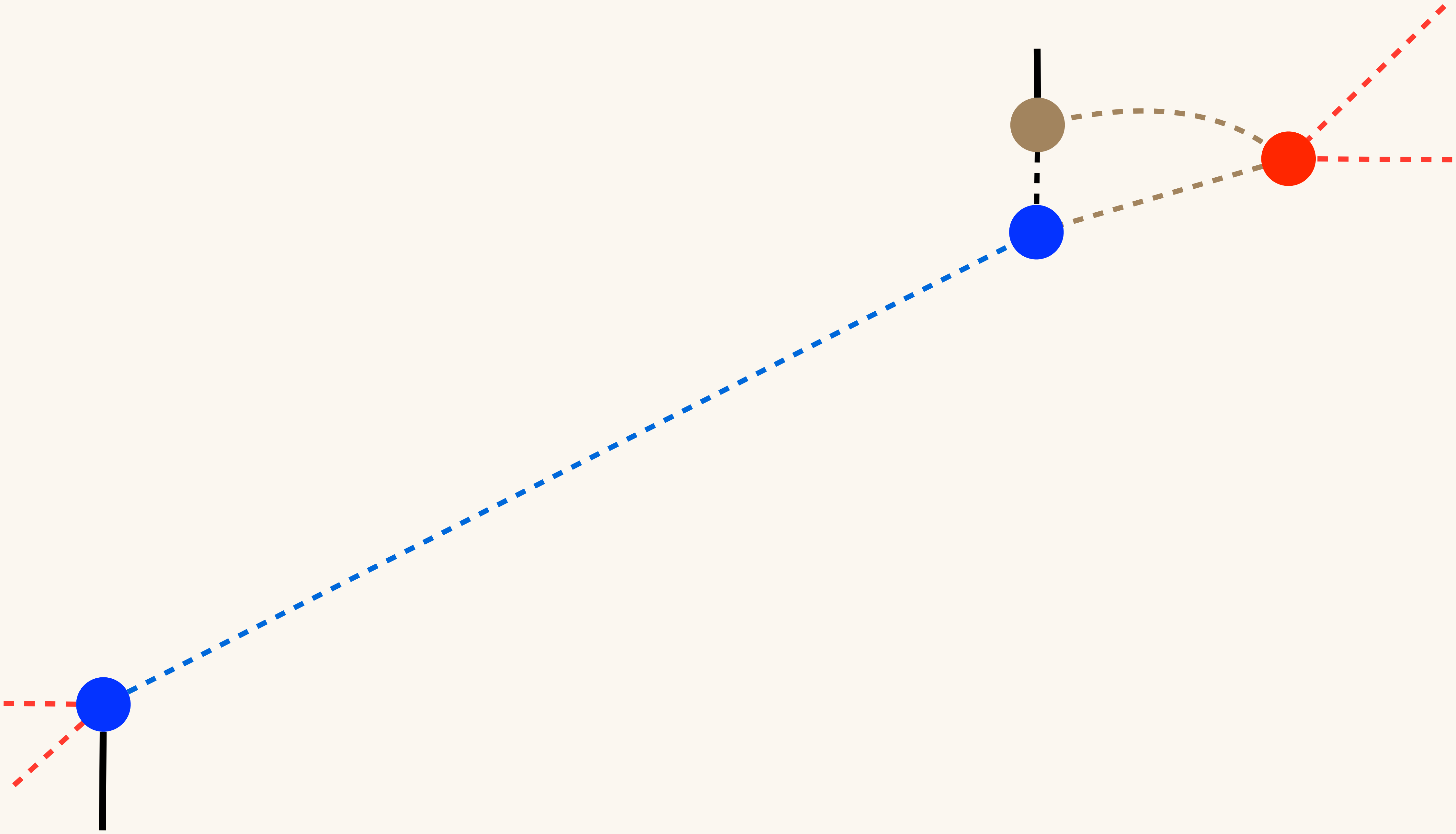


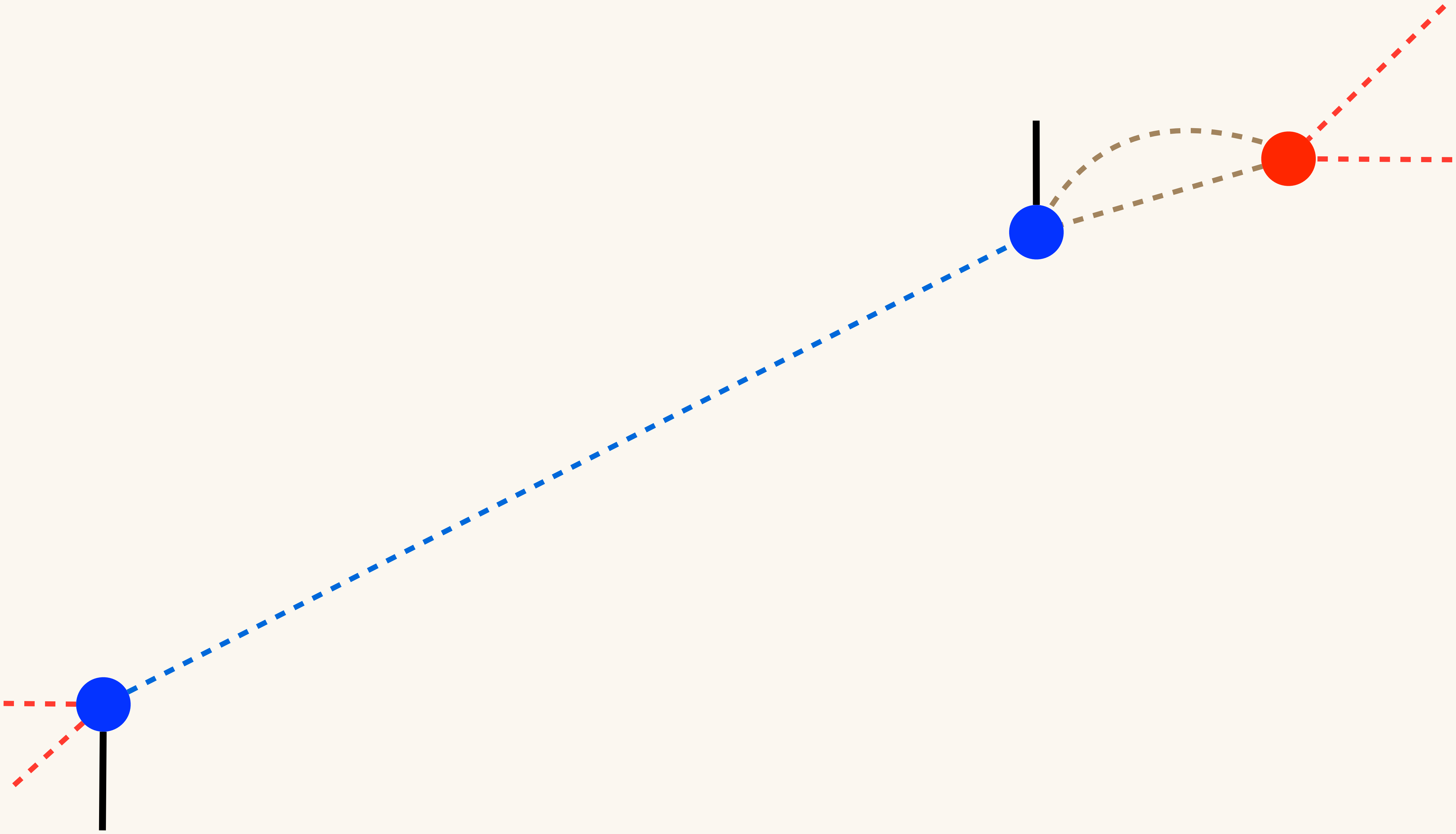


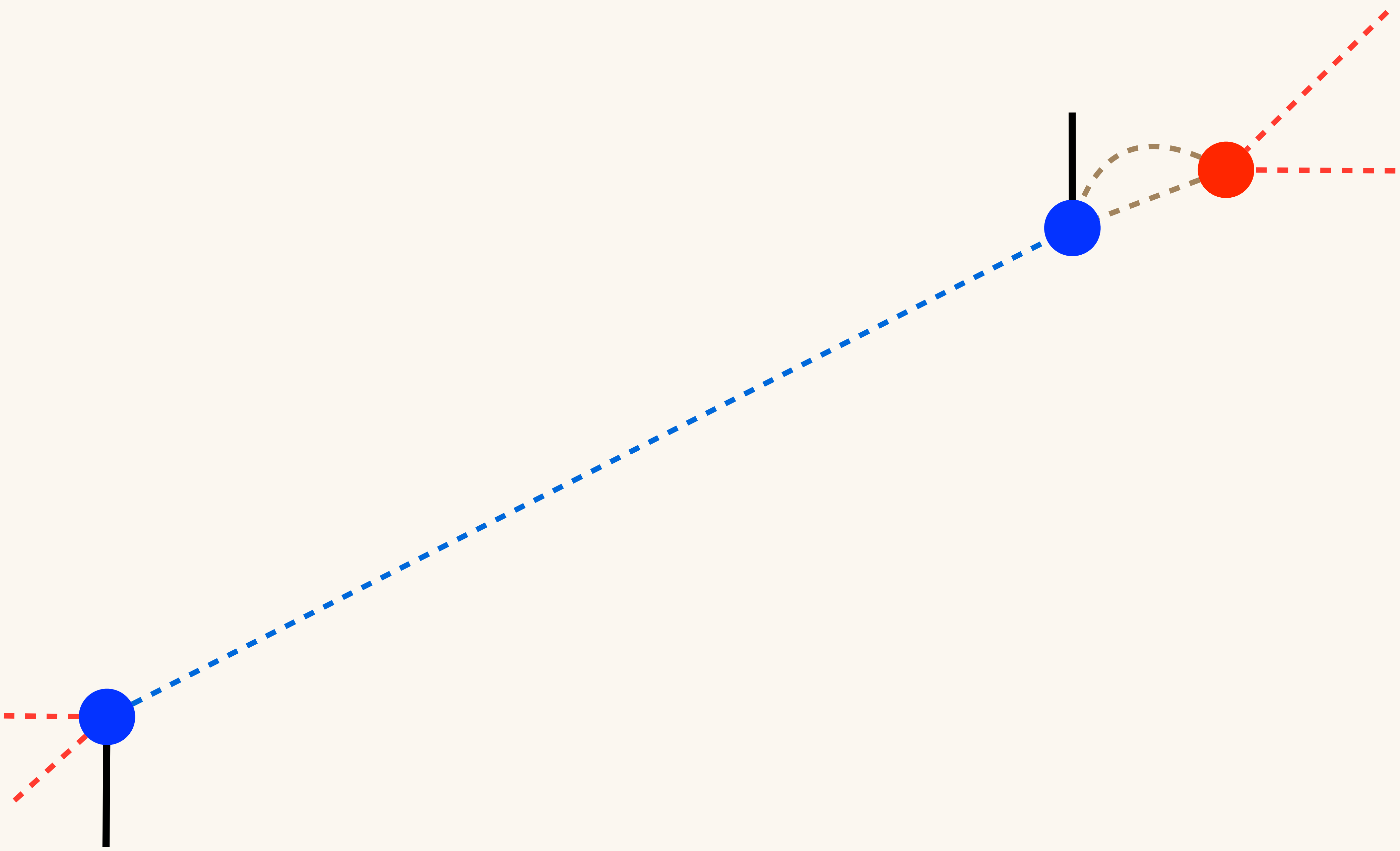


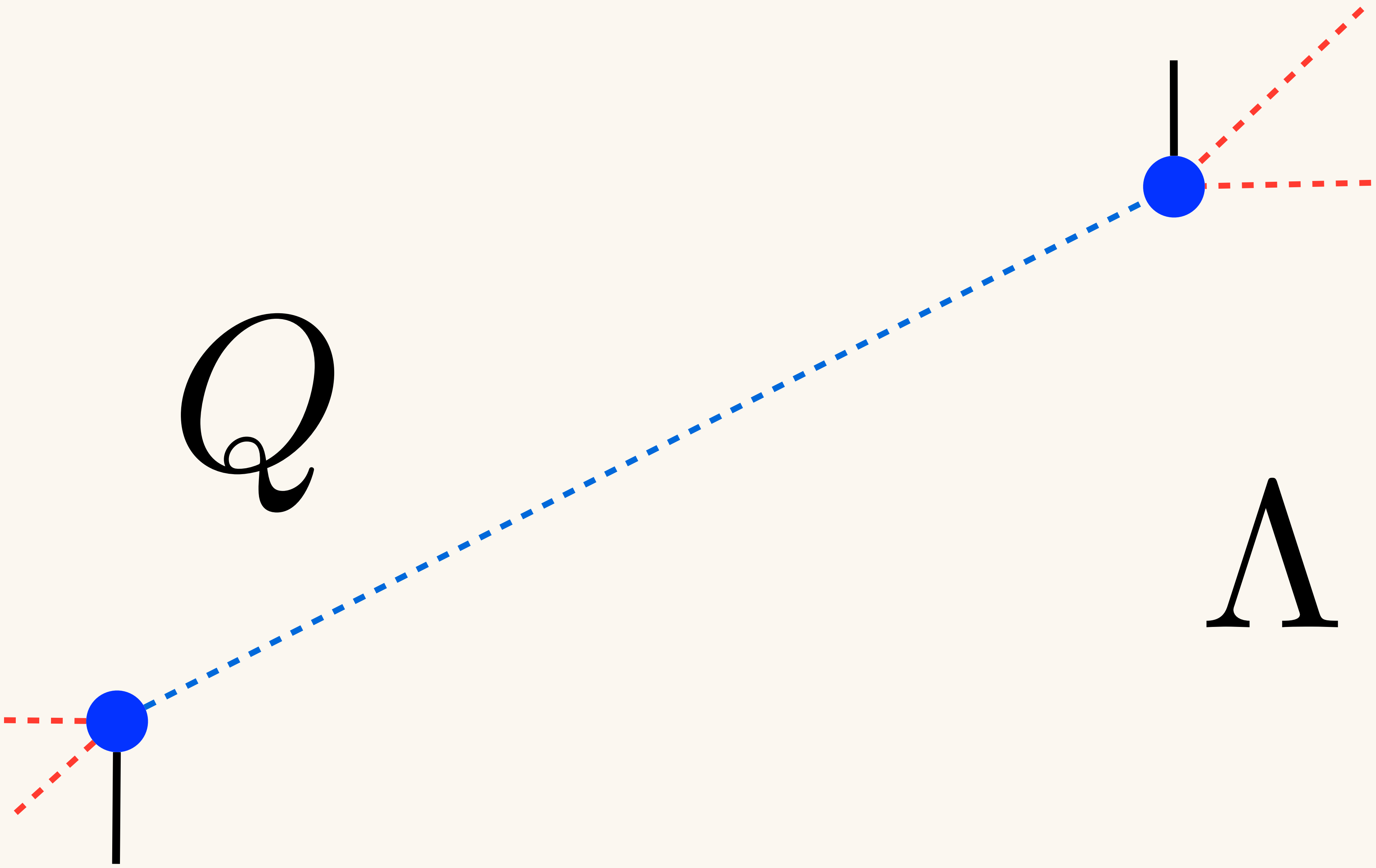




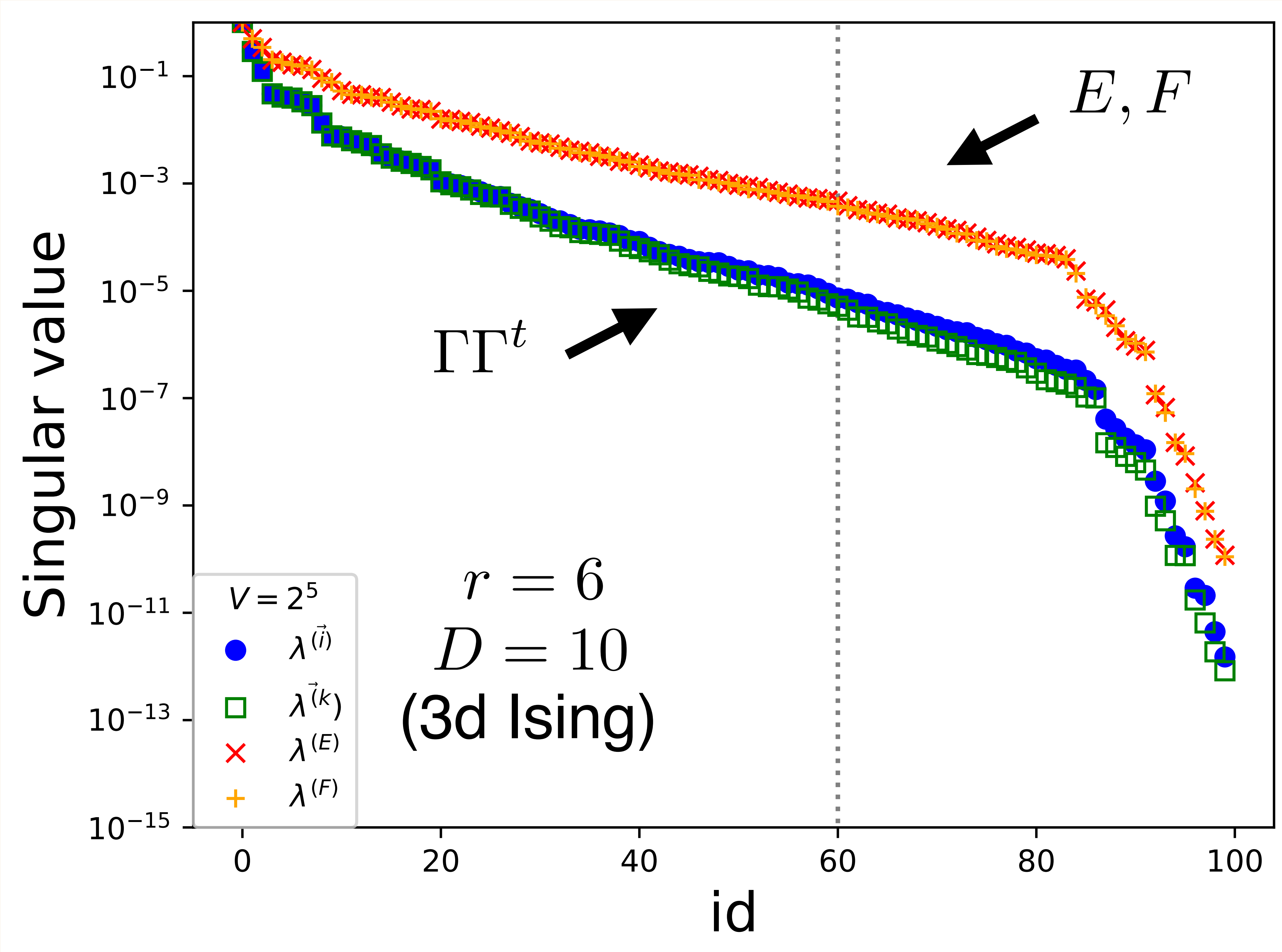


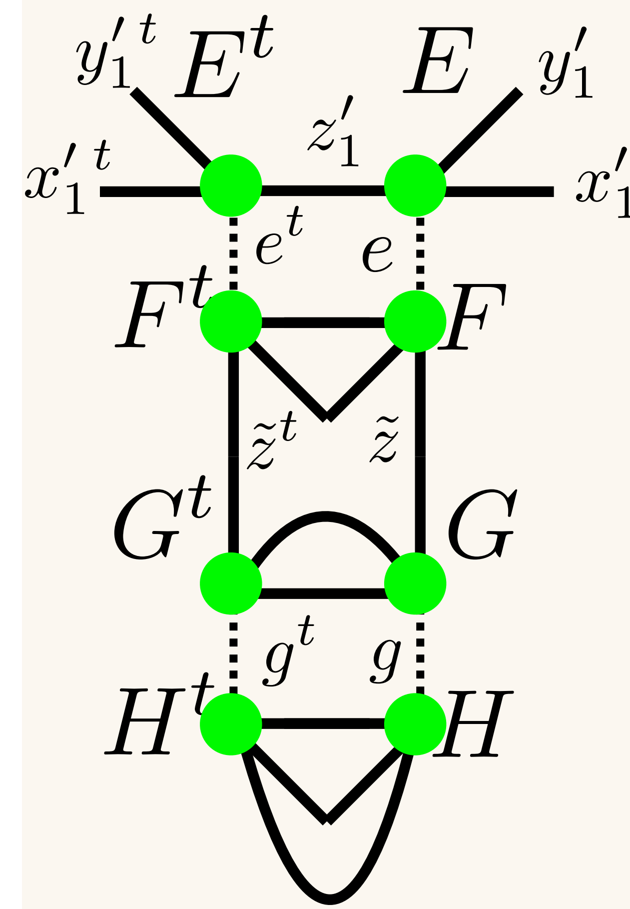





Q
Λ

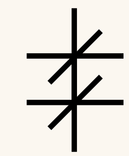
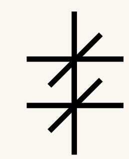
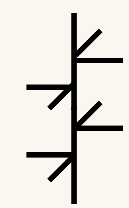
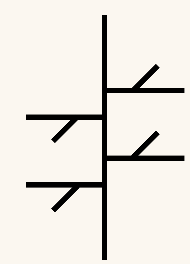
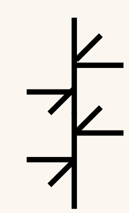
◉ Isometry for unit-cell tensor





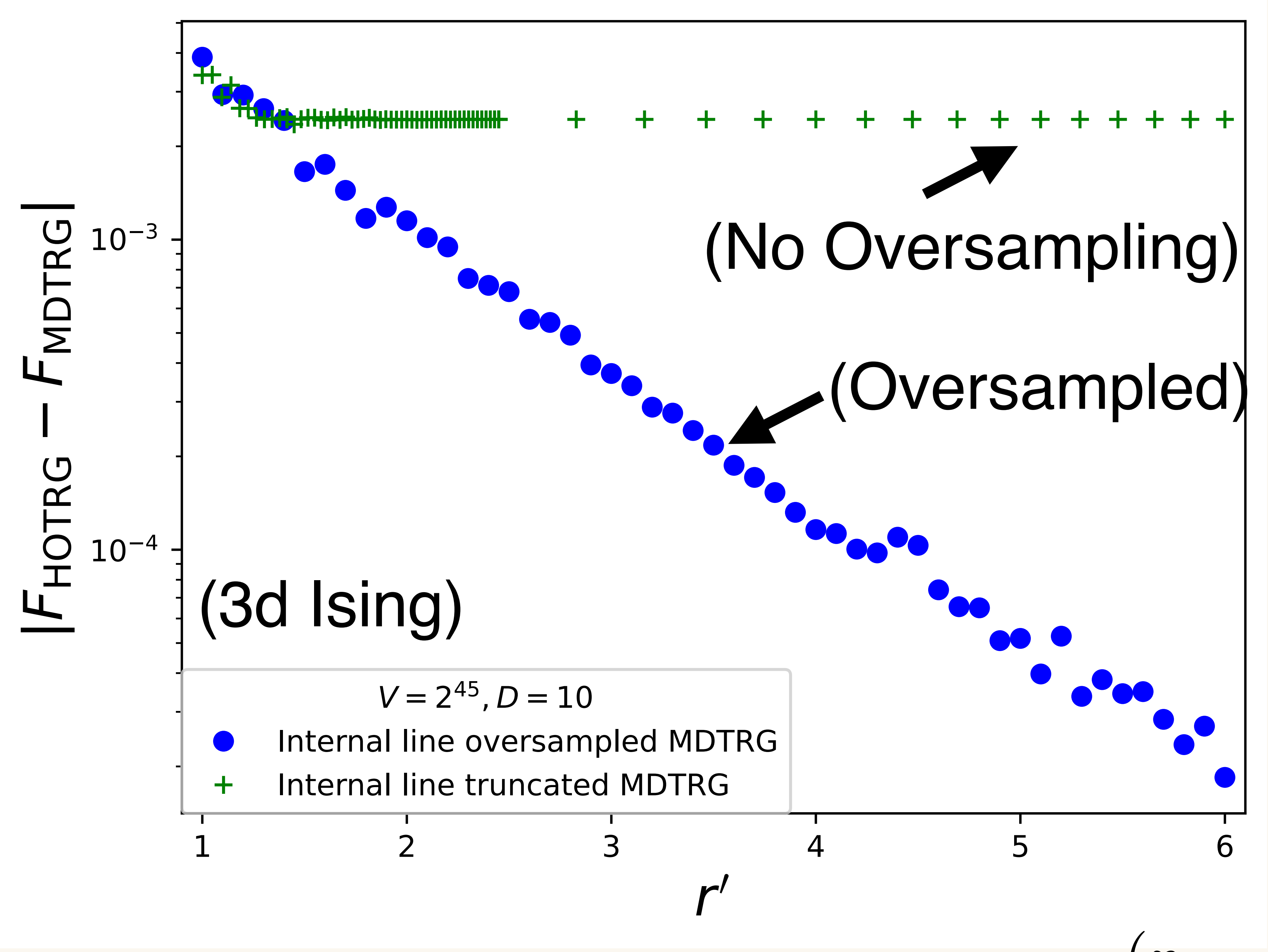
→ More global isometry reduces systematic error.

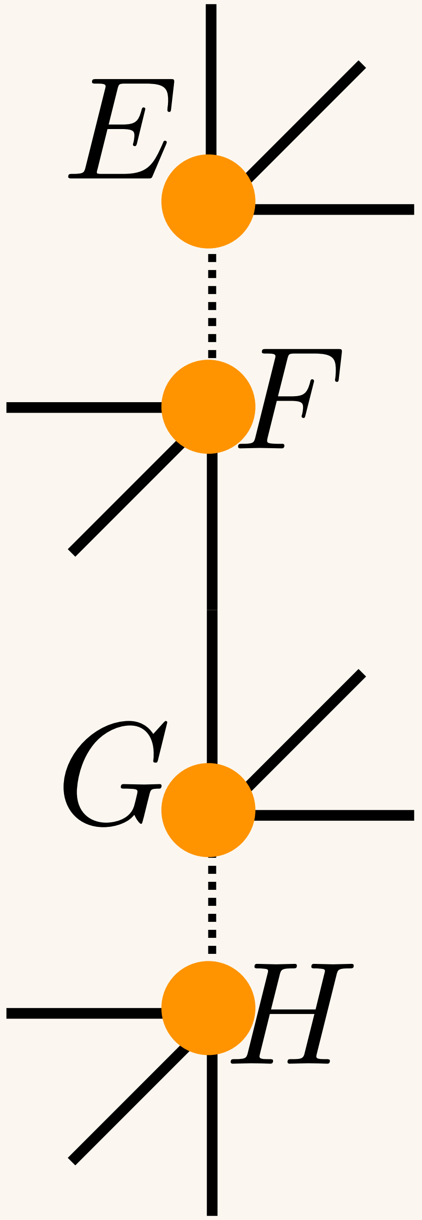
◉ HOTRG with R-SVD

	with R-SVD	w/o R-SVD	unit-cell order	
◇ HOTRG	$O(D^{3d})$	$O(D^{4d-1})$		$2d$
◇ ATRG	$O(D^{2d+1})$	$O(D^{3d})$		$2d$
◇ MDTRG	$O(D^{2d+1})$	$O(D^{3d})$		$d+1$
◇ TTRG	$O(D^{d+3})$	$O(D^{d+4})$		3
◇ Triad-MDTRG	$O(D^{d+3})$	$O(D^{3d})$		$d+1$

→ Comparable cost. We will check the systematic error.

Internal line oversampling

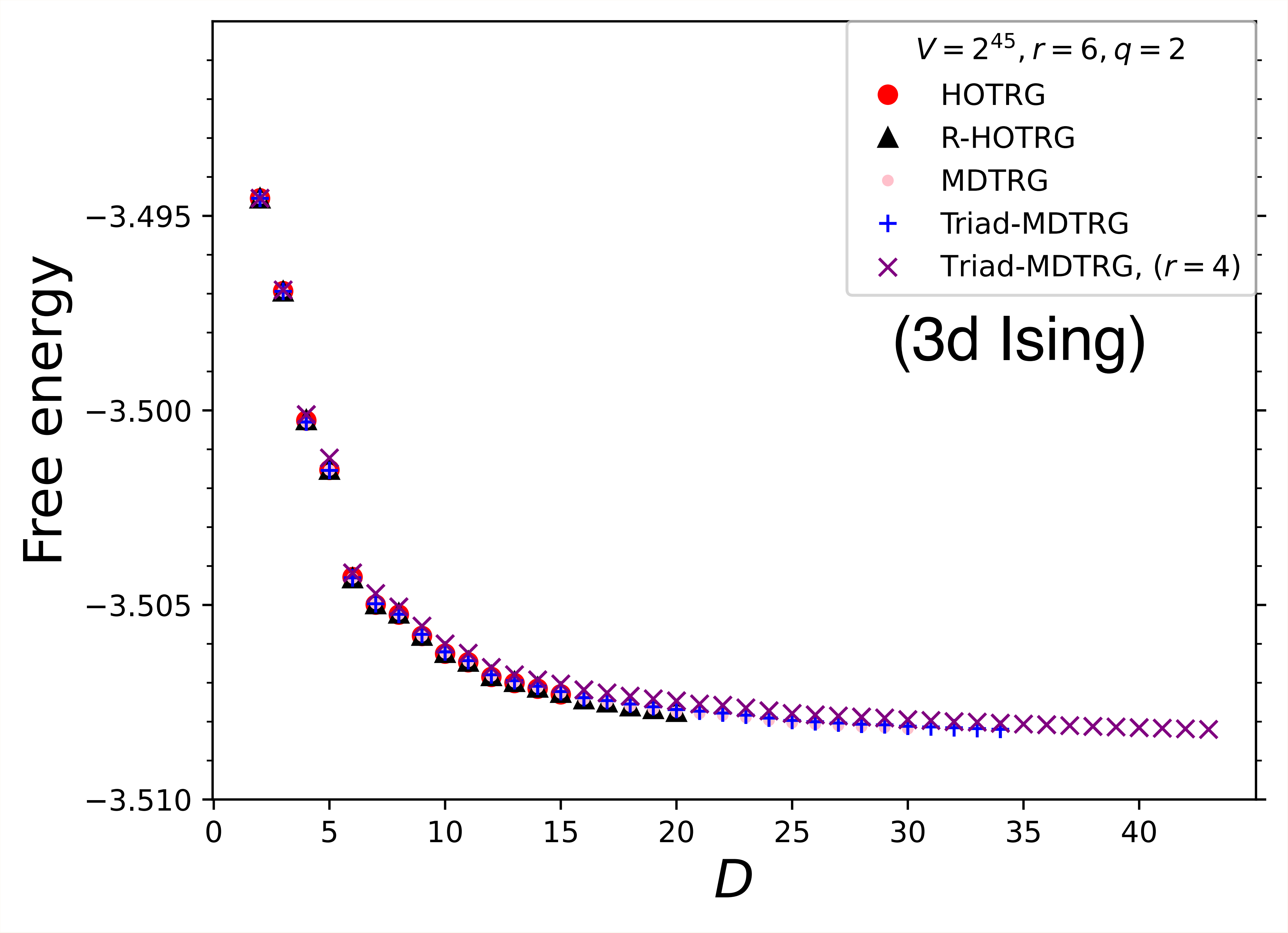




$(r = \mathrm{Const.})$

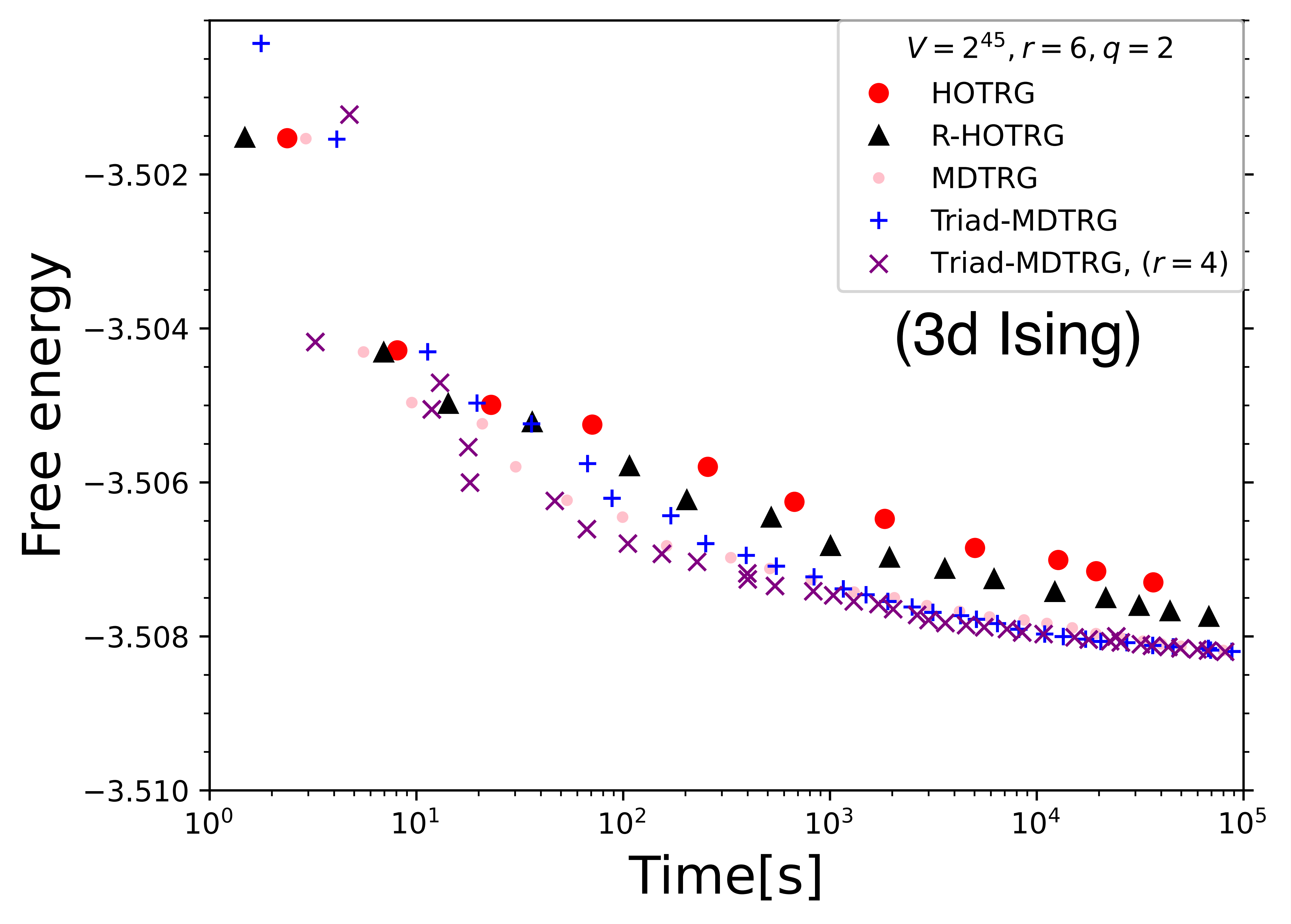
→ Internal lines (dotted lines) have to be oversampled $(D \to rD)$ to reduce the systematic error from R-SVD.

◉ Free energy density of 3d-Ising model



→ R-HOTRG, MDTRG, Triad-MDTRG results are consistent with HOTRG (additional systematic error is not dominant).

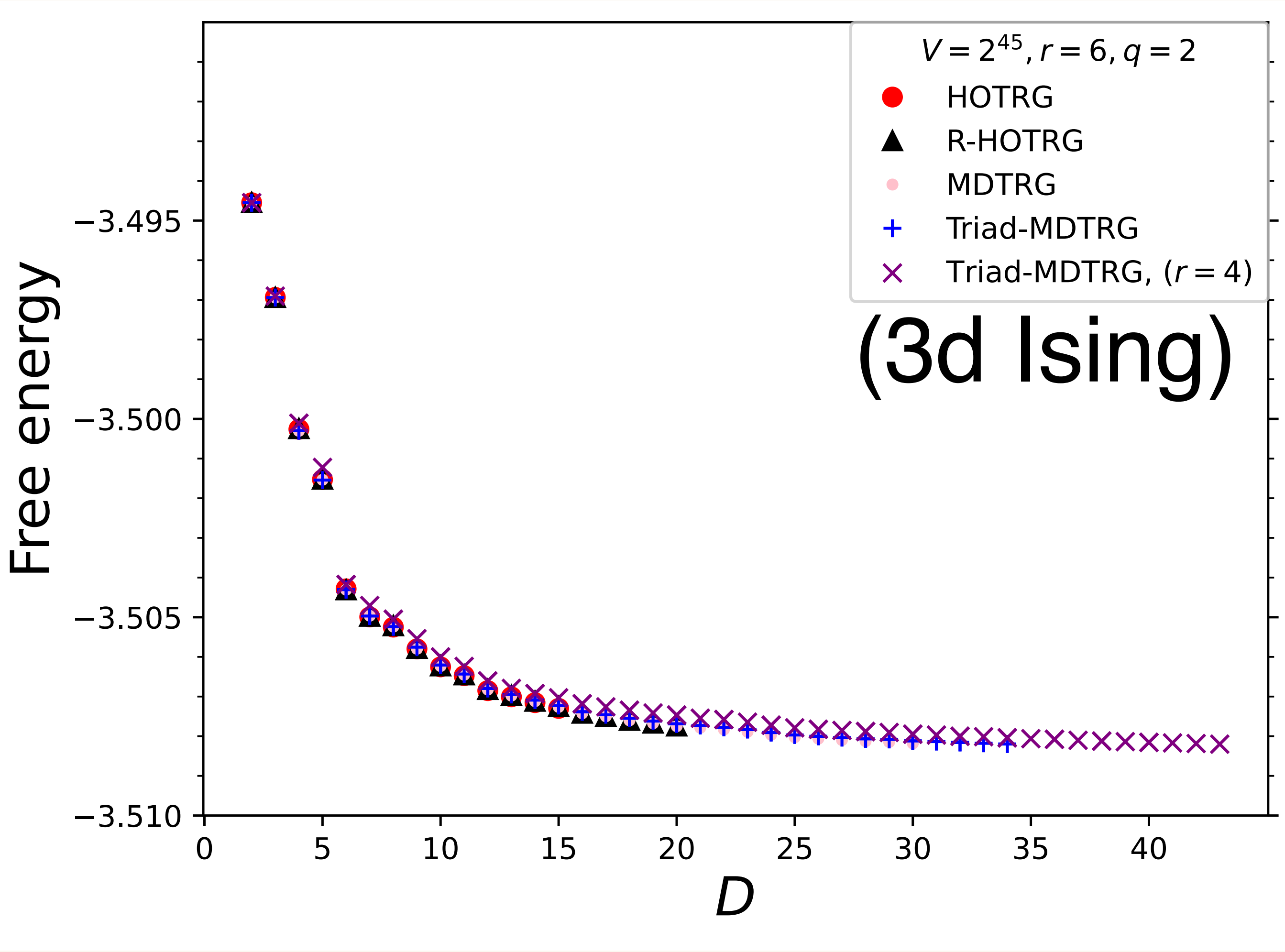
◉ Free energy density of 3d-Ising model



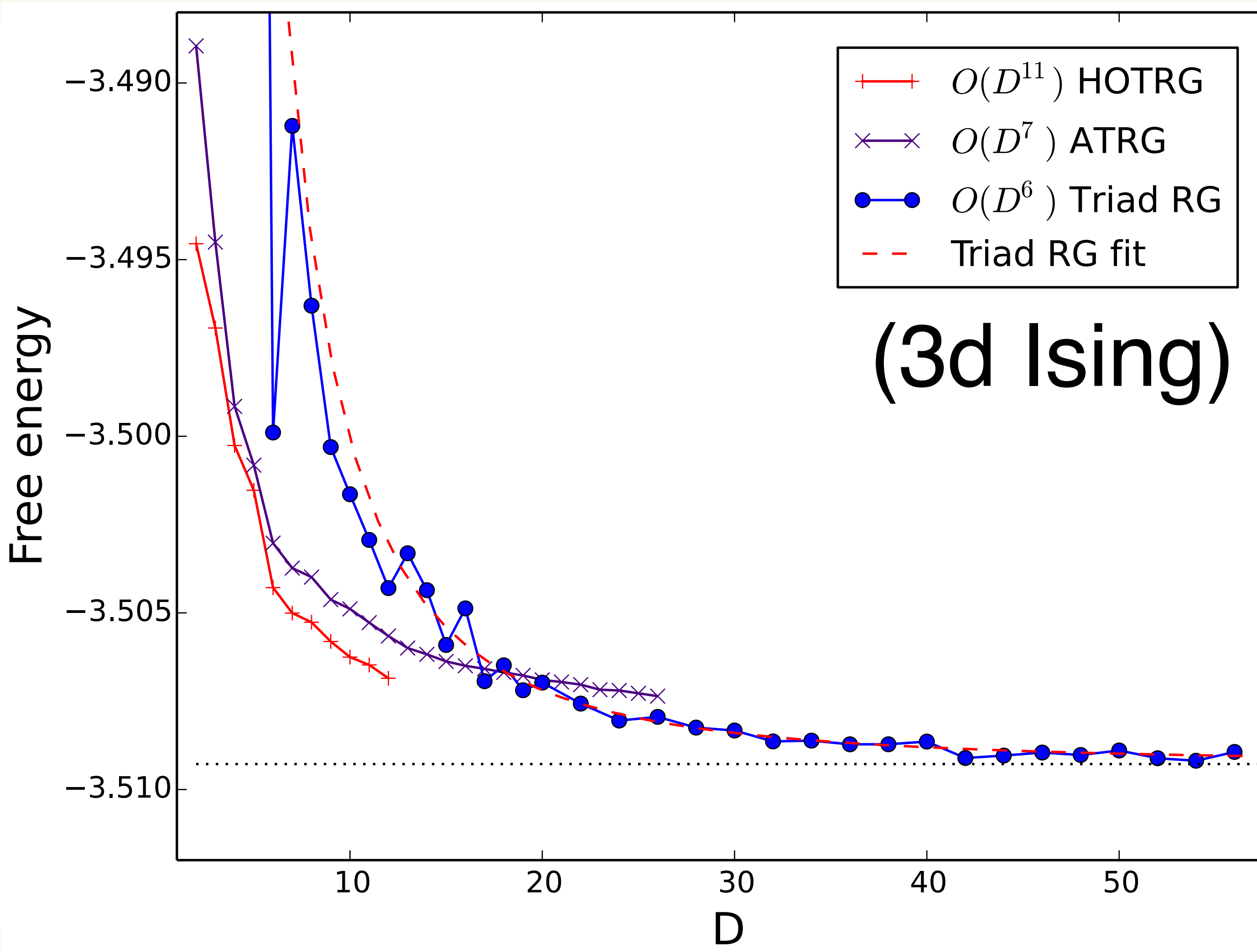
→ R-HOTRG, MDTRG, Triad-MDTRG results are consistent with HOTRG (additional systematic error is not dominant).

◉ Free energy density of 3d-Ising model

[K.N. arXiv:2307.14191]

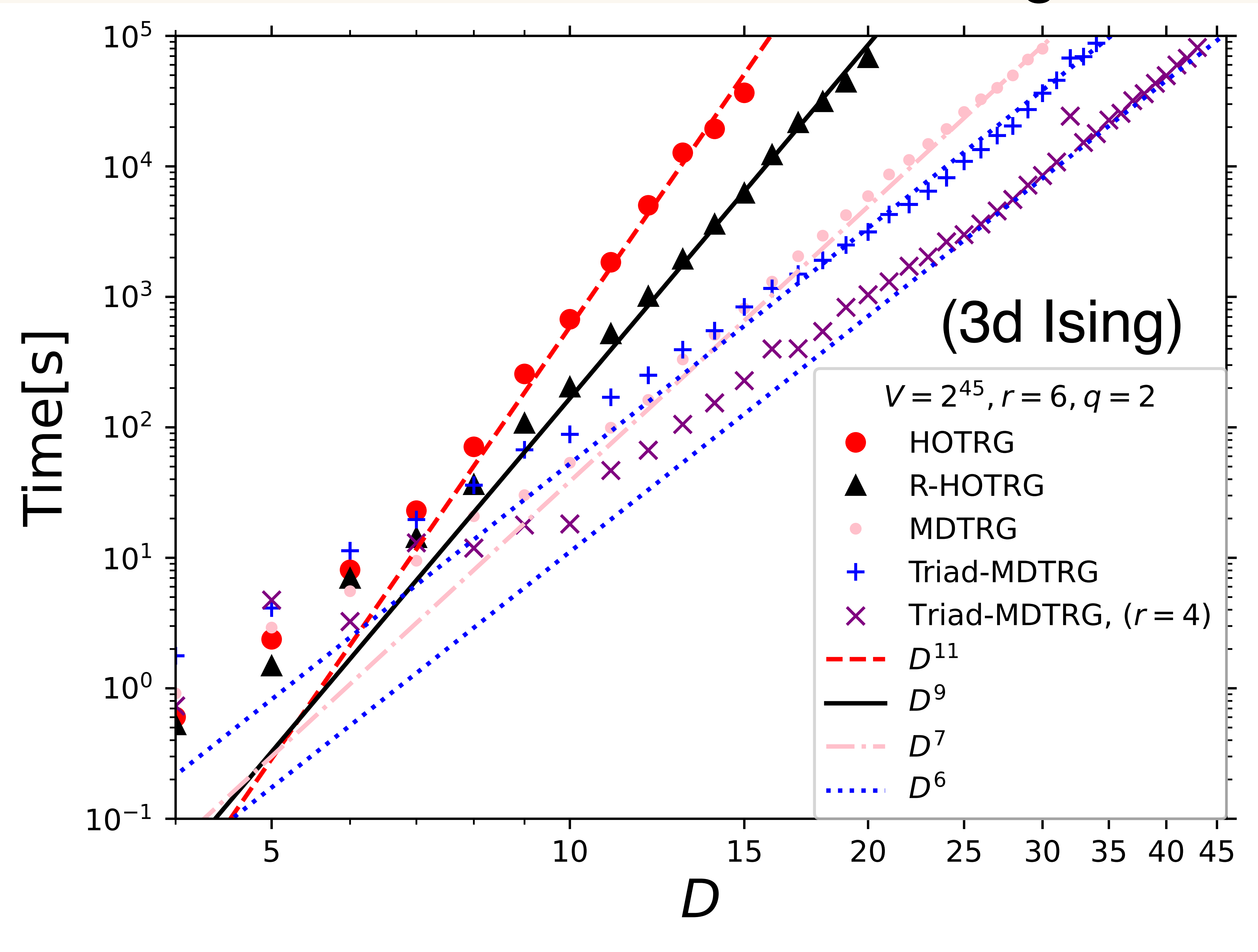


[D. Kadoh, K.N. arXiv:1912.02414]



→ R-HOTRG, MDTRG, Triad-MDTRG results are consistent with HOTRG (additional systematic error is not dominant).

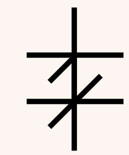
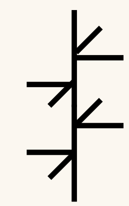
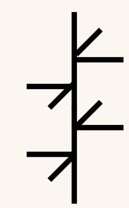
◉ Truncated bond dimension D scalings



→ The scalings of D are confirmed.

◉ Summary

◇ How about HOTRG with Randomized-SVD?

	with R-SVD	w/o R-SVD	unit-cell order	
◇ HOTRG	$O(D^{3d})$	$O(D^{4d-1})$		$2d$
◇ MDTRG	$O(D^{2d+1})$	$O(D^{3d})$		$d+1$
◇ Triad-MDTRG	$O(D^{d+3})$	$O(D^{3d})$		$d+1$

◇ Can we reduce the systematic error from decomposition?

→ R-HOTRG, MDTRG, and Triad-MDTRG produce consistent result with the HOTRG. The dominant systematic error is the truncation of the isometry.

Key ideas:

internal line oversampling, Isometry for unit-cell tensor.