Complex Langevin analysis of gauge theories with a theta term

Akira Matsumoto

The Graduate University for Advanced Studies (SOKENDAI)

@ RIKEN R-CCS (online)

March 3, 2021

Contents

- Introduction
- 2D U(1) gauge theory

  with M. Hirasawa, J. Nishimura, A. Yosprakob

  https://doi.org/10.1007/JHEP09(2020)023 [arXiv:2004.13982]
- 4D SU(2) gauge theory (ongoing work)

  with K. Hatakeyama, M. Hirasawa, M. Honda, Y. Ito, J. Nishimura, A. Yosprakob
- Summary and discussion

Contents

- Introduction
- 2D U(1) gauge theory
- 4D SU(2) gauge theory (ongoing work)
- Summary and discussion

Gauge theory with a θ term

☆ θ term: topological property of the gauge theory, nonperturbative

$$S_\theta = -i\theta Q = -\frac{i\theta}{32\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]$$

- strong CP problem of QCD
  The experimental bound of θ is extremely small: $|\theta| < 10^{-10}$
  → no reason for it theoretically

- phase structure of 4D SU(N) YM around $\theta = \pi$
  interesting prediction by the 't Hooft anomaly matching

Numerical study of the θ term

Monte Carlo simulation of the lattice gauge theory with a θ term

- θ term is purely imaginary → the action “S” is complex
- impossible to interpret Boltzmann weight “e^{-S}” as a probability
  → sign problem

- It arises in various cases…
  - finite density QCD
  - chiral fermion
  - real time dynamics
  - etc.

Approach to complex action systems

- Reweighting method
  - treat the phase of e^{-S} as an observable
  - does not work if the phase oscillates rapidly
- Lefschetz thimble method
  - reduce the phase oscillation by deforming the integral path in the complex plane
- Complex Langevin method
  - low computational cost
  - has to meet a condition to justify the result
- Tensor renormalization group
  - easy to increase the system size
  - higher dimension is difficult

Complex Langevin method

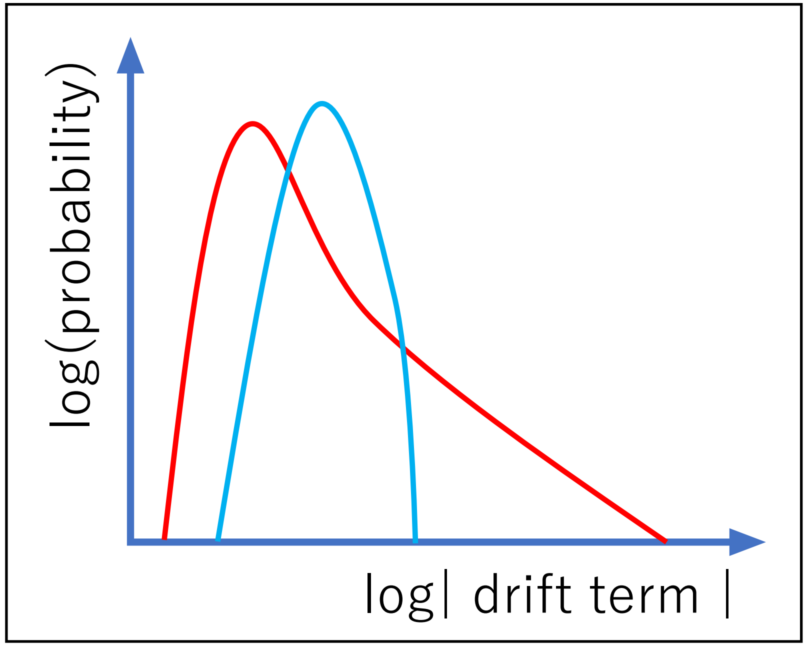
complex Langevin method (CLM) [G. Parisi (1983)] [J. R. Klauder (1983)]

- Langevin equation: fictitious time evolution of dynamical variables
- real variable → complex variable

$$\frac{dz(t)}{dt} = -\frac{\partial S(t)}{\partial z} + \eta(t) \qquad x \longmapsto z = x + iy$$

drift term: $-\frac{\partial S(t)}{\partial z}$

Gaussian noise: $\eta(t)$



- do not use "probability" → ~~sign problem~~
- condition required to be satisfied

The distribution of the drift term falls off exponentially or faster.

[K. Nagata, J. Nishimura, S. Shimasaki (2016)]

Contents

- Introduction
- 2D U(1) gauge theory
  - ➢Naive implementation of CLM
  - ➢Introducing a puncture
- 4D SU(2) gauge theory (ongoing work)
- Summary and discussion

Contents

- Introduction
- 2D U(1) gauge theory
  - ➢Naive implementation of CLM
  - ➢Introducing a puncture
- 4D SU(2) gauge theory (ongoing work)
- Summary and discussion

2D U(1) gauge theory

- total action = kinetic term + θ term

$$S_g = \frac{1}{4g^2}\int d^2x F_{\mu\nu}F_{\mu\nu} \qquad S_\theta = -i\theta Q \qquad Q = \frac{1}{4\pi}\int d^2x \epsilon_{\mu\nu}F_{\mu\nu}$$

- exactly solvable on a finite lattice → good test ground
- lattice gauge action

$$S_g = -\frac{\beta}{2}\sum_n \left(P_n + P_n^{-1}\right) \qquad P_n = U_{n,\hat{1}}U_{n+\hat{1},2}U^{-1}_{n+\hat{2},1}U^{-1}_{n,2} = e^{ia^2 F_{n,12}} \qquad \beta = \frac{1}{(ga)^2}$$

- topological charge ⋯ two types of definition

Topological charge on a 2D lattice

- exactly integer for a finite lattice spacing

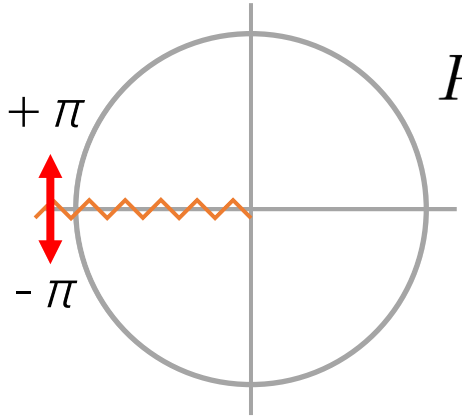
$$Q_{\log} := -\frac{i}{2\pi}\sum_n \log P_n = \frac{1}{4\pi}\sum_n a^2 \epsilon_{\mu\nu} F_{n,\mu\nu}$$

← changes discontinuously at the branch cut (arg $P_n = \pi$)

$$\log z = \log|z| + i \arg z \qquad -\pi < \arg z \leq \pi$$

$$\sum_n \log P_n = \log \prod_n P_n + 2\pi i \mathbb{Z} \qquad \prod_n P_n = 1$$

$$P_n = e^{ia^2 F_{n,12}}$$



- approaches integer in the continuum limit

$$Q_{\sin} = -\frac{i}{4\pi}\sum_n \left(P_n - P_n^{-1}\right) = \frac{1}{2\pi}\sum_n \sin\left(a^2 F_{n,12}\right)$$

← analogy of "clover leaf" in the 4D lattice

CLM for the lattice gauge theory

- discretized complex Langevin equation for the link variable $U_{n,\mu}$

$$U_{n,\mu}(t+\epsilon) = \exp\left[-i\epsilon D_{n,\mu} S(t) + i\sqrt{\epsilon}\eta_{n,\mu}(t)\right] U_{n,\mu}(t)$$

$$U_{n,\mu} \in \mathbb{C} \setminus \{0\}$$

drift term

- gauge group is extended: $\mathrm{U}(1) \to \mathbb{C} \setminus \{0\}$

$$U_{n,\mu}^{\dagger} \to U_{n,\mu}^{-1}$$

- drift term and observables have to respect holomorphicity
- control the non-unitarity by gauge cooling
  - gauge transformation to keep the link variable close to unitary
  - not affect gauge invariant observables

[E. Seiler, D. Sexty, I.-O. Stamatescu (2013)] [K. Nagata, J. Nishimura, S. Shimasaki (2016)]

Drift term from S_θ

on a torus…

definition	value	drift term
log : Q_{log}	integer for a finite lattice spacing	ill-defined
sin : Q_{sin}	approaches an integer in the continuum limit	well-defined

If we use $S_\theta = -i\,\theta\, Q_{log}$ … $D_{n,\mu} S_\theta$ is ill-defined on the branch cut and is always zero elsewhere.

→ We use $S_\theta = -i\,\theta\, Q_{sin}$ naively for CLM on a 2D torus.

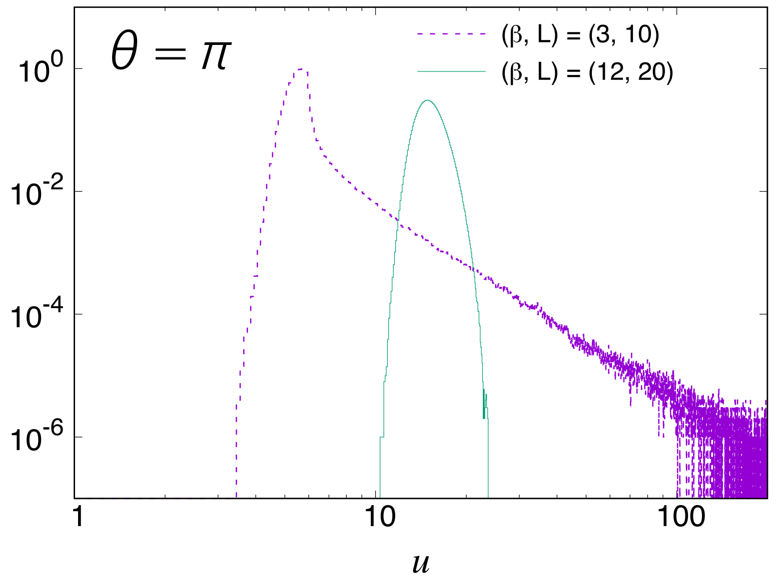
Result of the naive implementation

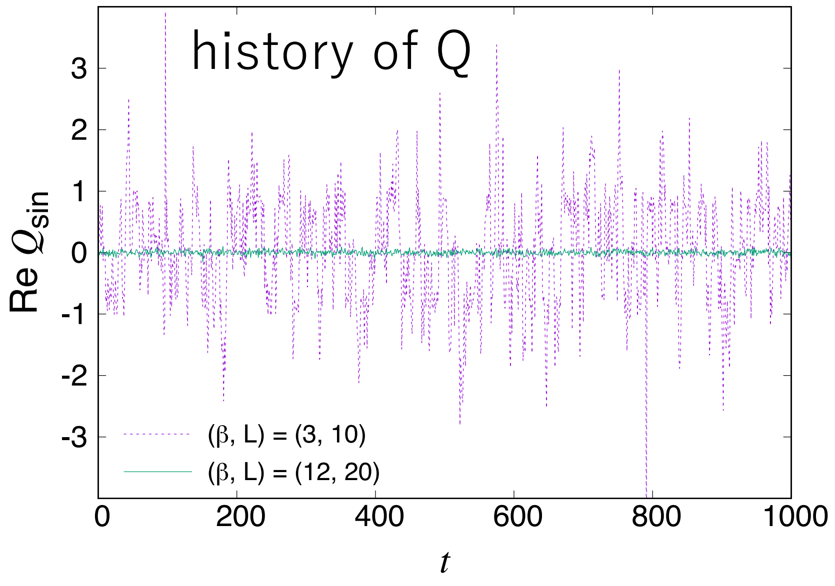
- small β (coarse lattice): wrong convergence of CLM
  - The condition for correct convergence is not satisfied.

⇕ trade-off

- large β (fine lattice): "freezing" of the topological charge
  - The configuration is confined in a single topological sector.

distribution of the drift term





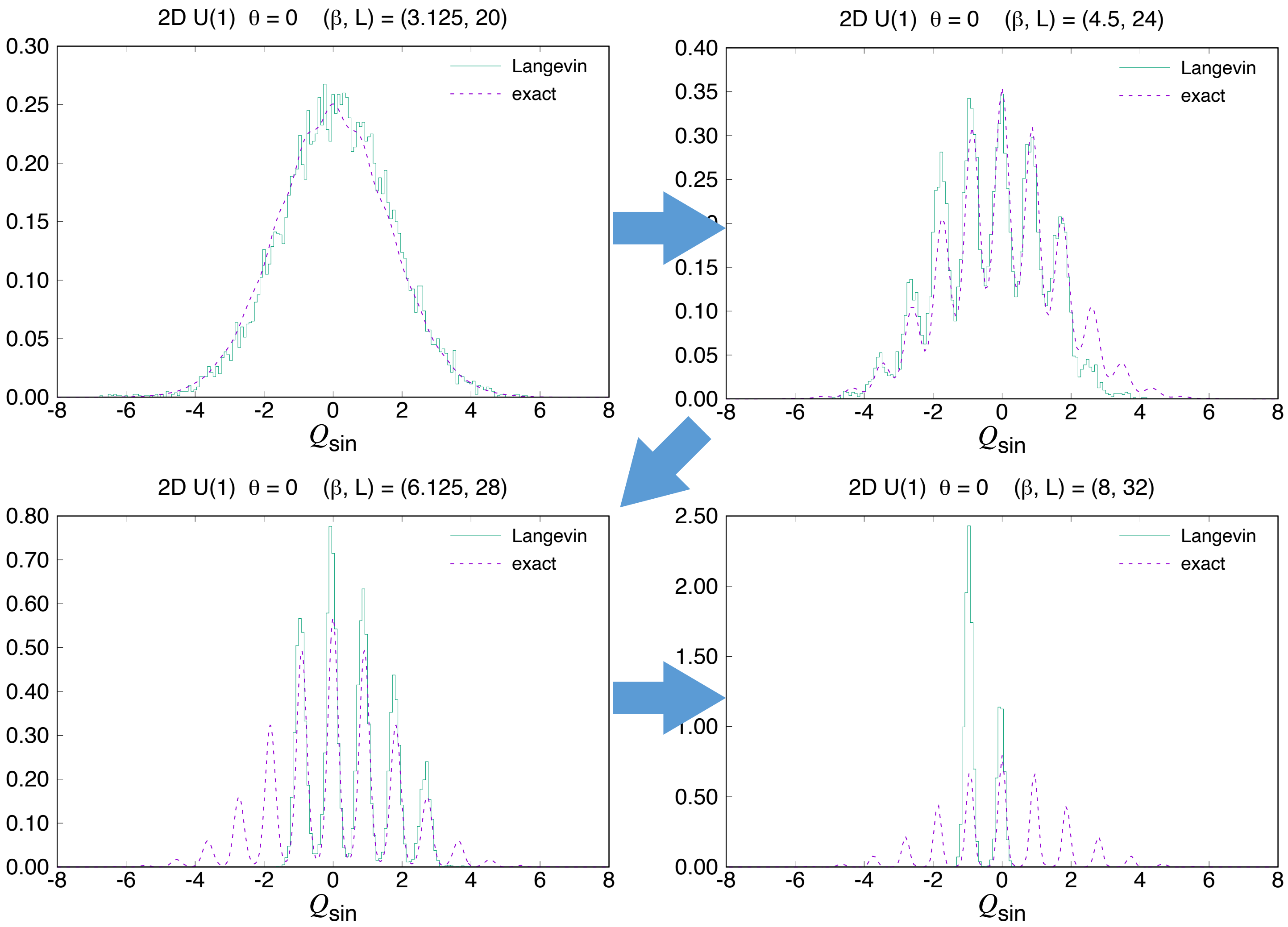
Behavior of the topological charge

- distribution of $Q_{\sin}$ at $\theta = 0$ for the fixed physical volume $V / \beta = 128$

$$Q_{\sin} = -\frac{i}{4\pi}\sum_n (P_n - P_n^{-1})$$

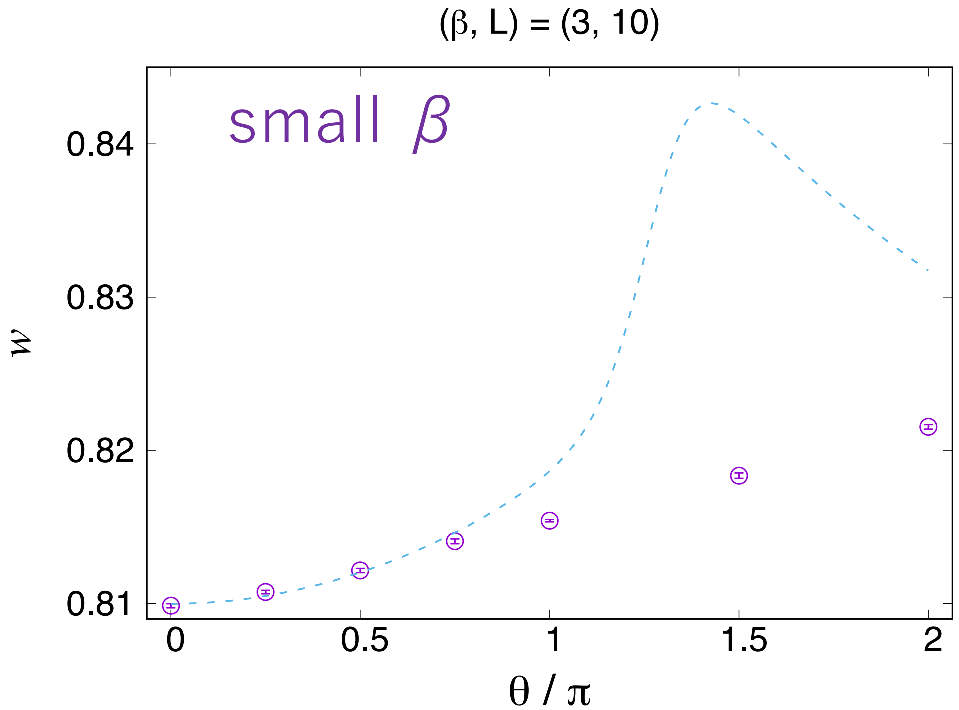
in the continuum limit

- $Q_{\sin} \rightarrow$ integer
- topology freezing

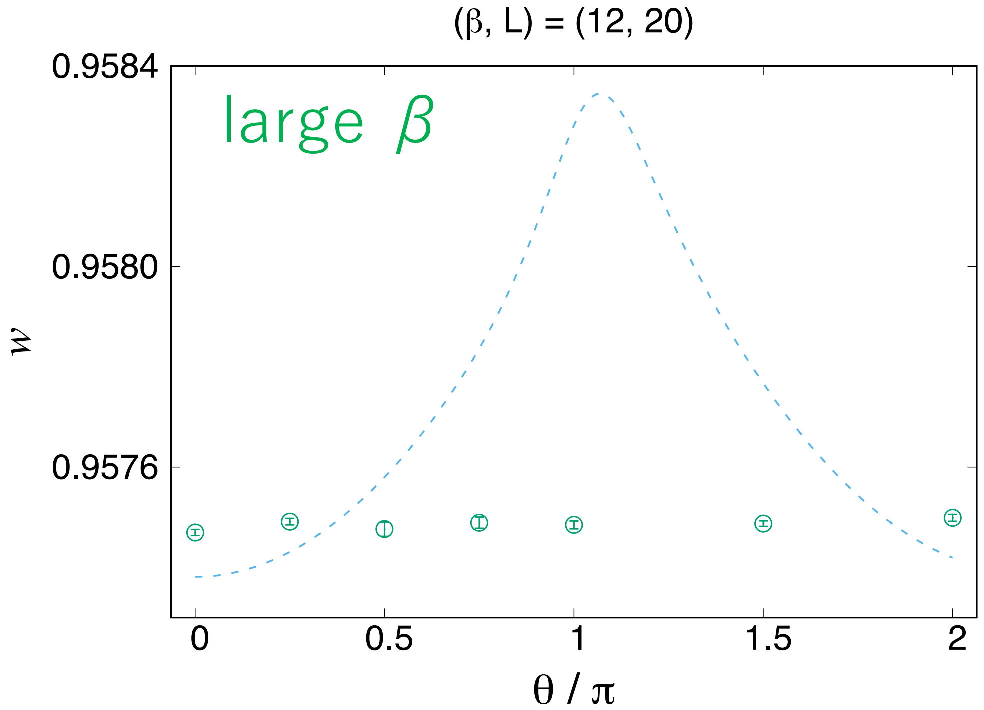


Observable

average plaquette $$w = \frac{1}{V}\frac{\partial}{\partial\beta}\log Z = -\frac{1}{\beta V}\langle S_g\rangle$$



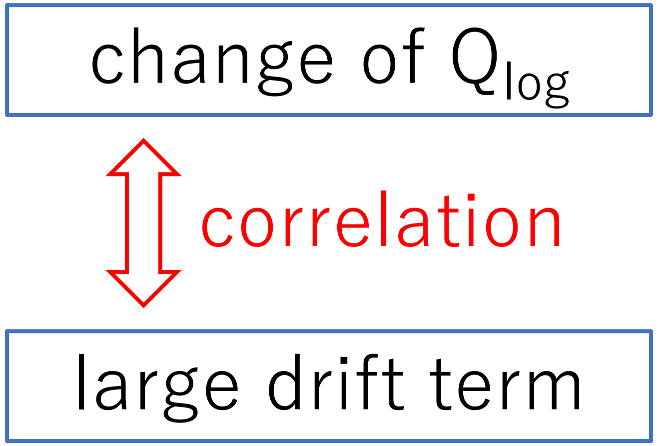
condition is not satisfied for $\theta \neq 0$

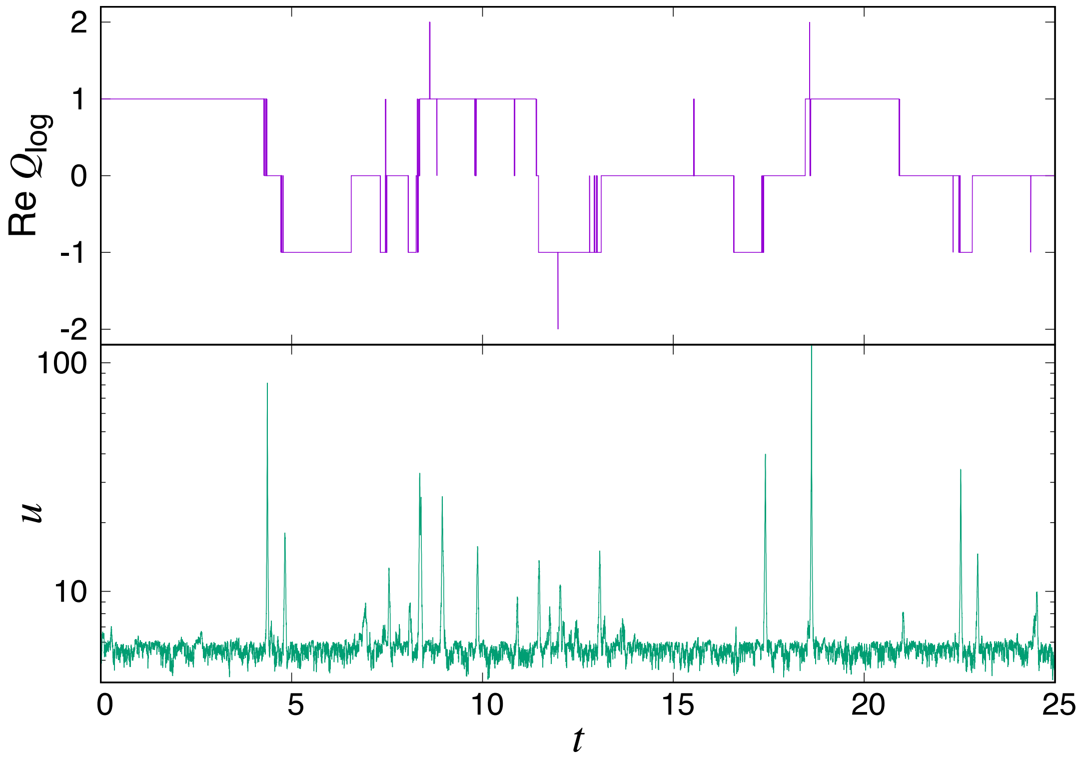


freezing of topological charge

Large drift term and topology change

- Each configuration can be classified into topological sectors by measuring Q_{log}.
- transition among topological sectors = change of Q_{log}



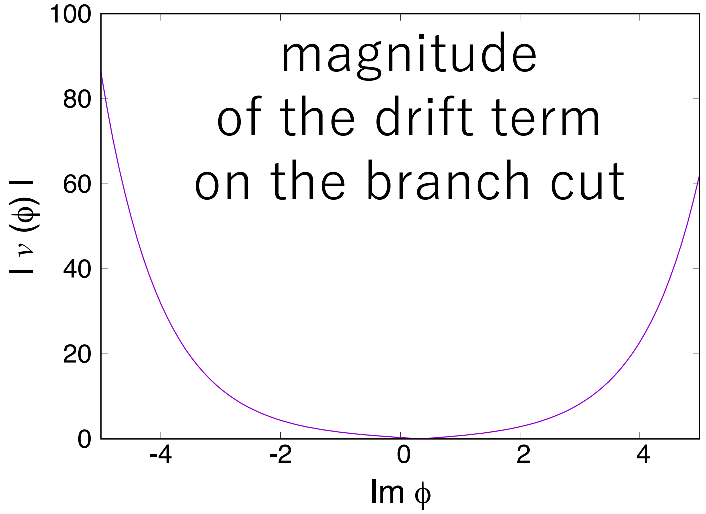


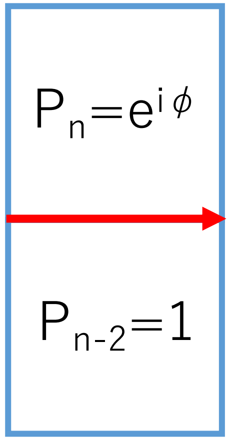
history of Re Q_{log}

history of max |drift term|

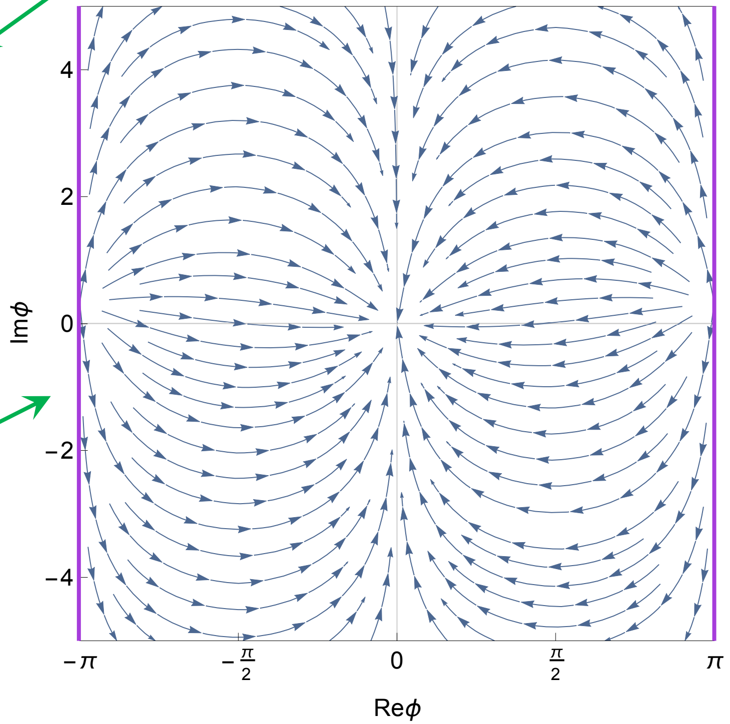
Origin of large drift terms

- Transition among topological sectors is caused by branch crossing of the phase of plaquette $\phi = -i \log P_n$.
- When ϕ approaches the branch cut, it flows to the imaginary direction.
- As Im ϕ increases, |drift term| increases exponentially.





flow diagram of ϕ
(drift of $U_{n,1}$)



drift force pushing
ϕ away from the
real axis

Contents

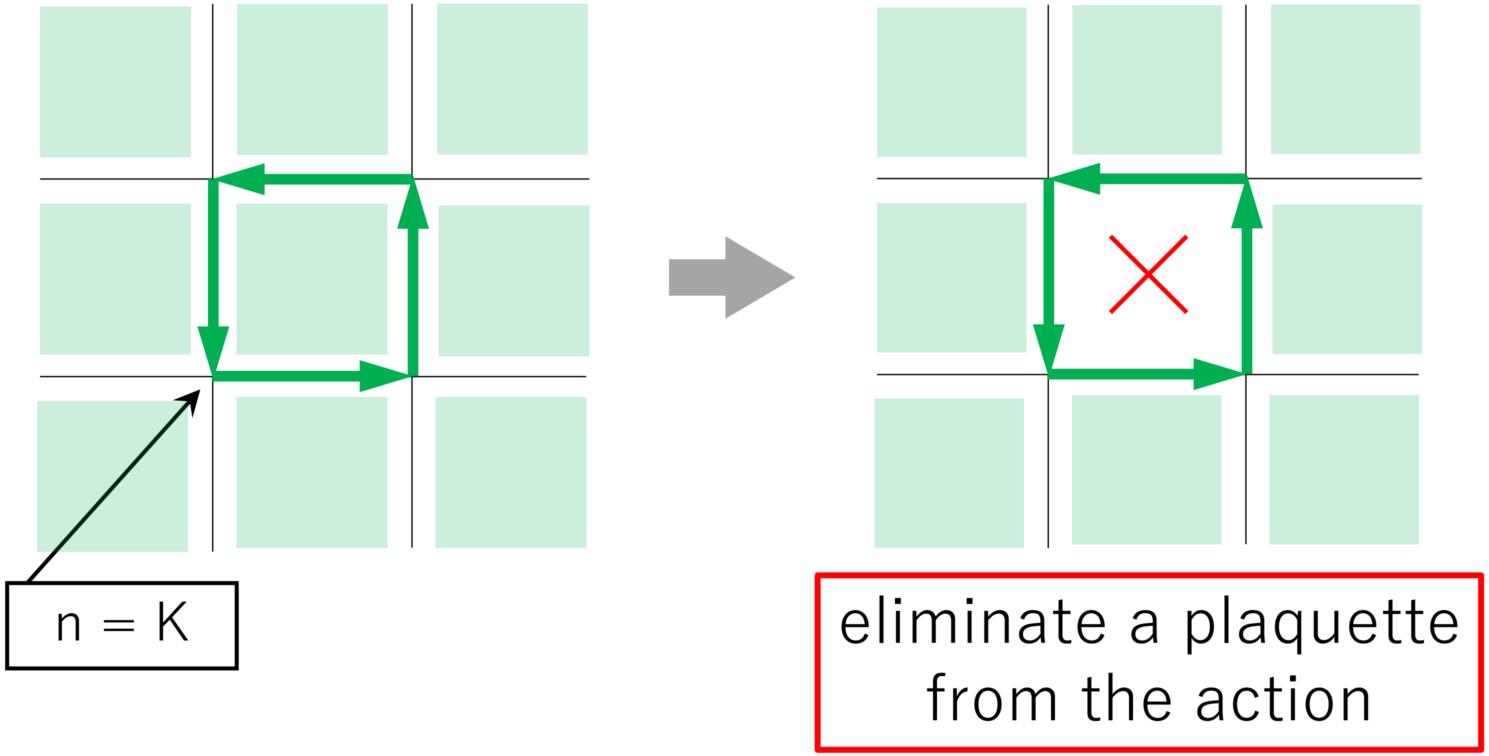
- Introduction
- 2D U(1) gauge theory
  - ➢Naive implementation of CLM
  - ➢Introducing a puncture
- 4D SU(2) gauge theory (ongoing work)
- Summary and discussion

Introducing a puncture on the torus

prescription for avoiding the freezing of Q

☆ introduce a puncture on the torus → Q is no longer an integer

topological charge can change frequently → freezing is resolved



$$S_g = -\frac{\beta}{2} \sum_{n \neq K} (P_n + P_n^{-1})$$

$$Q_{\sin} = -\frac{i}{4\pi} \sum_{n \neq K} (P_n - P_n^{-1})$$

$$Q_{\log} = -\frac{i}{2\pi} \sum_{n \neq K} \log P_n$$

Drift term for the punctured model

on a punctured torus...

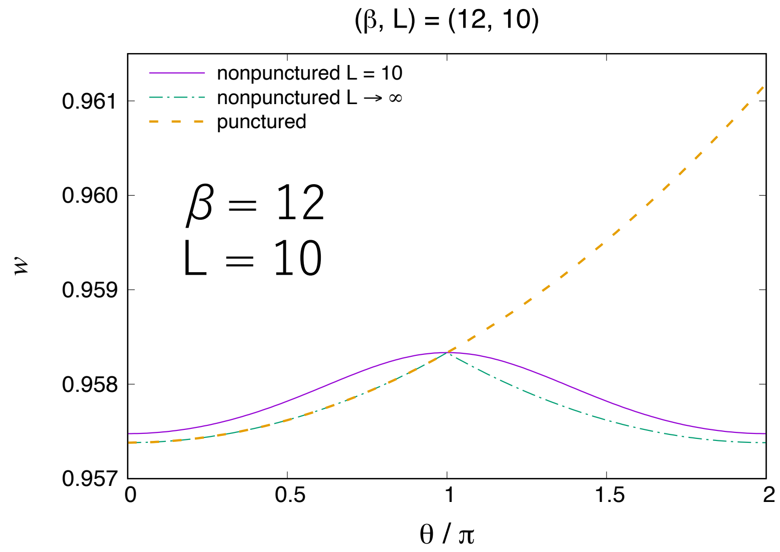
definition	value	drift term
log : $Q_{\log}$	not an integer	well-defined (if β is large enough)
sin : $Q_{\sin}$	not an integer	well-defined

- We can use $S_\theta = -i\,\theta\, Q_{\log}$ on the punctured torus.
- The links around the puncture get non-zero drift terms.

Property of the punctured model

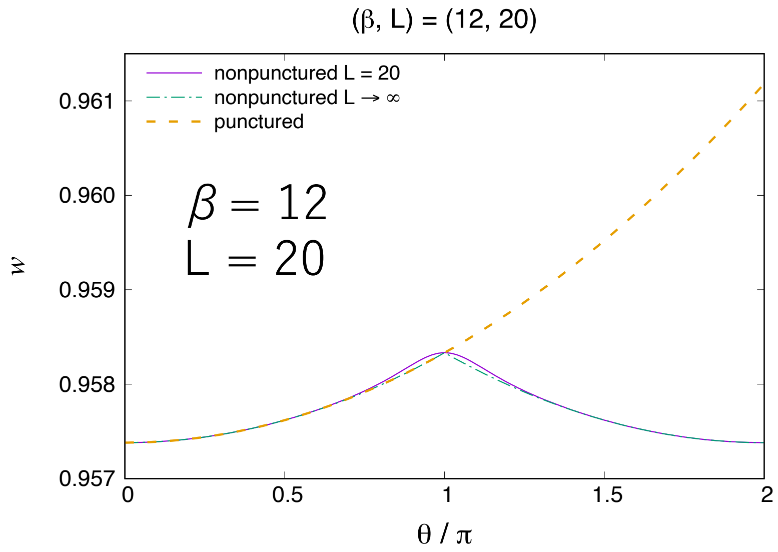
- The punctured model does not have 2π-periodicity of θ.
  - since the topological charge is not an integer
- The punctured model is equivalent to the infinite volume limit of the original (non-punctured) model for $|\theta| < \pi$.

$$Z_{\mathrm{nonpunc}} = \sum_{n=-\infty}^{+\infty} \left[\mathcal{I}(n,\theta,\beta)\right]^V \qquad Z_{\mathrm{punc}} = \left[\mathcal{I}(0,\theta,\beta)\right]^V \qquad \mathcal{I}(n,\theta,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{\beta\cos\phi + i\left(\frac{\theta}{2\pi}-n\right)\phi}$$



analytic solution of the average plaquette

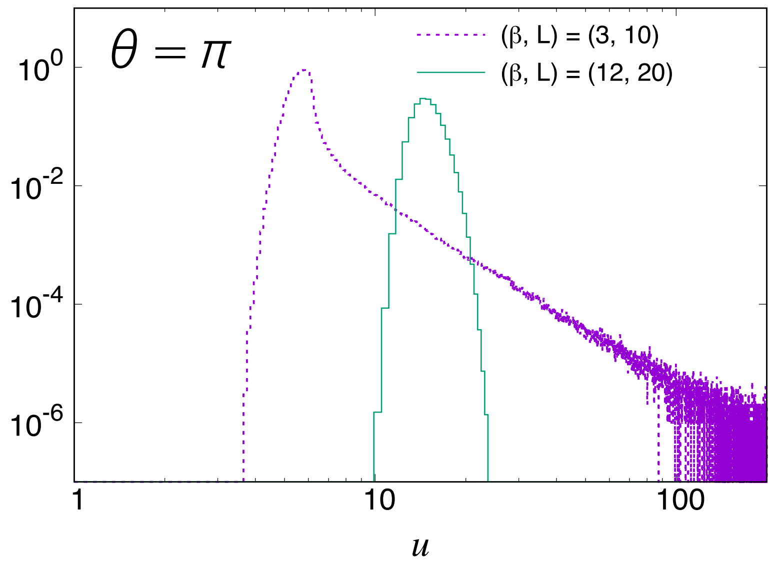
increase the volume



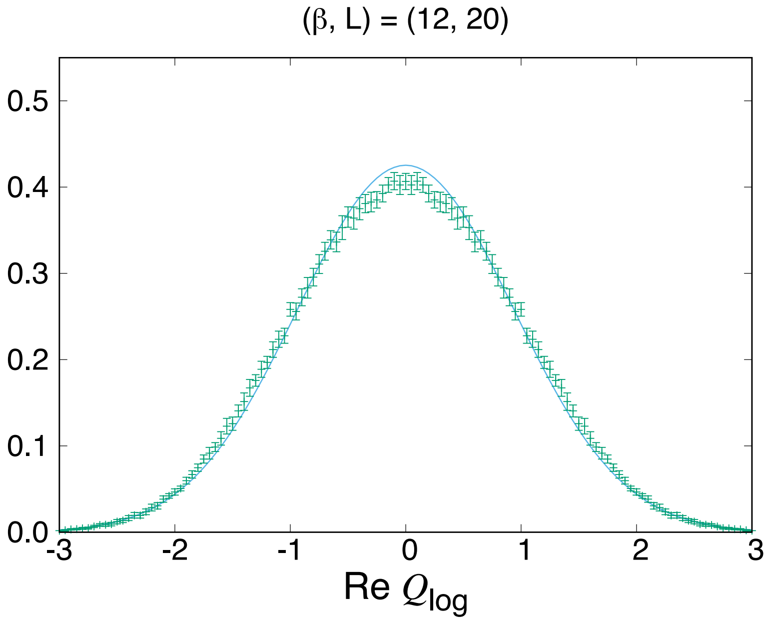
Result of the punctured model

- The condition for the correct convergence is satisfied for large β .
- The topology freezing does not occur.

☆ Both of the problem are resolved if β is large enough.



distribution of the drift term

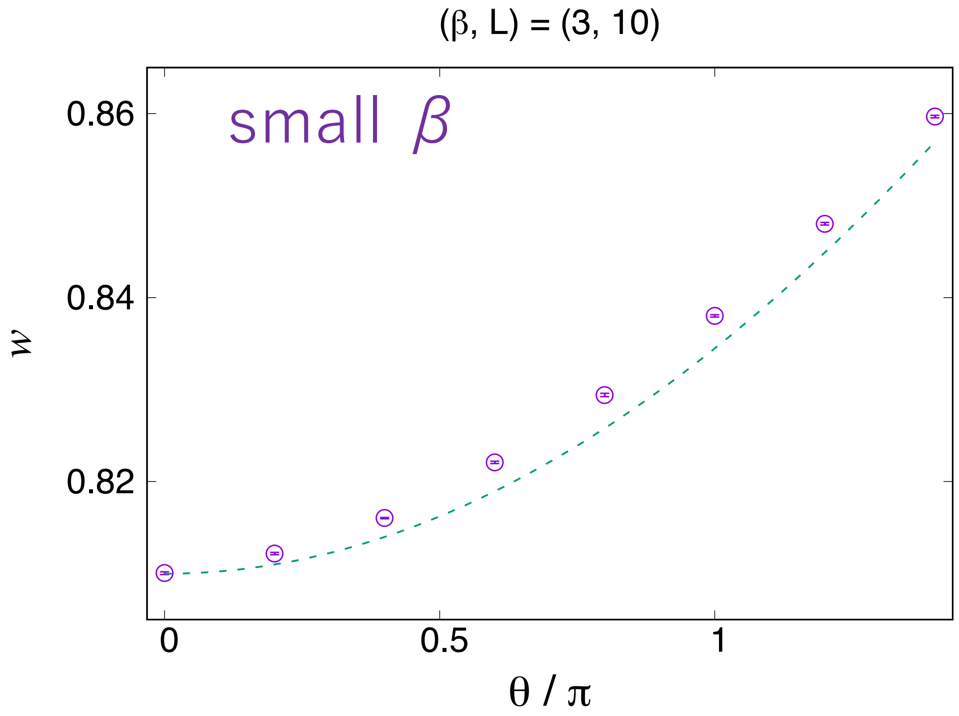


distribution of Q_{sin}

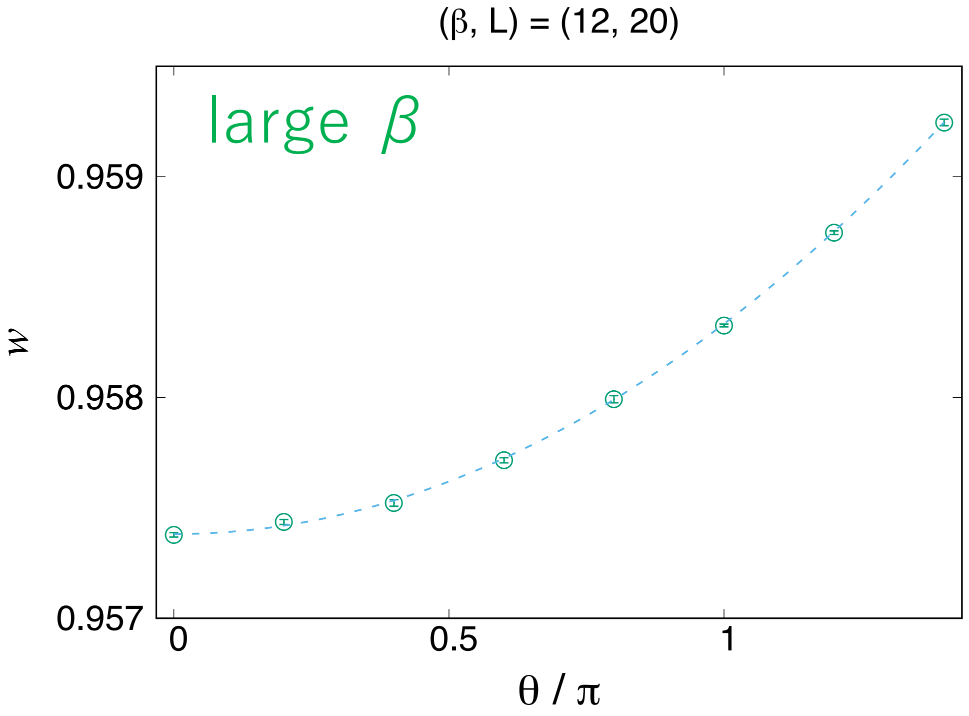
agrees with
the analytical result
(at θ =0)

Result of the punctured model

average plaquette $w = \frac{1}{V}\frac{\partial}{\partial\beta}\log Z = -\frac{1}{\beta V}\langle S_g \rangle$



condition is not satisfied for $\theta \neq 0$



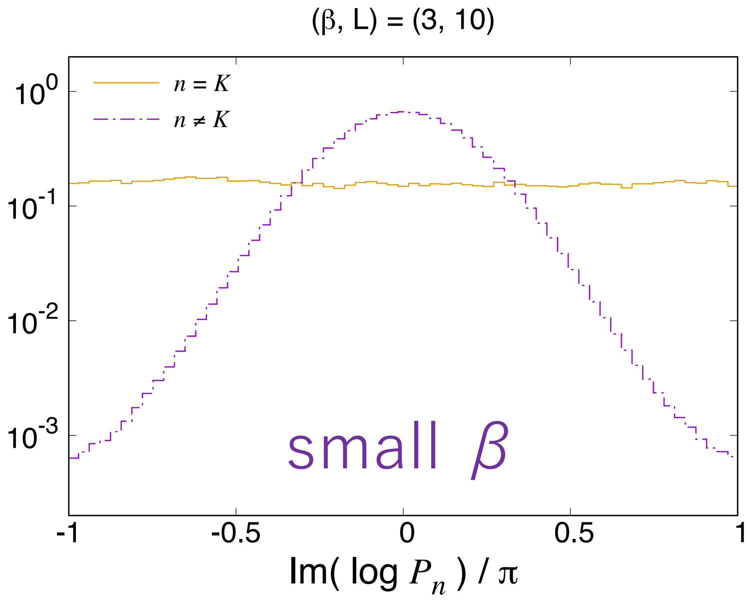
CLM works well

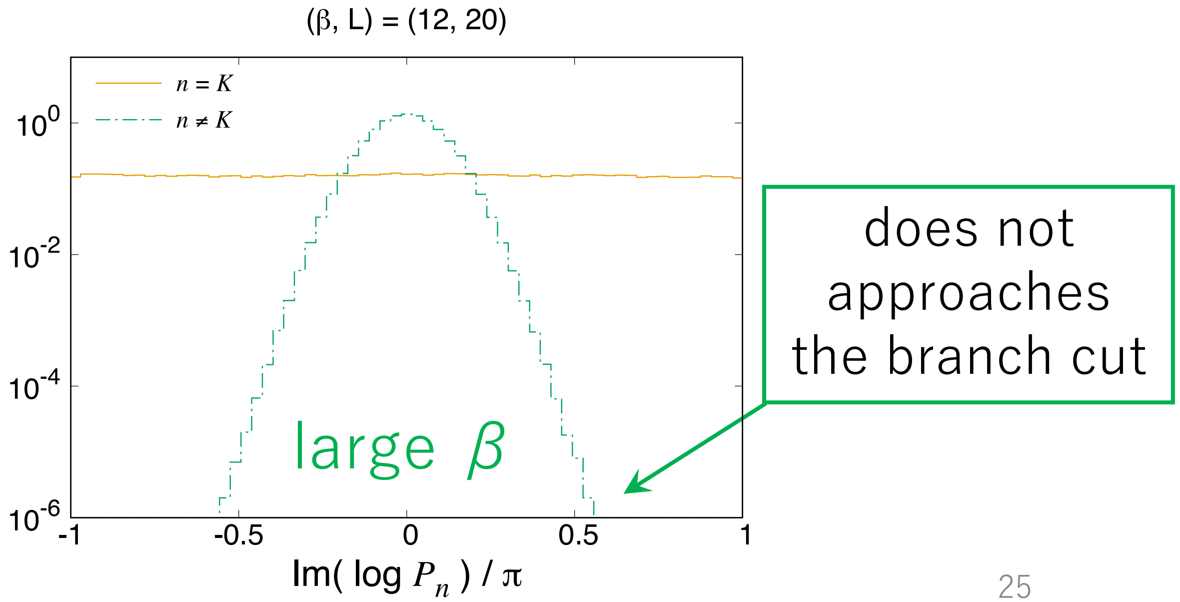
Effect of the puncture

- The phase of the puncture (eliminated plaquette) changes freely.
  → The topological charge can change easily.
- If β is large enough, branch crossing of the plaquettes included in the action is suppressed.
  → The source of large drift is absent.

distribution of the phase of plaquette/puncture

$\theta = \pi$





Contents

- Introduction
- 2D U(1) gauge theory
- 4D SU(2) gauge theory (ongoing work)
- Summary and discussion

Phase structure at $\theta = \pi$

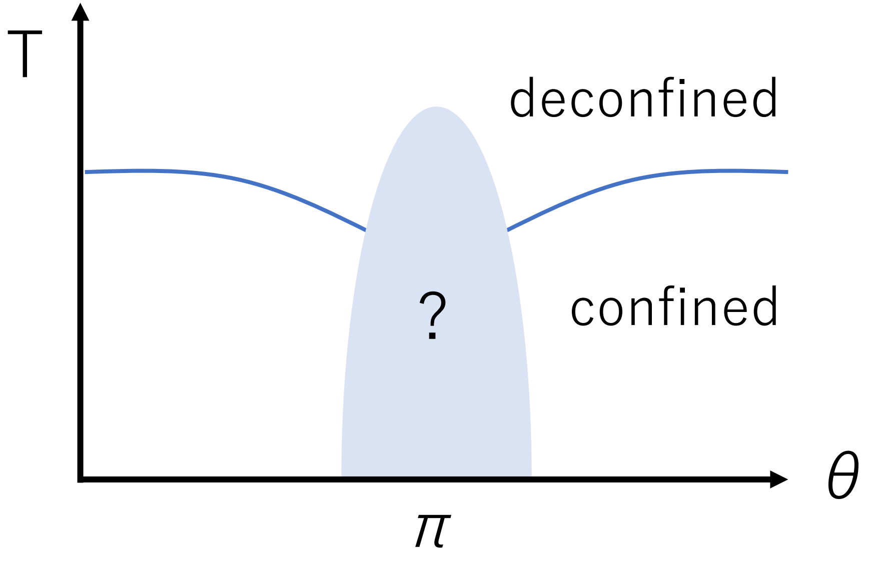
☆ 't Hooft anomaly matching of 4D SU(2) YM

→ constrain the phase structure at $\theta = \pi$

mixed 't Hooft anomaly between
CP symmetry & Z_2 1-form center symmetry at $\theta = \pi$

[D. Gaiotto, A. Kapustin, Z. Komargodski, N. Seiberg (2017)]

- SSB of CP
- SSB of $Z_2^{(1)}$
- gapless
- topological QFT

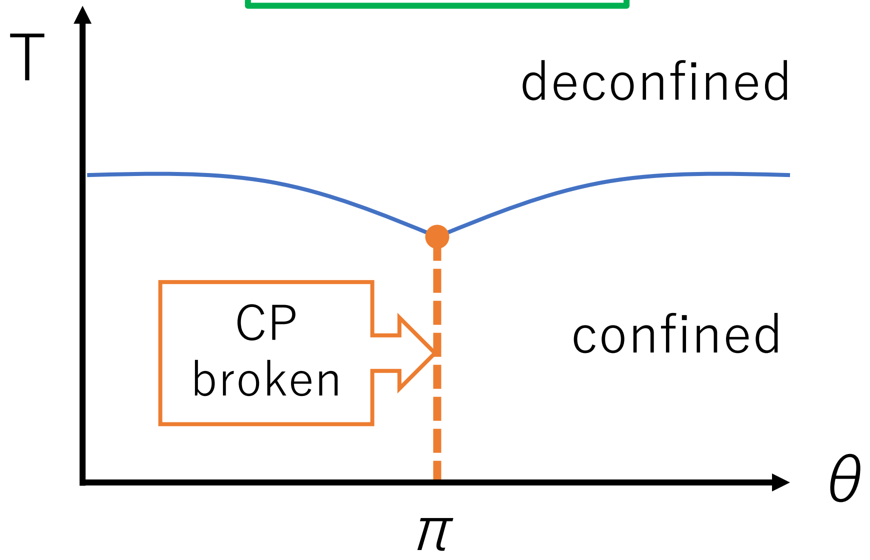


T_{dec} VS T_{CP}

☆ anomaly matching → $T_{dec} \leqq T_{CP}$ (assuming SSB of CP at T = 0)

examples of possible (θ , T) phase diagram

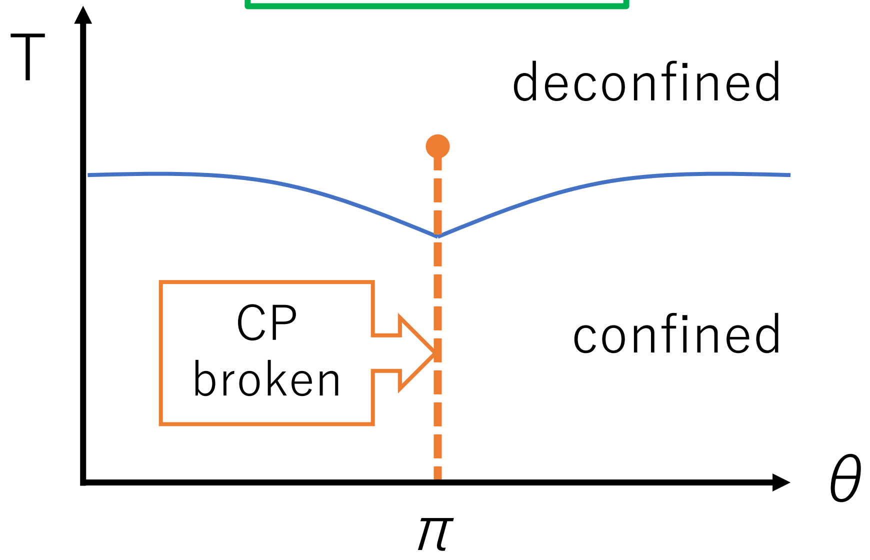
$T_{dec} = T_{CP}$



holography for large N supports

[F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]

$T_{dec} < T_{CP}$



soft SUSY breaking of SYM supports

[S. Chen, K. Fukushima, H. Nishimura, Y. Tanizaki (2020)]

4D SU(2) lattice gauge theory

- kinetic term : standard Wilson action

$$S_\beta = -\frac{\beta}{4}\sum_n \sum_{\mu\neq\nu} \mathrm{Tr}\left[P_n^{\mu\nu}\right]$$

$P_n^{\mu\nu}$: plaquette

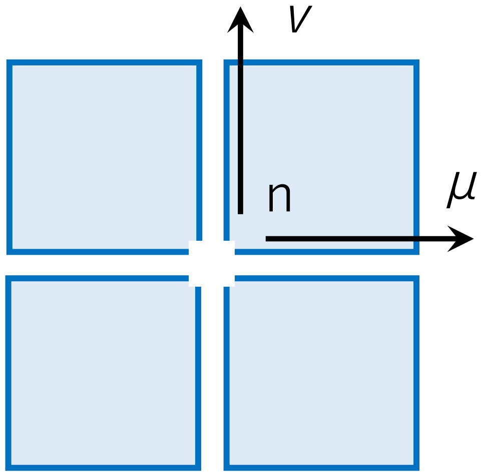
$$\beta = \frac{4}{g^2}$$

- topological charge : clover leaf

[P. Di Vecchia, K. Fabricius, G. C. Rossi, G. Veneziano (1981)]

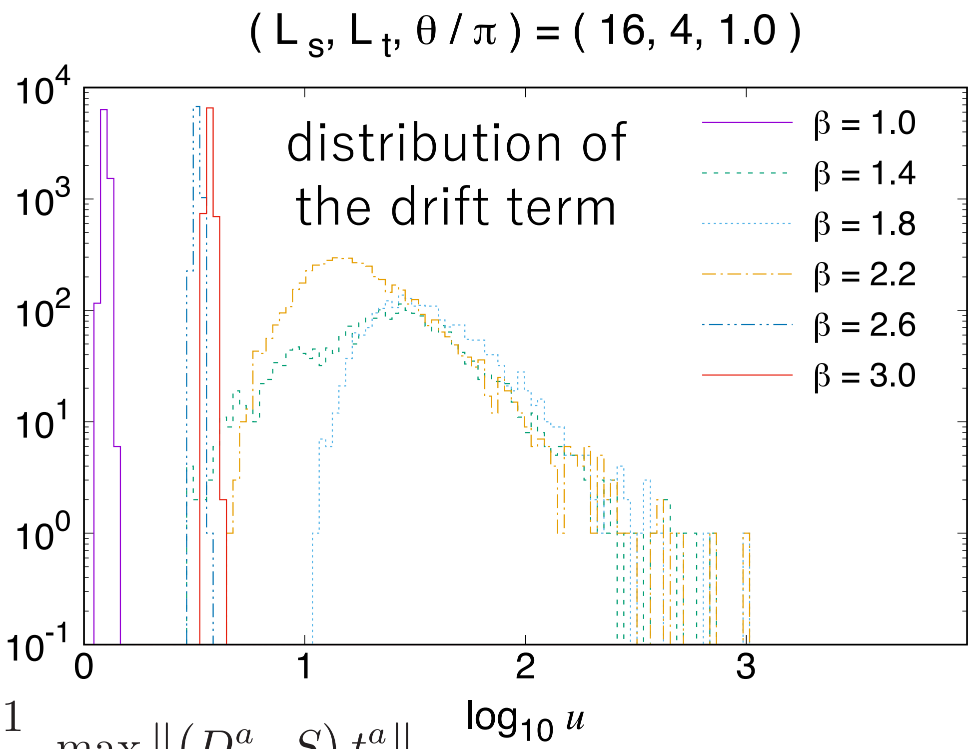
$$Q_{\mathrm{clov}} = -\frac{1}{32\pi^2}\sum_n \frac{1}{16}\sum_{\mu,\nu,\rho,\sigma=1}^{4} \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\bar{P}_n^{\mu\nu}\bar{P}_n^{\rho\sigma}\right]$$

$$\bar{P}_n^{\mu\nu} = P_n^{\mu\nu} - P_n^{-\mu\nu} - P_n^{\mu-\nu} + P_n^{-\mu-\nu}$$

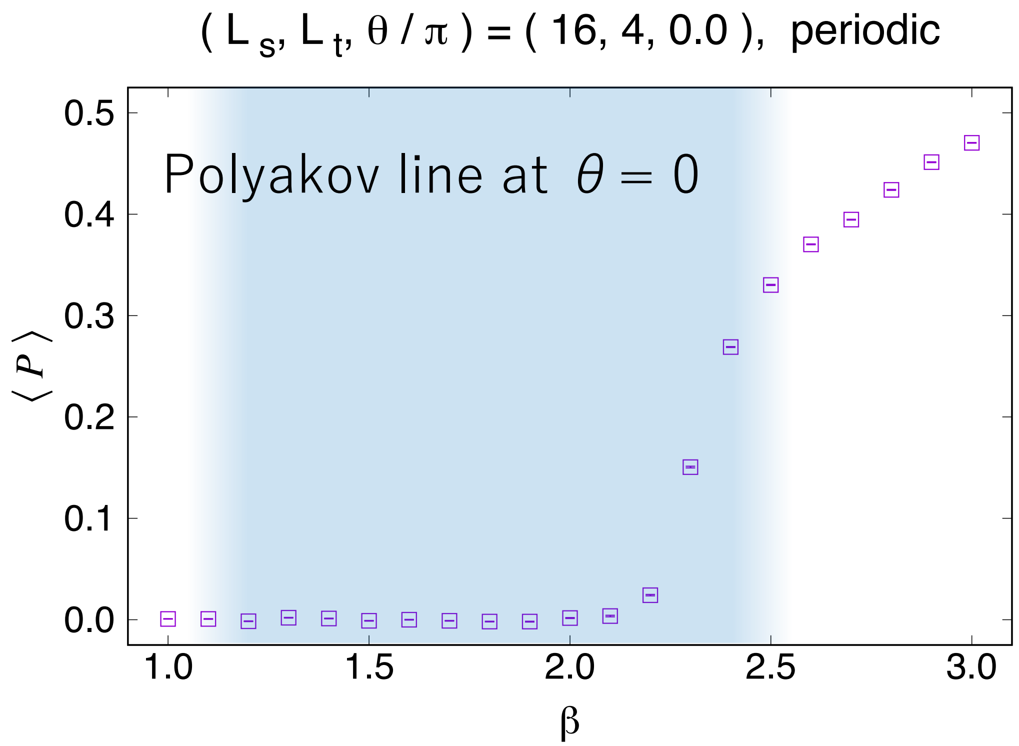


CLM on the 4D periodic lattice

- The condition for the correct convergence is not satisfied around the critical β.
  → try to see the high temperature (large β) region first



$$u = \frac{1}{\sqrt{2}} \max_{n,\mu} \left\| \left(D^a_{n,\mu} S \right) t^a \right\|$$



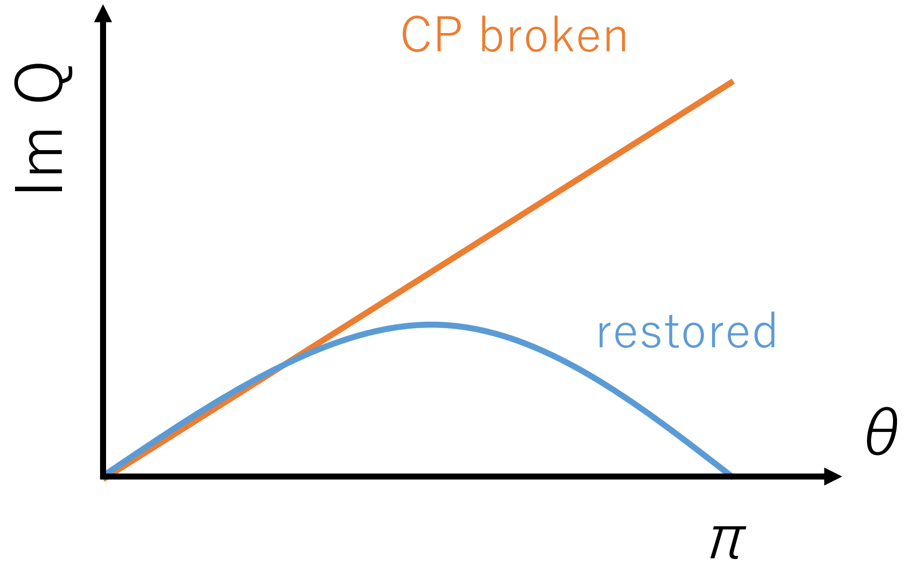
CP symmetry at $\theta = \pi$

- In the high temperatures region, CP is expected to be restored at $\theta = \pi$.
- The topological charge is CP odd.
  → $\langle Q \rangle = 0$ if CP is restored
- dilute instanton gas approximation

$$\langle Q \rangle = -i \frac{1}{Z} \frac{\partial Z}{\partial \theta}$$

$$\mathrm{Im}\langle Q \rangle \simeq \chi \sin\theta$$

$$\chi = \frac{1}{V} \left(\langle Q^2 \rangle - \langle Q \rangle^2 \right) = -\frac{1}{V} \frac{\partial^2}{\partial \theta^2} \log Z$$



Boundary condition of the 4D lattice

- large β → topology freezing

☆ consider three types of boundary condition

① periodic

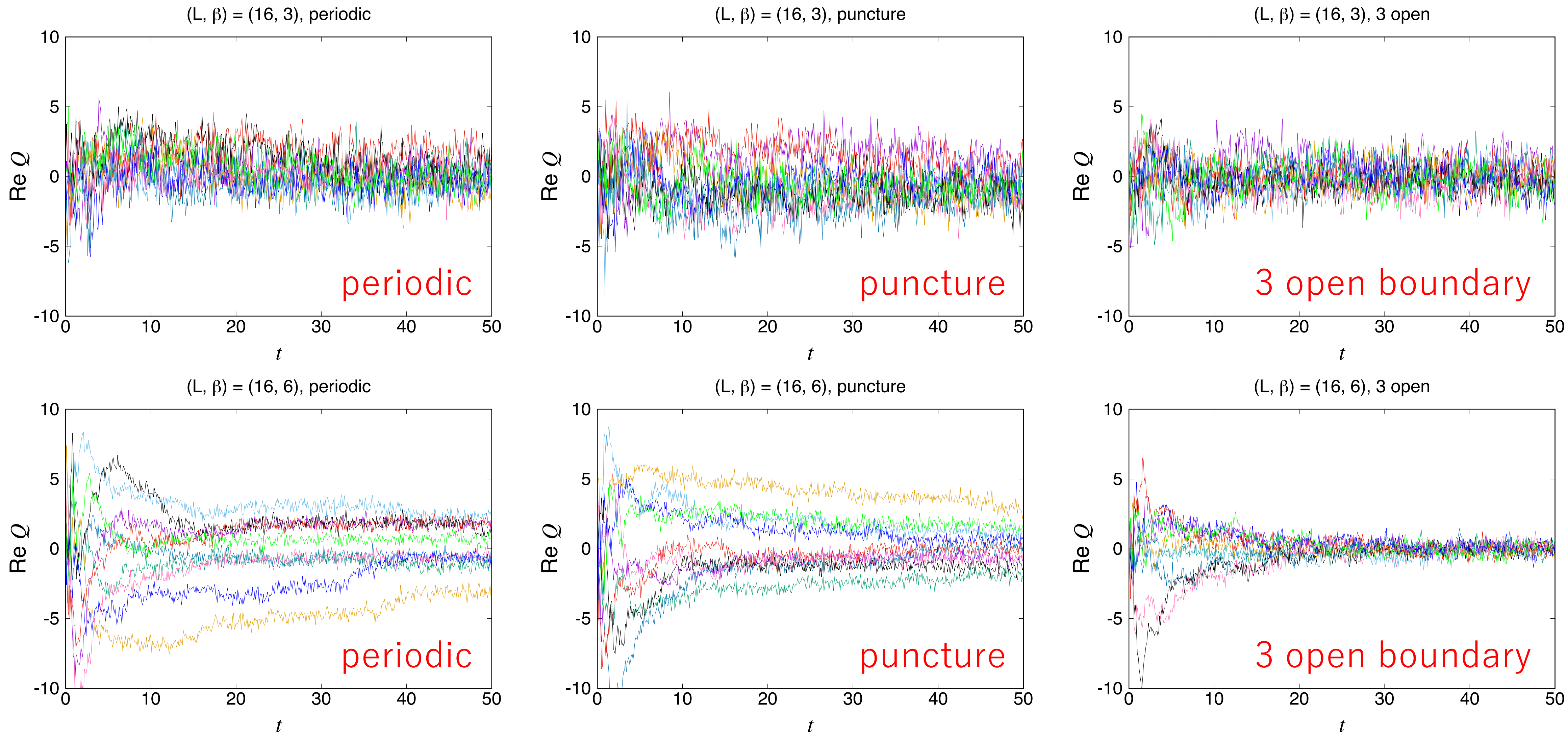
② spatially localized 2^3 puncture

③ open boundary for three spatial directions

The translational symmetry for the temporal direction is respected.

- start from random configurations (hot start)
- check the initial configuration dependence of Q at $\theta = 0$

Behavior of the topological charge



Behavior of the topological charge

- $\beta = 3.0$
- Q is far from an integer
- no significant difference

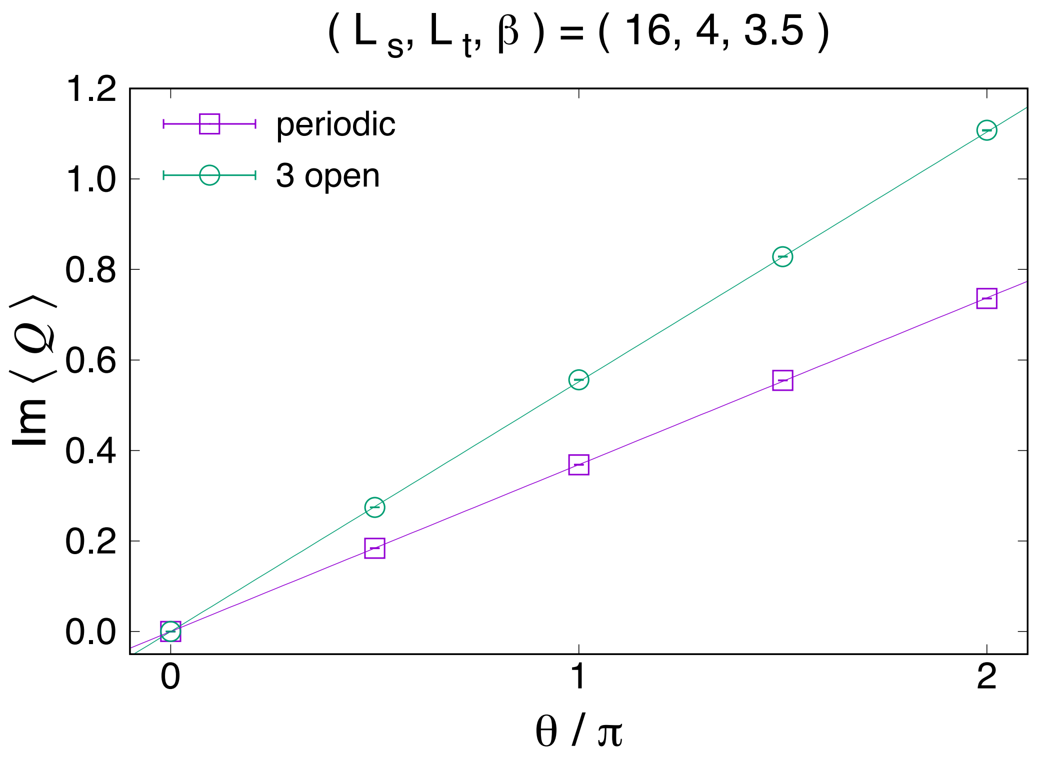
- $\beta = 6.0$
- periodic → frozen in different topological sectors
- puncture → severe autocorrelation still exists
- open boundary → thermalize slowly

Result for large β

- We found Im Q $\propto \theta$ in most cases.
- Im Q is related to the topological susceptibility

$$\frac{\partial}{\partial\theta}\mathrm{Im}\langle Q\rangle = \chi V$$

- The result for open boundary satisfies this relation.
- 2π-periodicity is absent in both cases.



	periodic	3 open
gradient of Im Q	0.11733(14)	0.17567(48)
χ V at $\theta = 0$	0.1015(47)	0.184(14)

Problem

- CLM works only in the large β region (and $\beta \leqq 1.0$).

<u>periodic lattice</u>

- topology freezing for large β

<u>open lattice</u>

- severe finite volume effect
- Q is not an integer → CP symmetry at $\theta = \pi$ is manifestly broken

→ a new technique is necessary

Contents

Summary

- The recent work on 't Hooft anomaly matching for 4D SU(2) YM predicted a nontrivial phase structure at $\theta = \pi$.
- We use the complex Langevin method to simulate the theory with the θ term, avoiding the sign problem.
- For 2D U(1), CLM works on the punctured torus where the topology freezing is absent.
- For 4D SU(2), CLM works in some cases, but it seems to be difficult to investigate the phase structure using the usual periodic or open lattice.

Discussion

<u>new technique</u>

- The distinct topological sectors need be smoothly connected.
  → avoid the topology freezing
  → well-defined drift terms
- The 2π-periodicity of θ should be respected.
  → CP symmetry at $\theta = \pi$

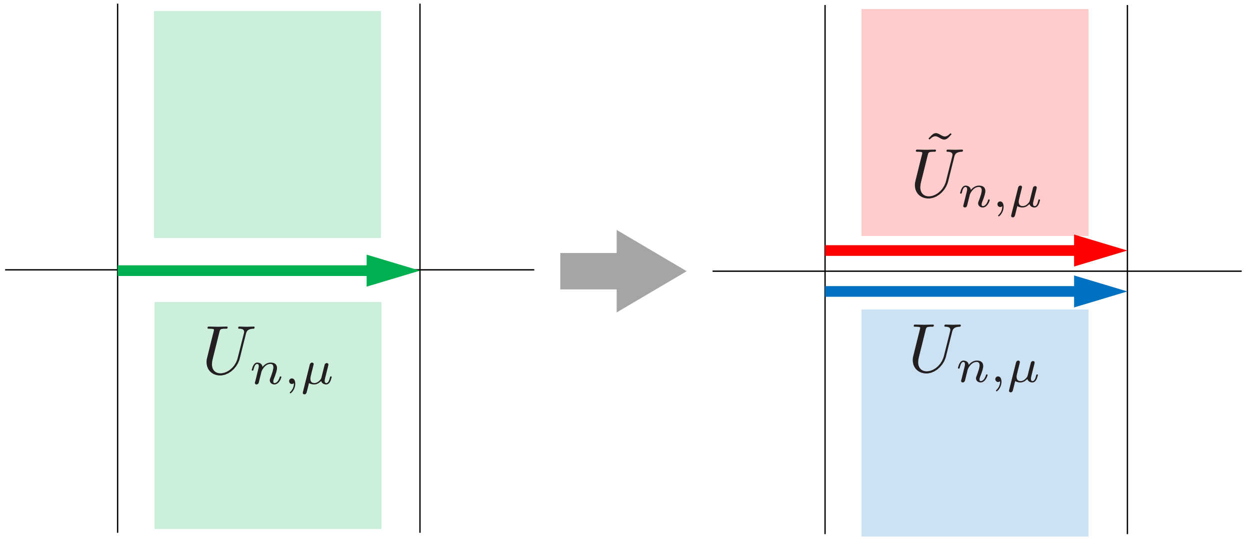
- Modify the boundary condition
  but recover the 2π-periodicity somehow ?

New technique

- Introduce an additional d.o.f. x which makes two adjacent plaquettes independent.
- Constrain x by the delta function.

$$\tilde{U}_{n,\mu} = M(x)\, U_{n,\mu} \qquad M(0) = 1$$

$$\delta(x) = \int \frac{dk}{2\pi}\, e^{ikx}$$



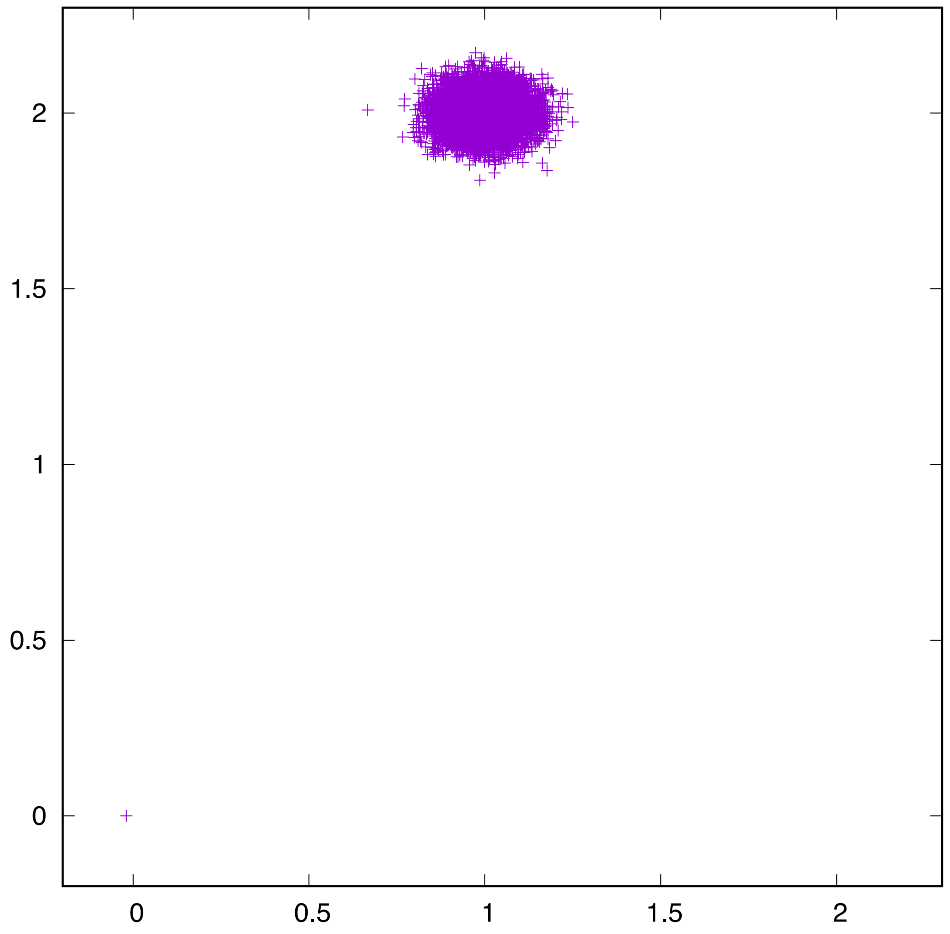
$$\int dU\, e^{-S(0)} = \int dU\, dx\, e^{-S(x)}\, \delta(x)$$
$$= \int dU\, dx dk\, e^{-S(x)+ikx}$$

Toy model

- Gaussian + delta function

$$S = mx^2 + ik(x - a)$$

- m = 1.0
- a = 1 + 2 i



Thank you!