Conserved non-Noether charge in general relativity : Physical definition vs. Noether's 2nd theorem

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References:

S. Aoki, T. Onogi and S. Yokoyama,
“Conserved charge in general relativity”,
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S. Aoki, T. Onogi and S. Yokoyama,
“Charge conservation, Entropy, and Gravitation”,
Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

S. Aoki and T. Onogi, “Conserved non-Noether charge in general relativity:
Physical definition vs. Noether’s 2nd theorem”,
arXiv:2201.09557[hep-th].

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I. Introduction

Energy in general relativity

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$$

$$T^{\mu\nu}(x) := \frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{matter}}}{g_{\mu\nu}(x)}$$

gravity matter $\kappa := 4\pi G$ energy momentum tensor (EMT)

$g_{\mu\nu}(x)$: metric $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

∇_μ: covariant derivative $\nabla_\mu v^\nu = v^\nu{}_{,\mu} + \Gamma^\nu_{\mu\alpha} v^\alpha$ $f_{,\mu} := \partial_\mu f$

$\Gamma^\nu_{\mu\alpha}$: Christoffer symbol $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[g_{\nu\beta,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu} \right] \longleftarrow \nabla_\mu g_{\alpha\beta} = 0$

$R_{\mu\nu\alpha}{}^\beta$: Reimann curvature $[\nabla_\mu, \nabla_\nu] v_\alpha = R_{\mu\nu\alpha}{}^\beta v_\beta$

$R_{\mu\nu}$: Ricci tensor $R_{\mu\nu} = R_{\mu\alpha\nu}{}^\alpha$ R: scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$

Λ: cosmological constant $g := \det g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$$

Bianchi identity

$$\nabla_\mu \left(R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R \right) = 0 \quad \Longrightarrow \quad \nabla_\mu T^\mu{}_\nu = 0$$

covariant conservation

However what we need for a conserved charge is $\partial_\mu(\sqrt{-g}T^\mu{}_\nu) = 0$

but in general $\partial_\mu(\sqrt{-g}T^\mu{}_\nu) \neq 0$ even though $\nabla_\mu T^\mu{}_\nu = 0$

What is a (conserved) energy in GR ?

(Textbook) answers

Traditional give up covariance Landau-Lifshitz, Weber, 内山, Weinberg, Misner-Thorne-Wheeler, ⋯

$$\partial_\mu \left[\sqrt{-g} (T^\mu{}_\nu + \underline{t^\mu{}_\nu}) \right] = 0$$

modify the EMT to satisfy the conservation law. Einstein

$t_{\mu\nu}$ is not covariant (pseudo-tensor). gravitational energy ? (Einstein)

$$E = \int_V d^3x \sqrt{-g} (T^\mu{}_\nu + t^\mu{}_\nu)$$

more than 2 particles, Cartesian, asymptotic flat, ⋯

This violates the fundamental principle of GR.

Modern give up a local definition of energy Wald, ⋯

$$E = \int_{r \to \infty} dS \,(\text{quasi-local energy}) \longleftrightarrow E = \int dV \,\cancel{(\text{local energy})}$$

Komar, Bondi, Arnowitt-Deser-Misner, Gibbons-Hawking, Brown-York

No local gravitational energy ?

No unified definition (case by case). asymptotic behavior, subtraction

Both are not satisfactory. Alternative ?

Plan of my talk

I. Introduction

II. Noether's 2nd theorem and conserved charges in general relativity

- II.1. Noether's 2nd theorem in general relativity
- II.2. Non-covariant conserved charge: pseudo-tensor
- II.3. Covariant conserved charge: Komar integral
- II.4. Cautions on charges from Noether's 2nd theorem

III. Our physical definition vs. Noether's 2nd theorem in general relativity

- III.1. Energy conservation by symmetry
- III.2. Energy conservation without symmetry
- III.3. Conserved charge in the absence of energy conservation

IV. Conclusion and discussions

II. Noether's 2nd theorem and conserved charges in general relativity

S. Aoki and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", arXiv:2201.09557[hep-th]..

II.1. Noether's 2nd theorem in general relativity

内山龍雄「一般相対性理論」　R. Utiyama, PTP 72 (1984) 83.

Lagrangian density $L = L_G + L_M$

$$L_G = \frac{1}{2\kappa}\sqrt{-g}(R - 2\Lambda)$$ Einstein-Hilbert action

$$L_M = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$ scalar theory

Action $S_\Omega = \int_\Omega d^d x\, L$

variation $\delta_{g_{\mu\nu},\phi} S_\Omega = 0 \longrightarrow$ equation of motion (EOM)

$$E_G^{\mu\nu} := -\frac{\sqrt{-g}}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}(R - 2\Lambda) - 2\kappa T^{\mu\nu}\right) = 0$$

$$E_\phi := \sqrt{-g}\left(\nabla_\mu \nabla^\mu \phi - V'(\phi)\right) = 0$$

Invariance under "gauge" transformation

(infinitesimal) general coordinate transformation

$$\delta x^\mu := (x')^\mu - x^\mu = \xi^\mu(x)$$

$$\delta\phi(x) := \phi'(x') - \phi(x) = 0$$

$$\delta g_{\mu\nu}(x) := g'_{\mu\nu}(x') - g_{\mu\nu}(x) = -\xi^\alpha{}_{,\mu}(x) g_{\alpha\nu}(x) - \xi^\alpha{}_{,\nu}(x) g_{\mu\alpha}(x)$$

Invariance $\quad \delta S_\Omega = \int_\Omega d^dx\, \xi^\alpha \left[2\partial_\mu(E_G^{\mu\nu} g_{\nu\alpha}) - E_G^{\mu\nu} g_{\mu\nu,\alpha} - E_\phi \partial_\alpha \phi\right] + \int_\Omega d^dx\, \partial_\mu J^\mu[\xi] = 0$

(1) boundary of Ω $\quad x \in \partial\Omega \quad \xi^\mu(x) = \xi^\mu{}_{,\alpha}(x) = \xi^\mu{}_{,\alpha\beta}(x) = 0$

arbitrary choice of Ω

$$\left[2\partial_\mu(E_G^{\mu\nu} g_{\nu\alpha}) - E_G^{\mu\nu} g_{\mu\nu,\alpha} - E_\phi \partial_\alpha \phi\right] = 0 \qquad d \text{ constraints}$$

(2) arbitrary ξ^μ and Ω

$$\boxed{\partial_\mu J^\mu[\xi] = 0}$$

conserved

conserved current from Noether's 2nd theorem

$$J^\mu[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right] = A^\mu{}_\nu\xi^\nu + B^\mu{}_\nu{}^\alpha\xi^\nu{}_{,\alpha} + C^\mu{}_\nu{}^{\alpha\beta}\xi^\nu{}_{,\alpha\beta}$$

$$\partial_\mu A^\mu{}_\nu = 0$$ conserved

$$\partial_\mu J^\mu[\xi] = 0$$

arbitrary ξ^ν

$$A^\mu{}_\nu + \partial_\alpha B^\alpha{}_\nu{}^\mu = 0$$

$$B^\mu{}_\nu{}^\alpha + B^\alpha{}_\nu{}^\mu + 2\partial_\beta C^\beta{}_\nu{}^{\mu\alpha} = 0$$

$$C^\mu{}_\nu{}^{\alpha\beta} + C^\beta{}_\nu{}^{\mu\alpha} + C^\alpha{}_\nu{}^{\beta\mu} = 0$$

$$A^\mu{}_\nu = -\partial_\alpha\tilde{B}^\alpha{}_\nu{}^\mu, \quad \tilde{B}^\alpha{}_\nu{}^\mu := \frac{1}{2}B^{[\alpha}{}_\nu{}^{\mu]} - \frac{1}{3}\partial_\beta C^{[\alpha}{}_\nu{}^{\mu]\beta}$$ total derivative

These conservation equations hold for arbitrary off-shell $g_{\mu\nu}, \phi$

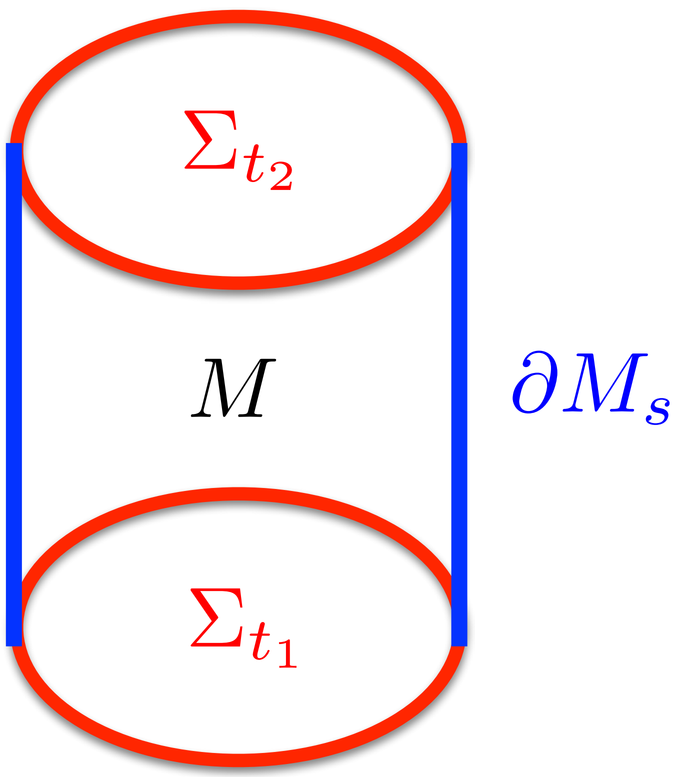
without using EOM.

II.2. Non-covariant conserved charge: pseudo-tensor

S. Aoki and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", arXiv:2201.09557[hep-th]..

non-covariant off-shell conserved current density

$$A^\mu{}_\nu = \frac{\sqrt{-g}}{2\kappa}\left[2R^\mu{}_\nu + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right]$$



$$\partial M = \partial M_s \oplus \Sigma_{t_2} \ominus \Sigma_{t_1}$$ boundaries

$$0 = \int_M d^dx\,\partial_\mu A^\mu{}_\nu = \underline{\int_{\Sigma_{t_2}} (d^{d-1}x)_\mu A^\mu{}_\nu - \int_{\Sigma_{t_1}} (d^{d-1}x)_\mu A^\mu{}_\nu} + \int_{\partial M_s} (d^{d-1}x)_\mu \underbrace{A^\mu{}_\nu}_{= 0 \text{ on } \partial M_s}$$

$$\Rightarrow \quad Q_{\text{pseudo},\nu}(t) := \int_{\Sigma_t} (d^{d-1}x)_\mu A^\mu{}_\nu$$ conserved

EOM $E_G^{\mu\nu} = 0$ $\Rightarrow$ $A^\mu{}_\nu = \sqrt{-g}\,(T^\mu{}_\nu + t^\mu{}_\nu)$

$$t^\mu{}_\nu := \frac{1}{2\kappa}\left[R^\mu{}_\nu + \frac{\delta^\mu_\nu}{2}(R-2\Lambda) + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right]$$

Einstein's pseudo-tensor

energy of gravitational field ?

conserved energy from pseudo-tensor

$$E_{\rm pseudo} = -\int_\Sigma [d^{d-1}x]_0 \sqrt{-g}\left(T^0{}_0 + t^0{}_0\right) = \int_{\partial\Sigma}[d^{d-2}x]_{0k}\tilde{B}^k{}_0{}^0$$

quasi-local expression

$$A^\mu{}_\nu = -\partial_\alpha \tilde{B}^\alpha{}_\nu{}^\mu$$

II.3. Covariant conserved charge: Komar integral

S. Aoki and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", arXiv:2201.09557[hep-th]..

Covariant off-shell conserved current density

$$J^\mu[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right] \qquad \partial_\mu J^\mu[\xi] = 0$$

Conserved charge (Komar integral)

Komar, PR113(1959)934

$$Q_{\rm Komar}[\xi] = \int_\Sigma [d^{d-1}x]_\mu J^\mu[\xi] = \frac{1}{2\kappa}\int_{\partial\Sigma}[d^{d-2}x]_{\mu\nu}\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}$$

quasi-local expression

A different choice of ξ gives a different conserved charge.

II.3.1 Komar energy

$$J^\mu[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right]$$

ξ^μ : Killing vector $\quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

isometry: metric is invariant under the general coordinate transformation by ξ^μ

$$E_{\rm Komar} = \frac{1}{\kappa}\int_\Sigma [d^{d-1}x]_\mu \sqrt{-g} R^\mu{}_\nu \xi^\nu = \int_\Sigma [d^{d-1}x]_\mu \sqrt{-g}\left[2T^\mu{}_\nu\xi^\nu - \frac{2}{d-2}T\xi^\mu + \frac{2}{\kappa(d-2)}\Lambda\xi^\mu\right]$$

$$T := T^\mu{}_\mu$$

Einstein equation $\quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$

Killing vector

$$\nabla_\nu\nabla^{[\mu}\xi^{\nu]} = 2g^{\mu\alpha}\nabla_\nu\nabla_\alpha\xi^\nu = 2g^{\mu\alpha}[\nabla_\nu,\nabla_\alpha]\xi^\nu = 2R^\mu{}_\nu\xi^\nu$$

$$\Rightarrow \quad \lim_{\kappa\to 0} E_{\rm Komar} \neq E_{\rm flat}$$

II.3.2 Wald entropy

$S_{\rm Wald} = Q_{\rm Komar}[\xi = t + \Omega_H \varphi]$

Wald, PRD48 (1993)R3427

t^μ : stationary Killing vector φ^μ : axial Killing vector Ω_H : angular velocity

II.3.3 Energy from asymptotic symmetry

$E_{\rm asym} = Q_{\rm Komar}[\eta]$

η^ν : asymptotic time-like Killing vector($\nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu = 0$ at spacial infinity).

asymptotically flat: Poincare group ADM energy

asymptotically de Sitter: SO(d,1)

asymptotically Anti de Sitter: SO(d,2)

II.4 Cautions on charges from Noether's 2nd theorem

The current associated with local symmetries always conserved without using equations of motion.

Thus conservations of “energy” are merely identities and do not reflect dynamical properties of the system.

In addition, these charges from the Noether’s 2nd theorem are easily modified by an arbitrary total derivative, which can be added without changing equations of motion.

$$L \to L + \partial_\mu(\sqrt{-g}K^\mu) \quad \Longrightarrow \quad J^\mu[\xi] \to J^\mu[\xi] + \sqrt{-g}\left[\xi^{[\mu}\nabla_\nu K^{\nu]} - K^{[\mu}\nabla_\nu \xi^{\nu]}\right]$$

pseudo-tensor — non-covariant, ambiguity by total derivative

Komar integral — covariant, ambiguity by total derivative and ξ^μ

In $\kappa \to 0$ limit, they DON’T reduce to energy in the flat spacetime.

III. Our physical definition vs. Noether's 2nd theorem in general relativity

S. Aoki and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", arXiv:2201.09557[hep-th]..

III.1 Our proposal for conserved non-Noether charge

S. Aoki, T. Onogi and S. Yokoyama,
"Charge conservation, Entropy, and Gravitation",
Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

construct covariantly conserved current from EMT $J^\mu := T^\mu{}_\nu \zeta^\nu$ $\nabla_\mu T^\mu{}_\nu = 0$

condition $\boxed{\nabla_\mu J^\mu = T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0}$ solution ζ^ν always exists

$$\because \quad \nabla_\mu J^\mu = \underbrace{(\nabla_\mu T^\mu{}_\nu)}_{=0}\zeta^\nu + T^\mu{}_\nu \nabla_\mu \zeta^\nu$$

conserved charge $\boxed{Q[\zeta] = \int_\Sigma [dx^{d-1}]_\mu \sqrt{-g} T^\mu{}_\nu \zeta^\nu}$ covariant

$$\because \quad \partial_\mu(\sqrt{-g} T^\mu{}_\nu \zeta^\nu) = \sqrt{-g} \nabla_\mu (T^\mu{}_\nu \zeta^\nu) = 0$$

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu)$$ holds for an arbitrary vector

III.2. Energy conservation by symmetry

S. Aoki, T. Onogi and S. Yokoyama,
“Conserved charge in general relativity”,
Int. J. Mod. Phys. A36 (2021) 2150098, arXiv:2005.13233[gr-qc].

Lie derivative with vector ξ^μ $\mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

Killing vector $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ $\longrightarrow$ Symmetry (isometry) in GR

take $\zeta^\mu = \xi^\mu$ $\longrightarrow$ $T^\mu{}_\nu \nabla_\mu \zeta^\nu = \frac{1}{2} T^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 0$

energy $E = \int_\Sigma [dx^{d-1}]_\mu \sqrt{-g} T^\mu{}_\nu \xi^\nu$ ξ^ν : “time-like” Killing vector

This energy is conserved as a charge from Noether’s 1st theorem for a symmetry generated by ξ^ν acting on a fixed background metric $g_{\mu\nu}$

This gives a covariant and universal definition of a total energy.
In $\kappa \to 0$, it reduces to the standard one in the flat spacetime.

$$E = \int_\Sigma [dx^{d-1}]_\mu \sqrt{-g} T^\mu{}_\nu \xi^\nu$$ Conserved energy with a Killing vector

This definition has been found in some literature:

1. V. Fock, *TheTheory of Space, Time and Gravitation* (Pergamon Press, New York 1959)

2. A. Tautman, Kings Collage lecture notes on general relativity,
mimeographed note (unpublished), May-June 1958; Gen. Res. Grav. **34** (2002), 721-762,
Fock's book was cited.

3. A. Tautman's lecture notes was cited by Komar in PRD127(1962)1411.

However, this definition was forgotten in major textbooks (e.g. Landau-Lifshitz)
except a few.

4. R. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984),
p.286, footnote 3.

See also Hawking Ellis, lecture notes by Blau; Shiromizu, Sekiguchi (Japanese).

No applications. Let's try using this.

III.2.1 Vacuum energy

Metric $$ds^2 = -f(r)(dx^0)^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-2}^2 \qquad f(r) = 1 - \frac{2\Lambda r^2}{(d-2)(d-1)}$$

$$\longrightarrow \qquad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

"time-like" Killing vector $\xi^\nu = -\delta_0^\nu$

by definition $$E_{\text{our}}^{\text{vac}} = 0$$

on the other hand

$$E_{\text{pseudo}}^{\text{vac}} = E_{\text{Komar}}^{\text{vac}} = -\frac{2\Lambda\Omega_{d-2}}{(d-2)\kappa}\int_0^\infty dr\, r^{d-2} \begin{cases} = 0 & \Lambda = 0 \\ = -\Lambda \times \infty & \Lambda \neq 0 \end{cases}$$

$$\Omega_{d-2} := \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$$

III.2.1 Schwarzschild black hole energy

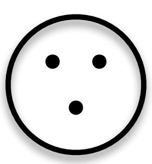
Metric
$$ds^2 = -(1+u)d\tau^2 - 2ud\tau dr + (1-u)dr^2 + r^2 d\Omega^2_{d-2}$$

Eddington-Finkelstein coordinates

$$u(r) := \delta u(r) - \frac{2\Lambda r^2}{(d-2)(d-1)}$$

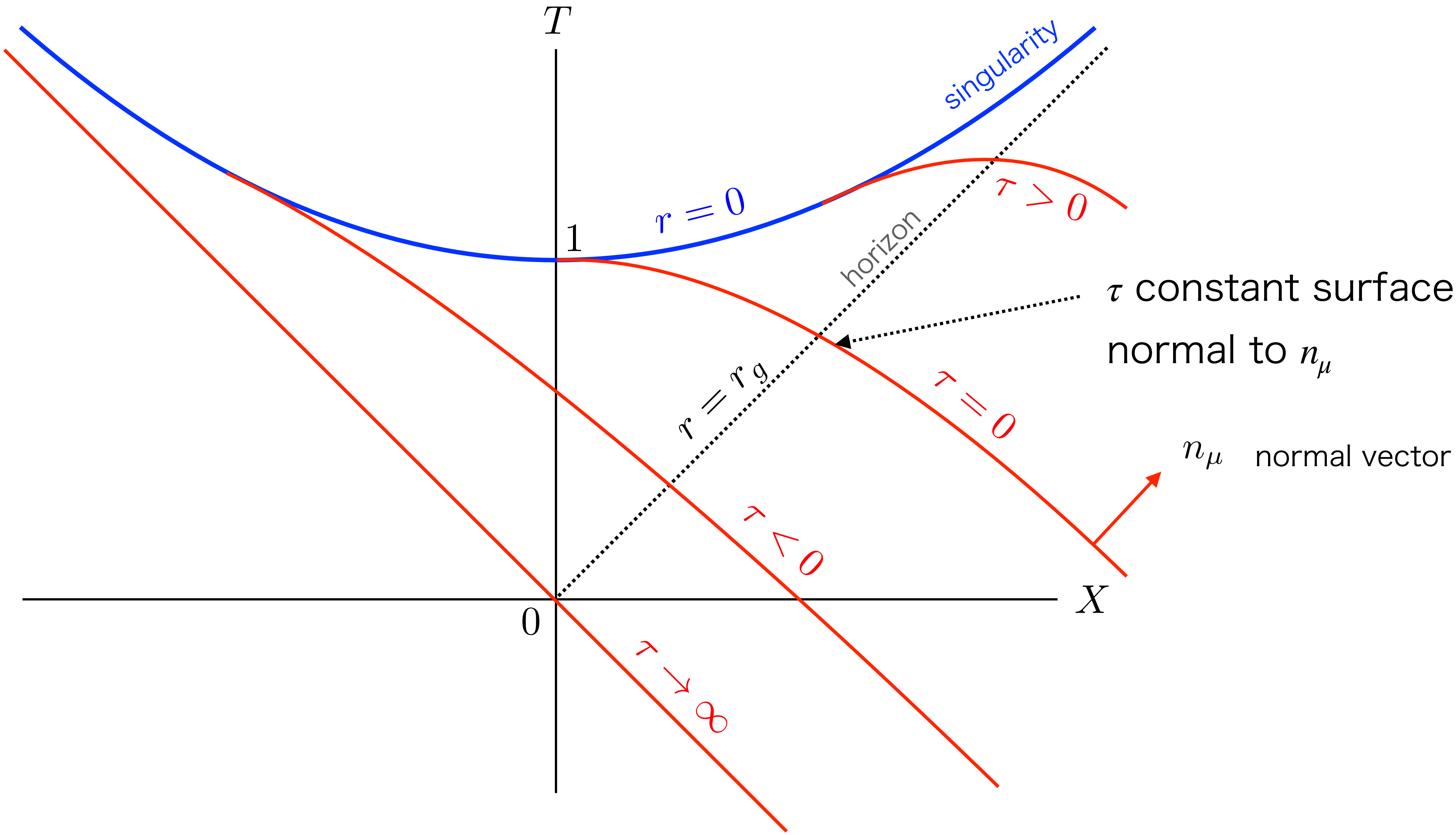
$$\delta u(r) := -\frac{2GM\theta(r)}{r^{d-3}}$$
$\theta(r)$ with $\theta(0)=0$ handles singularity at $r=0$

constant τ surface is always space-like even inside the horizon



normal vector $n_\mu = -\frac{1}{\sqrt{1-u}}\delta^\tau_\mu$ $\quad n_\mu n^\mu = -1$ $\quad$ time-like

$\Lambda = 0$ Kruskal-like coordinate



$\boxed{\text{"time-like"Killing vector}}$ $\xi^\mu = -\delta^\mu_\tau$ $\xi^\mu \xi_\mu = -(1+u) = \frac{2GM}{r^{d-3}} + \frac{2\Lambda r^2}{(d-2)(d-1)} - 1$

Remarks 1. Killing vector becomes space-like inside the horizon

2. In dS space, Killing vector also becomes space-like outside the cosmological horizon.

$\boxed{\text{Energy of black hole}}$

$$E_{BH} = -\int_{\tau:\mathrm{fix}} d^{d-1}x\,\sqrt{-g}T^\tau{}_\tau$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_d T_{\mu\nu}$$

$$T_{\mu\nu} = 0 \text{ at } r \neq 0$$

EMT of black hole ?

EMT for black hole

$$T^\tau{}_\tau = \frac{d-2}{16\pi G}\frac{\partial_r(r^{d-3}\delta u)}{r^{d-2}} = -\frac{(d-2)M}{8\pi}\frac{\delta(r)}{r^{d-2}} = T^r{}_r \qquad r^{d-3}\delta u(r) = -2GM\theta(r)$$

$$T^i{}_i = \frac{1}{16\pi G}\frac{\partial_r^2(r^{d-3}\delta u)}{r^{d-3}} = -\frac{1}{8\pi}\frac{\partial_r\delta(r)}{r^{d-3}}$$

1. This result is know at d=4 in the distributional approach $\theta(r) \to r^\lambda \ (\lambda \to 0)$

Balasin-Nachbagauer, Class. Quant. Grav. 10 (1993) 2271.

2. EMT is well-defined in the distributional sense without a product of δ functions

c.f. Geroch-Traschen, Conf. Proc. C 861214 (1986) 138.

3. black hole has non-perfect fluid EMT at $r=0$

$\longrightarrow$ black hole is not a vacuum solution to the Einstein equation

cf. Coulomb potential by a point charge is NOT a vacuum solution to Maxwell eq.

$$\nabla^2\left(\frac{1}{r}\right) = 0 \quad r \neq 0 \quad \longrightarrow \quad \nabla^2\left(\frac{1}{r}\right) \propto \delta(x)$$

energy of black hole

$$E^{\mathrm{BH}}_{\mathrm{our}} = -\int d^{d-1}x\,\sqrt{-g}\,T^{\tau}{}_{\tau} = \frac{(d-2)\Omega_{d-2}}{8\pi}\int_0^\infty dr\partial_r(M\theta(r)) = \frac{(d-2)\Omega_{d-2}}{8\pi}M$$

cf. pseudo/Komar(ADM) energy

$$E^{\mathrm{BH}}_{\mathrm{pseudo}} = E^{\mathrm{BH}}_{\mathrm{Komar}} = \frac{(d-3)\Omega_{d-2}}{4\pi}M + E^{\mathrm{vac}}_{\mathrm{Komar}}$$

$$\longrightarrow \qquad \frac{E^{\mathrm{BH}}_{\mathrm{Komar}} - E^{\mathrm{vac}}_{\mathrm{Komar}}}{E^{\mathrm{BH}}_{\mathrm{our}}} = \frac{2(d-3)}{d-2}$$

=1 only at d=4

II.3. Energy conservation without symmetry

S. Aoki, T. Onogi and S. Yokoyama,
“Charge conservation, Entropy, and Gravitation”,
Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

If $\xi^\mu = -\delta^\mu_0$ is not a Killing vector but the metric satisfies

$$T^\mu{}_\nu \nabla_\mu \xi^\nu = -T^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0$$

$$E = -\int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, T^0{}_0$$

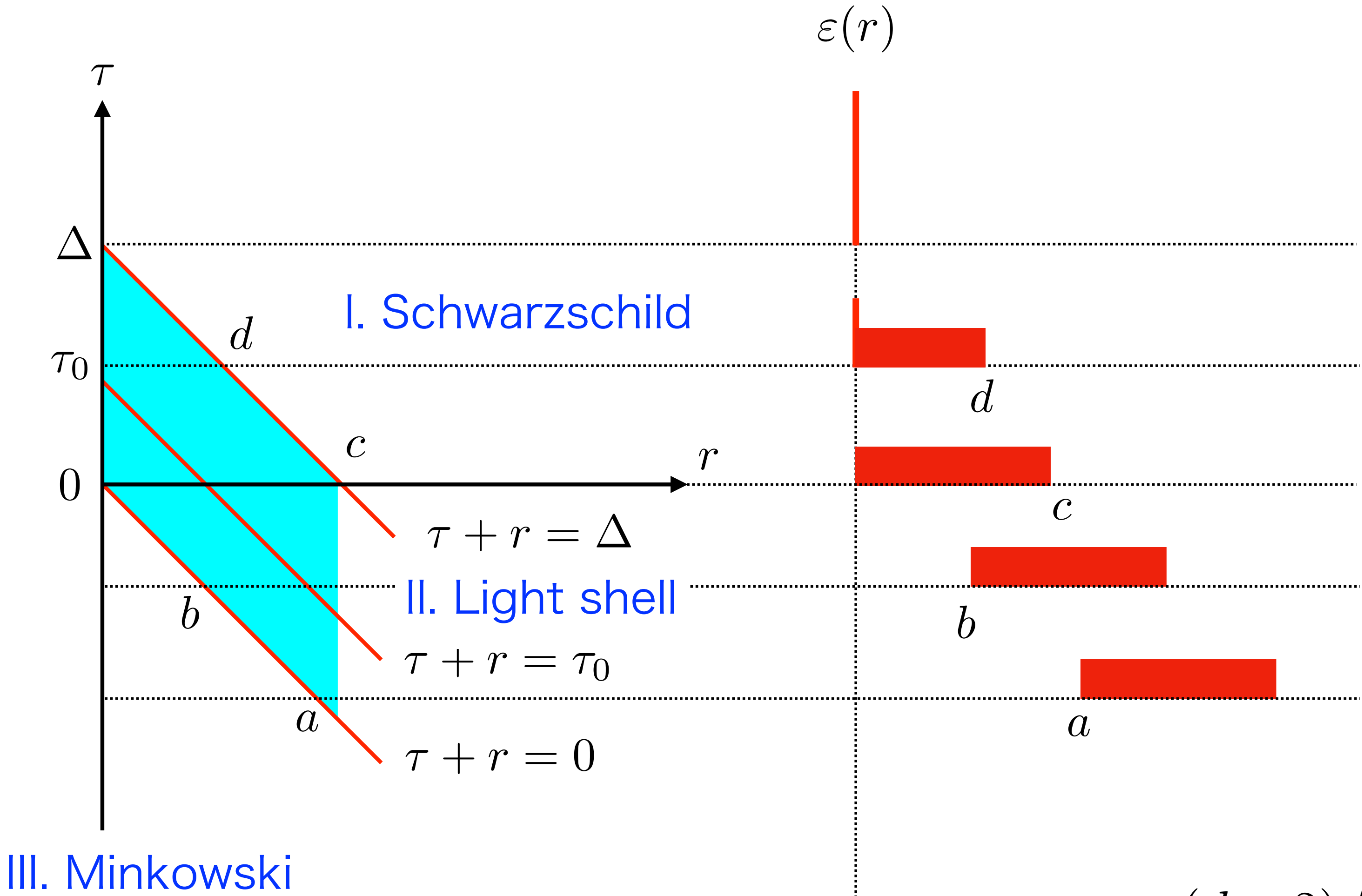
gives a conserved energy without symmetry.

Gravitational collapse for thick light shell

$$g_{\mu\nu}dx^\mu dx^\nu = -(1+u)d\tau^2 - 2ud\tau dr + (1-u)dr^2 + r^2\bar{g}_{ij}dx^i dx^j,$$ Eddington-Finkelstein

$$u(r,\tau) := -\frac{m(r,\tau)}{r^{d-3}}, \quad m(r,\tau) := \begin{cases} M\theta(r), & \tau + r > \Delta, \quad \mathrm{I} \\ M\theta(r)F\left(\frac{\tau+r}{\Delta}\right), & 0 \le \tau + r \ge \Delta, \quad \mathrm{II} \\ 0, & \tau + r < \Delta, \quad \mathrm{III} \end{cases}$$

- I: Schwarzschild
- II: Light shell
- III: Minkowski



$$F(0) = 0, F(1) = 1$$

In II, $\xi^\mu = -\delta^\mu_0$ is not Killing,

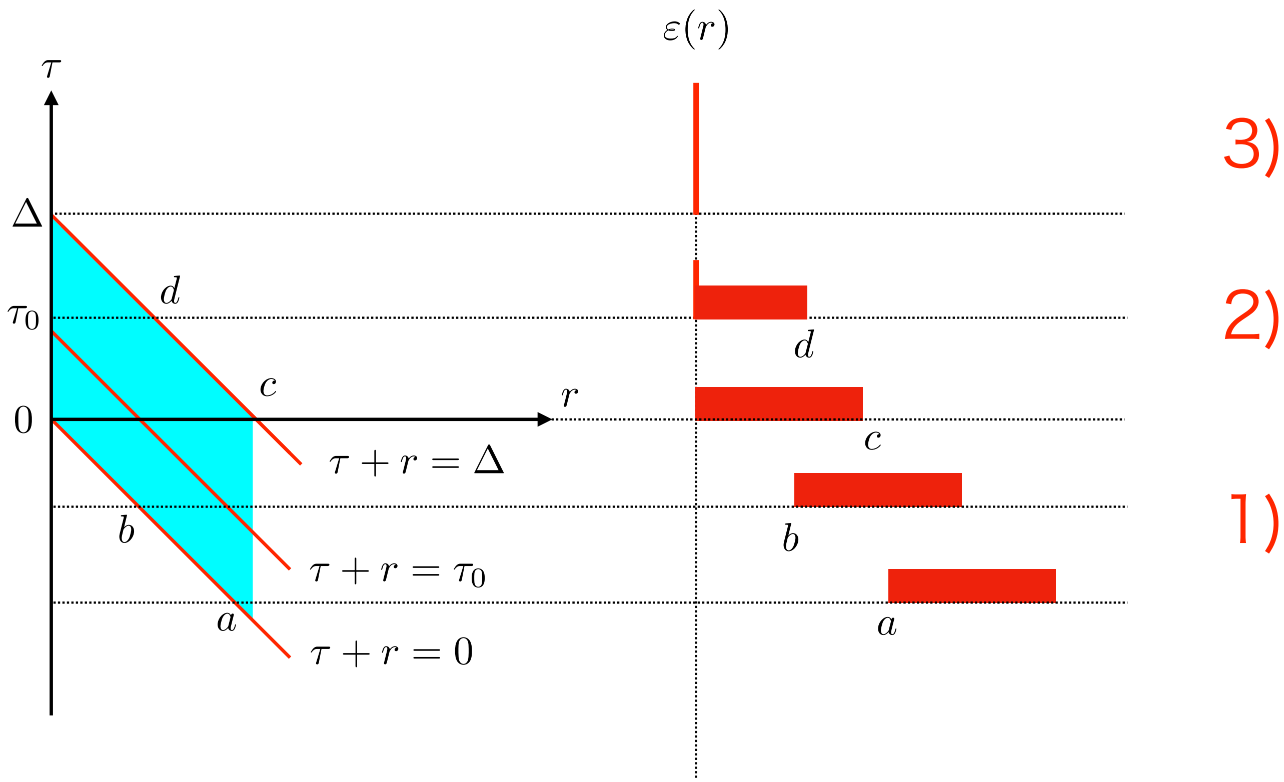
but $T^\mu{}_\nu \nabla_\mu \xi^\nu = -T^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0.$

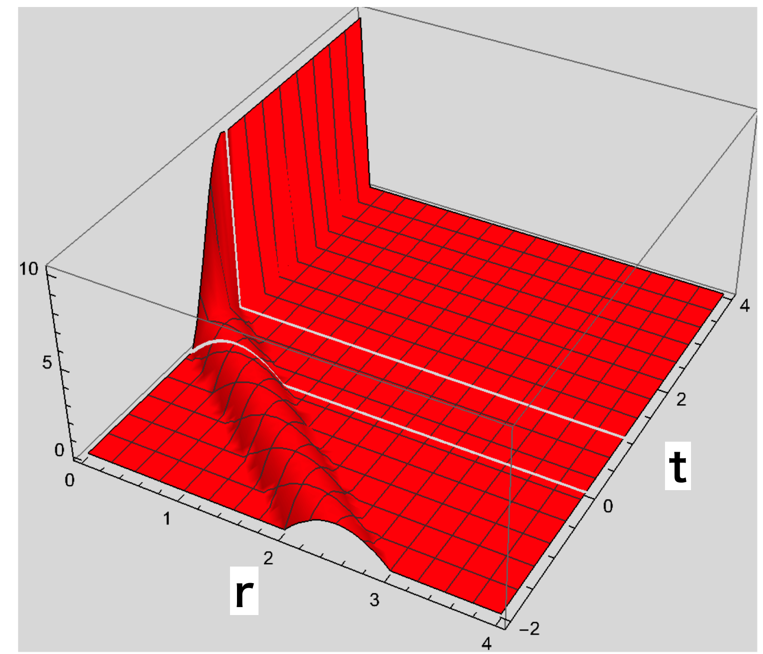
$$T^0{}_0 = \frac{(d-2)}{16\pi G}\frac{\left(r^{d-3}u\right)_r}{r^{d-2}}, \quad T^r{}_r = \frac{(d-2)}{16\pi G}\left[\frac{\left(r^{d-3}u\right)_r}{r^{d-2}} - \frac{2(u)_\tau}{r}\right],$$

$$T^0{}_r = \frac{(d-2)}{16\pi G}\frac{(u)_\tau}{r} = -T^r{}_0,$$

Conserved energy

$$E(\tau) = -\int d^{d-1}x\,\sqrt{-g}\,T^{0}{}_{0} = \frac{(d-2)V_{d-2}}{16\pi G}\int_0^\infty dr\;[m(r,\tau)]_r\,,\quad V_{d-2} := \int d^{d-2}x\,\sqrt{\bar{g}},$$





1) $\tau < 0$

$$E(\tau) = = \frac{(d-2)MV_{d-2}}{16\pi G}\int_{-\tau}^{\Delta-\tau} dr\,\partial_r(\theta F) = \frac{(d-2)MV_{d-2}}{16\pi G} := E_{\rm tot}.$$

light-shell

2) $0 \le \tau \le \Delta$

$$E(\tau) = E_{\rm tot}\int_0^{\Delta-\tau} dr\,\partial_r(\theta F) = E_{\rm tot}\left[1 - \theta(0)F\left(\frac{\tau}{\Delta}\right)\right] = E_{\rm tot},$$

$$\partial_r(\theta F) = \delta(r)F + \partial_r F \longrightarrow E(\tau) = E_{\rm tot}\left[F\left(\frac{\tau}{\Delta}\right) + \left\{F(1) - F\left(\frac{\tau}{\Delta}\right)\right\}\right] = E_{\rm tot},$$

BH light-shell

3) $\tau > \Delta$

$$E(\tau) = E_{\rm tot}\int_0^\infty dr\,\delta(r) = E_{\rm tot}$$

BH

$$E_{\mathrm{our}} = \frac{(d-2)\Omega_{d-2}}{8\pi} M$$

cf. energy from Noether's 2nd theorem

$$E_{\mathrm{2nd}} = \frac{(d-3)\Omega_{d-2}}{4\pi} M + E_{\mathrm{2nd}}^{\mathrm{vac}}$$

II.4. Conservation charge in the absence of energy conservation

S. Aoki, T. Onogi and S. Yokoyama,
“Charge conservation, Entropy, and Gravitation”,
Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

A solution ζ^ν to $T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0$

Introduce a parameter η to define a time direction as $v^\mu := \dfrac{dx^\mu(\eta)}{d\eta}$.

Take $\zeta^\nu = \beta v^\nu$ $\Rightarrow$

$$A^0(x)\partial_0\beta(x) + \sum_{\mu\neq 0} A^\mu(x)\partial_\mu\beta(x) + B(x)\beta(x) = 0.$$

1st order linear PDE

$$A^\mu(x) := T^\mu{}_\nu(x)v^\nu(x), \quad B(x) := T^\mu{}_\nu(x)\partial_\mu v^\nu(x) + T^\mu{}_\nu(x)\Gamma^\nu_{\mu\alpha}v^\alpha(x)$$

If an initial value $\beta(x^0, {}^\forall\vec{x})$ is given at some x^0

$\longrightarrow$ A solution exists (unless $A^0(x)$ identically vanishes).

For a spherically symmetric system, a solution is known as a Kodama vector.

Kodama'80

There exists a generalized conserved charge without symmetry in GR.

Homogeneous and Isotropic expanding Universe

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j$$ Freedman-Lemaitre-Robertson-Walker metric

$$\tilde{R}_{ij} = (d-2)k\tilde{g}_{ij}$$ $k = 1$(sphere), 0(flat), -1(hyperbolic)

EMT (perfect fluid) $T^0{}_0 = -\rho(x^0),\ T^i{}_j = P(x^0)\delta^i_j,\ T^0{}_j = T^i{}_0 = 0$

conservation $\nabla_\mu T^\mu{}_\nu = 0 \longrightarrow \dot{\rho} + (d-1)(\rho + P)\frac{\dot{a}}{a} = 0$

Energy $E(x^0) := -\int d^{d-1}x\sqrt{-g}\,T^0{}_0 = V_{d-1}a^{d-1}\rho,\quad V_{d-1} := \int d^{d-1}x\sqrt{\tilde{g}}$

$$\longrightarrow \quad \frac{\dot{E}}{E} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} \neq 0$$ not conserved

$T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0 \qquad \zeta^\mu = -\beta(x^0)\delta^\mu_0 \longrightarrow -T^0{}_0\dot{\beta} - T^i{}_j\Gamma^j_{i0}\beta = \rho\dot{\beta} - (d-1)P\frac{\dot{a}}{a}\beta = 0$

charge $S(x^0) := \int d^{d-1}x\sqrt{-g}(-T^0{}_0)\beta = V_{d-1}a^{d-1}\rho\beta$

$$\longrightarrow \quad \frac{\dot{S}}{S} = \frac{\dot{E}}{E} + \frac{\dot{\beta}}{\beta} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} + (d-1)\frac{P}{\rho}\frac{\dot{a}}{a} = 0$$ conserved !

Physical interpretation of the generalized energy

charge density

$$s(x^0) = e(x^0)\beta(x^0)$$

energy density

$$e(x^0) = \rho(x^0)v(x^0)$$

volume density

$$v(x^0) = a(x^0)^{d-1}$$

$$\longrightarrow \quad \frac{ds}{dx^0} = \frac{de}{dx^0}\beta + e\frac{d\beta}{dx^0} = \left(\frac{de}{dx^0} + P\frac{dv}{dx^0}\right)\beta \quad \Longleftrightarrow \quad Tds = de + Pdv$$

1st law of thermodynamics

$\Rightarrow$ S entropy $\quad \beta = \frac{1}{T}$ inverse temperature

c.f. Gravity is an entropic force (emergent gravity) Jacobson'95, Verlinde'11

Entropy of the Universe is conserved during its expansion.

$$\frac{\dot\beta}{\beta} = (d-1)\frac{P}{\rho}\frac{\dot a}{a} > 0 \longrightarrow$$

Temperature of the Universe decreases as it expands, so as to conserve the total entropy.

Noether charge from 2nd theorem

pseudo-tensor

$$A^0{}_0 = \frac{\sqrt{-g}}{2\kappa}\left[2R^0{}_0 + g^{0\alpha}\Gamma^\beta_{\beta 0,\alpha} - g^{\alpha\beta}\Gamma^0_{\alpha\beta,0}\right] = 0$$

$$\longrightarrow \qquad E^{\rm FLRW}_{\rm pseudo} = 0$$

Komar integral

$$J^\mu[\xi] = \frac{1}{\kappa}\sqrt{-g}\nabla_\nu[\nabla^{[\mu}\xi^{\nu]}] \qquad \xi^\mu = \gamma(x^0, r)\delta^\mu_0$$

boundary condition at $r = r_\infty$ (spatial infinity/boundary)

$$\lim_{r\to r_\infty} r^{d-2}\sqrt{1-kr^2}\partial_r\gamma(x^0,r) = 0$$

$$\Rightarrow \quad E^{\rm FLRW}_{\rm Komar} = \int d^{d-1}x\, J^0(x) = -\frac{\Omega_{d-2}}{2\kappa}a^{d-3}(x^0)r^{d-2}\sqrt{1-kr^2}\partial_r\gamma(x^0,r)\Big|^{r=r_\infty}_{r=0} = 0$$

Both are conserved but physically trivial !

IV. Summary and discussion

Summary

Conserved charges from Noether's 2nd theorem are unphysical.

pseudo-tensor   non-covariant, ambiguity by total derivative

Komar integral   covariant, ambiguity by total derivative and ξ^μ

In $\kappa \to 0$ limit, they DON'T reduce to energy in the flat spacetime.

We have proposed a conserved charge without symmetry (non-Noether charge), in general relativity, which can be identified as entropy.

Thus, entropy is a source of gravity. c.f. Jacobson'95, Verlinde'11, " gravity is an entropic force".

A total entropy in the whole system is always conserved, as nothing can escape from a censorship of gravity.

天網恢恢疎にして漏らさず。

At the same time, (local) temperature can be defined so as to conserve entropy in a gravitational system.

Discussions (my personal view)

Since our entropy is defined from EMT, gravitational fields with $T_{\mu\nu} = 0$ can not carry entropy. Thus entropy is a quantity associated with matters.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$$

If $T_{\mu\nu} = 0$ in a system at an initial x^0, it remains so at late x^0 .

Further studies

initial temperature of matter   How can we determine it from EMT ?

binary stars   How do they loose “energy” to merge ?

colliding gravitational waves   How do they become BH ?

Quantum gravity

Gravitational fields classically carry no energy/entropy.

No exchange of energy/entropy between matters and gravitational fields.

Option 1

No quantum gravity.   Quantum matters and classical gravity.

Option 2

Quantum gravity: graviton without observed)on-shell) energy/entropy.

condition ? $$\langle \text{pure gravity} \,|\, T_{\mu\nu} \,|\, \text{pure gravity} \rangle = 0$$