Lattice QCD Precision Science for Muon g-2 and Running Coupling

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Muon Anomalous Magnetic Moment $a_{\ell=e,\mu,\tau}$

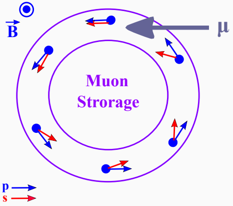
- **Dirac Eq. with B:**

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\boldsymbol{\alpha}\cdot\left(-i\hbar c\nabla - e\mathbf{A}\right) + \beta c^2 m_\ell + eA_0\right]\psi \,,$$

- **Nonlelativistic Limit, Pauli Eq.:**

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{(-i\hbar c\nabla - e\mathbf{A})^2}{2m_\ell c} - \mathbf{M}_\ell\cdot\mathbf{B} + eA_0\right]\phi \,,$$

- **Magnetic Moment:** $\mathbf{M}_\ell = g_\ell \frac{e}{2m_\ell c}\frac{\hbar\boldsymbol{\sigma}}{2}$,
- **In Dirac Theory:**
  $g_\ell = 2\,, \quad a_\ell \equiv (g_\ell - 2)/2 = 0\,, \quad \omega_{\rm cyc} = \omega_{\rm prec}$.
- **In QFT (with Loops) for Electron (M.Knecht ,NPPP2015):**
  $a_e^{\rm SM} = 1\,159\,652\,180.07(6)(4)(77)\times 10^{-12} \quad (\mathcal{O}(\alpha^5))\,,$
  $a_e^{\rm exp} = 1\,159\,652\,180.73(0.28)\times 10^{-12} \quad [0.24 ppb]\,.$



$a_\mu^{exp.} = a_\mu^{\rm SM}?$

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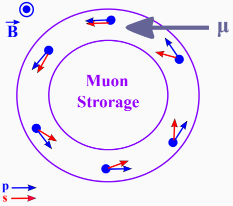


$$a_\mu^{exp.} = a_\mu^{\rm SM}?$$

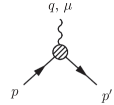
$a_\mu^{exp.}$ vs. a_μ^{SM}

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
LO-HVP($\mathcal{O}(\alpha^2)$) by pheno.	692.8 ± 2.4	[Keshavarzi et al '19]
	694.0 ± 4.0	[Davier et al '19]
	687.1 ± 3.0	[Benayoun et al '19]
	688.1 ± 4.1	[Jegerlehner '17]
NLO-HVP($\mathcal{O}(\alpha^3)$) by pheno.	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
	-9.83 ± 0.04	[KNT19]
NNLO-HVP($\mathcal{O}(\alpha^4)$) by pheno.	1.24 ± 0.01	[Kurz et al '14]
HLbyL($\mathcal{O}(\alpha^3)$)	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp − SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

$$a_\mu^{\text{LO-HVP}}\big|_{NoNewPhys} = a_\mu^{\text{ex.}} - (a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{(N)NLO-HVP}} + a_\mu^{\text{HLbL}}) \simeq (720 \pm 7) \times 10^{-10} \, ,$$

a_ℓ in QFT

- **QFT Def. for a_ℓ:**

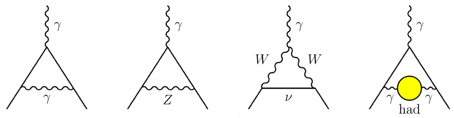


$$= \langle \bar{\ell}^-(p)|\mathcal{J}^\mu|\ell^-(p')\rangle = \bar{u}(p)\Gamma^\mu(p,p')u(p') \quad (1)$$

$$\Gamma^\mu(q=p-p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2) + \cdots\,, \quad (2)$$

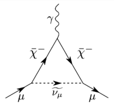
$$F_2(0) = a_\ell = (g_\ell - 2)/2\,. \quad (3)$$

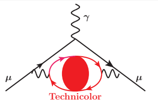
- **Standard Model, Loop Corr.:**



$$a_\ell = \alpha/(2\pi) + \cdots\,.$$

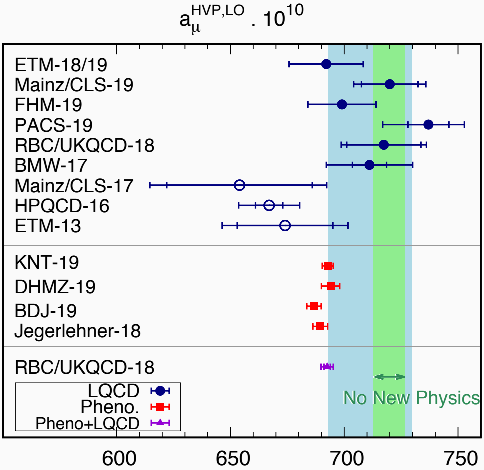
- **BSM = MSSM (Padley et.al.'15) or TC (Kurachi et.al. '13) etc.:**





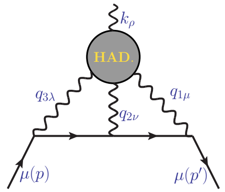
$$\propto (m_\ell/\Lambda_{BSM})^2.$$

Whitepaper (WP): Lattice QCD Consensus



- Muon g-2 Theory Initiative Whitepaper, arXiv:2006.04822.
- LQCD Concensus: $a_\mu^{\text{LO-HVP}} = 711.6(18.4) \cdot 10^{-10}$, BMW-2020 Not Yet Included.

Hadronic Light-by-Light (HLbL)



- $\mathcal{O}(\alpha^3)$ Contributions.
- Need investigate $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3, k)$.
- Not full related to experimental observables.

Current Status

- **LQCD:** $a_\mu^{\text{HLbL}} = 7.87(3.06)_{stat}(1.77_{sys}) \times 10^{-10}$. [RBC/UKQCD PRL2020.]
- **Pheno.:** $a_\mu^{\text{HLbL}} = 9.2(1.9) \times 10^{-10}$. [Whitepaper 2006.04822.]
- LQCD and Phenomenology are consistent. HLbL seems not to be a source of the muon g-2 discrepancy.

Motivation

Questions

- Really $a_\mu^{ex.} \neq a_\mu^{\text{SM}}$?
- More specifically, $a_\mu^{\text{LO-HVP}} \neq (720 \pm 7) \times 10^{-10}$?
- Impact for $\Delta^{\text{had}}\alpha(Q^2)$ at EW scale?
  [c.f. Crivellin et.al.(2003.04886), Keshavarzi et.al.(2006.12666).]

New Experiments

- $a_\mu^{ex.}$: FNAL-E989 $0.14ppm$ (soon $0.5ppm$), J-PARC-E34 $0.1ppm$ (2024).
- $\Delta^{\text{had}}\alpha(Q^2)$: MUonE, ILC.

THIS TALK

- Investigate $a_\mu^{\text{LO-HVP}}$ by Lattice QCD (BMW-2020, arXiv:2002.12347).
- Discuss $\Delta^{\text{had}}\alpha(Q^2)$ by Lattice QCD (Manz/CLS) compared with Data-Driven Dispersion (Jegerlehner et.al.).

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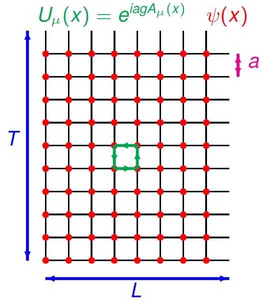
Lattice Gauge Theory I

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}[U,\psi,\bar\psi]\; e^{-S_G[U]-\bar\psi\cdot D[U,M]\cdot\psi}\, O[U,\psi,\bar\psi]\;,$$

$$= \frac{1}{Z}\int \mathcal{D}U\; e^{-S_G[U]}\mathrm{Det}\left[D[U,M]\right] O[U]_{\text{wick}}\;,$$

$$= \sum_{i=1}^{N} O[U^{(i)}]_{\text{wick}} + \mathcal{O}(N^{-1/2})\;,$$

$\{U^{(i)}\}$ created w. $P = e^{-S_G}\cdot \mathrm{Det}[D]/Z$.
Hybrid Monte Carlo (HMC) $\leftrightarrow$ Heat-Bath.



- Regulalization: UV cutoff a, IR cutoff $L^3 \times T$.
- Gauge Fields: $U_\mu \in SU(N_c)$.
- Action: $S_{LatGT} = S_G[U] - \bar\psi\cdot D[U,M]\cdot\psi$ possesses exact gauge symm. Formally taking $a\to 0$ reproduces the continuum theory action.
- Renormalization: $\mu = a \to 0$ w. $\frac{M_{\pi,K,\cdots}}{M_\Omega}$ fixed around the physical values.

Lattice Gauge Theory II

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}[U,\psi,\bar\psi]\; e^{-S_G[U]-\bar\psi\cdot D[U,M]\cdot\psi}\, O[U,\psi,\bar\psi]\;,$$

$$= \frac{1}{Z}\int \mathcal{D}U\; e^{-S_G[U]}\mathrm{Det}\big[D[U,M]\big]\, O[U]_{\rm wick}\;,$$

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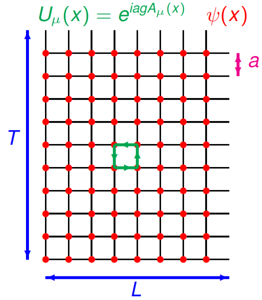

Lattice Gauge Theory

- Non-Perturbative Definition of asymptotic-free gauge theory.
  1. Regulalization: UV cutoff a, IR cutoff $L^3 \times T$.
  2. Renormalization: $\mu = a \to 0$ keeping $\frac{M_{\pi,K,\cdots}}{M_\Omega}$
  3. With a mass gap $\Lambda \sim F_\pi, M_\rho, ...,$ $a\Lambda \to 0$ and $L\Lambda \to \infty$ under controlled.
- First-Principle Calculations, i.e., No Approximation.

LQCD Meas. of HVP and $a_\mu^{\text{LO-HVP}}$

$\{U^{(i)}\}$: HMC

$\downarrow$

$D_f[U] \equiv D[U, m_f]$: Dirac Op.

$\downarrow$ $D_{XY}\phi_X = \eta_X^{(r)}$, $\sum_{r=1}^{N_r} \frac{\eta_X^{(r)}\eta_Y^{(r)}}{N_r}|_{N_r\to\infty} = \delta_{XY}$

$\downarrow$ with Conjugate Gradient Method,

$\downarrow$ Low-Mode Averaging (Lanczos, No $\eta_X^{(r)}$).

$D_f^{-1}[U]$: Quark Propagator.

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Vector Current Correlator

$$G^f_{\mu\nu}(x) = \langle(\bar\psi\gamma_\mu\psi)_x(\bar\psi\gamma_\nu\psi)_{y=0}\rangle \xrightarrow[\text{wick}]{}$$

$$C^f_{\mu\nu}(x) = -\langle \mathrm{ReTr}[\gamma_\mu D_f^{-1}(x,0)\gamma_\nu D_f^{-1}(0,x)]\rangle ,$$

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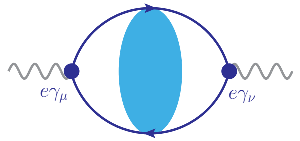

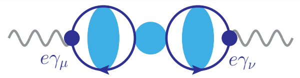

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$$C^f(t) = \frac{a^3}{3L^3}\sum_{i=1}^{3}\sum_{\vec{x}} C^f_{ii}(x) .$$

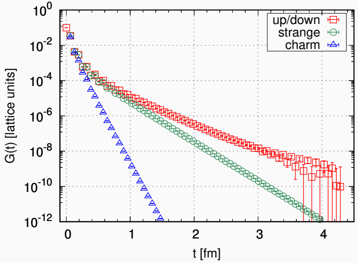

Figure: BMW2020 finest lattice ensemble.

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$\downarrow$

HVP: $\Pi^f_{\mu\nu}(Q) = \mathcal{F}.\mathcal{T}.[G^f_{\mu\nu}(x)]$.

$$\Pi_{\mu\nu}(Q) = \left(Q^2\delta_{\mu\nu} - Q_\mu Q_\nu\right)\Pi(Q^2) ,$$
$$\hat\Pi(Q^2) = \Pi(Q^2) - \Pi(0) .$$

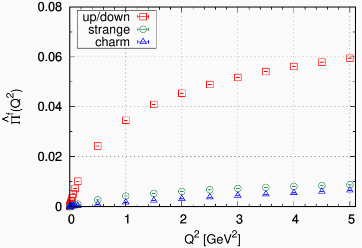


Figure: BMW2020 finest lattice ensemble.

HVP Phenomenology

- HVP in Pheno:

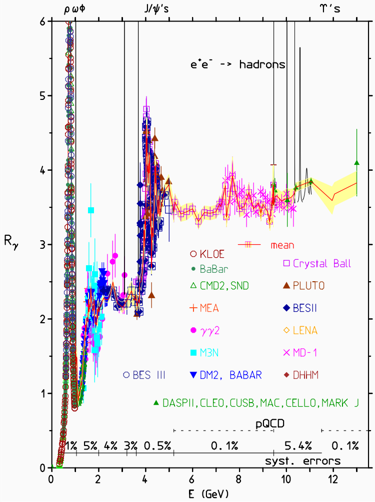
$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\mathrm{Im}\Pi(s)}{\pi} \quad \text{(dispersion)} ,$$

$$= \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s+Q^2)} \quad \text{(optical)} .$$

- R-ratio:

$$R(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to had.)}{4\pi\alpha^2(s)/(3s)} .$$

- Systematics is challenging to control. Some tension among experiments in $\sigma(e^+e^- \to \pi^+\pi^-)$.



[Jegerlehner EPJ-Web2016]

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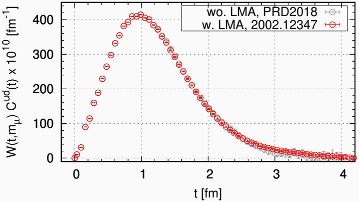
$\downarrow$

HVP: $\Pi^f_{\mu\nu}(Q) = \mathcal{F.T.}[G^f_{\mu\nu}(x)]$,

Muon g-2: $a^{\text{LO-HVP}}_{\mu,f} = (\frac{\alpha}{\pi})^2 \sum_t W(t, m_\mu^2) G^f(t)$.



$\Downarrow$ Tail Zoom

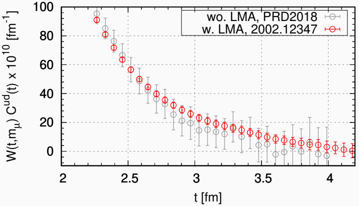
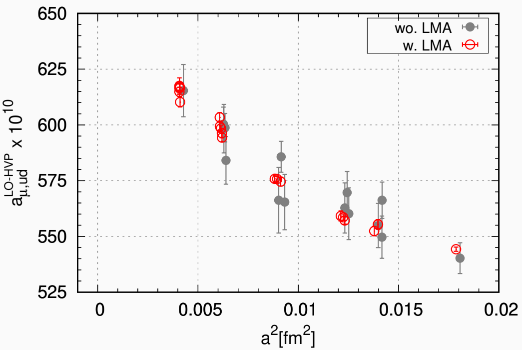


Figure: BMW2020 finest lattice ensemble.

Impact of Low-Mode Averaging (LMA)



- **Figure:** Red: BMW2020 with LMA. Gray: BMW2018 without LMA.
- LMA drastically reduces statistical error in up/down contributions into per-mil level.
- Various systematics from a^2, α, $(m_d - m_u)/\Lambda$, finite-volume effect, etc. must be controlled in per-mil level.

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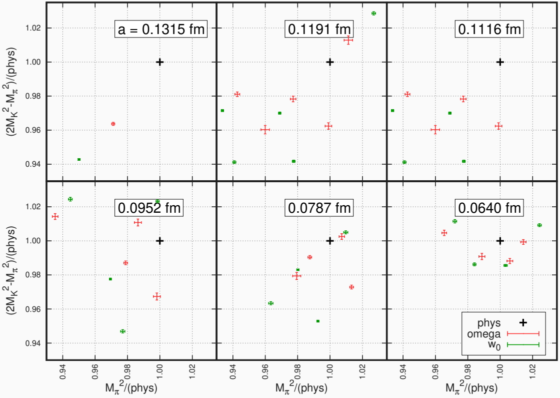
Budapest-Marseille-Wuppertal Collaboration

Sz. Borsanyi, Z. Fodor, J.N. Guenther, C. Hoelbling, S.D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato, K.K. Szabo, F. Stokes, B.C. Toth, Cs. Torok, and L. Varnhorst.

References

- arXiv:2002.12347. Submitted to Nature.
- Phys. Rev. Lett. **121**, no. 2, 022002 (2018).
- Phys. Rev. D **96**, no. 7, 074507 (2017).

BMW Simulation Setup



- 6 lattice spacings, 28 simulations around phys. pt.
- $N_f = (2+1+1)$ staggered quarks. Isospin Limit.
- Large Volume: $(L, T) \sim (6, 9-12) fm$.
- $\beta(a) = \frac{6}{g^2(a)} \leftrightarrow a[fm]$ via $M_\Omega^{lat} = M_{\Omega\_}^{phys} a[fm]/(\hbar c)$.

Input Quark Mass (m_{ud}^0, m_s, m_c) Tuning

$$\left[\frac{M_\pi^2}{M_\Omega^2}\right]_{lat} \simeq \left[\frac{M_{\pi^0}^2}{M_{\Omega\_}^2}\right]_{phys}, \quad \left[\frac{M_K^2 - M_\pi^2/2}{M_\Omega^2}\right]_{lat} \simeq \left[\frac{(M_{K^+}^2 + M_{K^0}^2 - M_{\pi^0}^2)/2}{M_{\Omega\_}^2}\right]_{phys}, \quad \frac{m_c}{m_s} = 11.85\,.$$

Control of Various Systematics

- Scale Setting in 0.2% Precision.
  M_Ω^{lat} in 0.1% precision.
  $M_\Omega^{lat}\big|_{w.isb} = 1672.45(29)[MeV] \cdot \frac{a[fm]}{\hbar c}$.
- Isospin Breaking.
- Finite a Effect: 15% correction at each simulation with XPT and window method. c.f. Staggered taste violation.
- Finite Volume: 2.74(34)% correction at continuum. Simulation based estimate (HEX fermions) as well as NNLO XPT. c.f. $(\frac{m_\mu}{2\hbar c})^{-1} \sim 4fm$, $L_{ref} = 6.274fm$.
- Fermion choice independence. Additional simulations with overlap valence quarks.

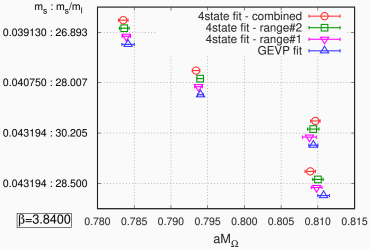


Fig: M_Ω^{lat} at $\beta = 3.8400$. We have 4 ensembles. For each, 4 estimates.

QED and Strong-Isospin Breaking Corrections

$$\mathcal{O}(\alpha) \sim \mathcal{O}\left(\frac{m_d - m_u}{\Lambda_{QCD}}\right) \sim 1\% \text{ Correction} \quad .$$

Isospin Breaking Perturbatively

- Iso-symm. LQCD (U) + Stochastic QED (A_μ with $P \propto e^{-S_\gamma}$).

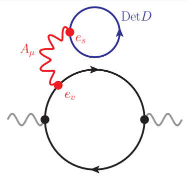
$$Z = \int \mathcal{D}U\, e^{-S_g[U]} \int \mathcal{D}A\, e^{-S_\gamma[A]} \prod_{f=u,d,s,c} \mathrm{Det}\, D[Ue^{ieq_f A}, m_f]\,. \qquad (4)$$

- QED_L [Hayakawa PTP2008] in Coulomb gauge.
  - Remove spatial zero-mode, $a^3 \sum_{\vec{x}} A_{\mu,x} = 0$. c.f. Gauss's Law.
  - Preserve reflection positivity, i.e. well-defined charged particles. (no constraint like $\lim_{\xi\to\infty} \exp[-a^4 \sum_{t,\vec{x}} A_{\mu,x}/\xi^2]$.)

- Expand w.r.t. $\alpha = e^2/(4\pi)$ and $\delta m = m_d - m_u$:

$$\langle O[Ue^{ie_v q_f A}, m_f]\rangle = \langle O[U, m_f^0]\rangle_U + \frac{\delta m}{m_{ud}^0}\langle O\rangle'_m + e_v^2\langle O\rangle''_{20} + e_v e_s \langle O\rangle''_{11} + e_s^2 \langle O\rangle''_{02}\,,$$

e.g. $\langle O\rangle''_{11} = \left\langle \left\langle \frac{\partial O}{\partial e_v}\Big|_{e_v\to 0} \frac{\partial}{\partial e_s} \prod_f \frac{\mathrm{Det}\ D[Ue^{ie_s q_f A}, m_f^0]}{\mathrm{Det}\ D[U, m_f^0]} \right\rangle_A \Big|_{e_s\to 0} \right\rangle_U$



- Larger num. of stochastic A_μ with sea-quarks. for noise control.

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- Iso-symm. LQCD (U) + Stochastic QED (A_μ with $P \propto e^{-S_\gamma}$).

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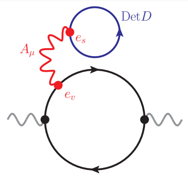

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- QED_L [Hayakawa PTP2008] in Coulomb gauge.
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  - Preserve reflection positivity, i.e. well-defined charged particles. (no constraint like $\lim_{\xi\to\infty} \exp[-a^4 \sum_{t,\vec{x}} A_{\mu,x}/\xi^2]$.)

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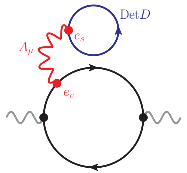
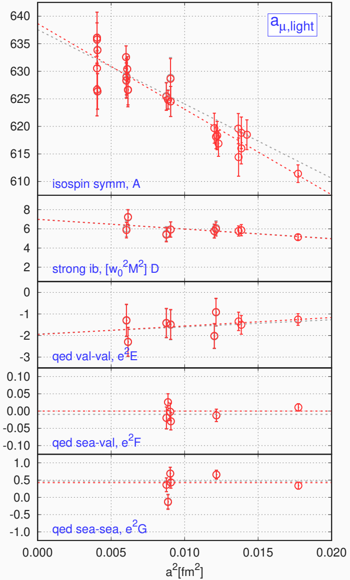
e.g. $\langle O\rangle''_{11} = \left\langle\left\langle \frac{\partial O}{\partial e_v}\Big|_{e_v\to 0} \frac{\partial}{\partial e_s} \prod_f \frac{\mathrm{Det}\ D[Ue^{ie_s q_f A}, m_f^0]}{\mathrm{Det}\ D[U, m_f^0]}\right\rangle_A \Big|_{e_s\to 0}\right\rangle_U$



- Larger num. of stochastic A_μ with sea-quarks. for noise control.

Continuum Global Fit



- Mass Corrections:

$$M^2 = [M_{dd}^2 - M_{uu}^2]_{dat} \; ,$$

$$\Delta M_{\pi_\chi}^2 = \left[\frac{M_{uu}^2 + M_{dd}^2}{2}\right]_{dat'} - \left[\frac{M_{uu}^2 + M_{dd}^2}{2}\right]_{phys} \; ,$$

$$\Delta M_{ss} = [M_{ss}]_{dat'} - [M_{ss}]_{phys} \; .$$

- Fit Model:

$$a_{\mu,light}^{dat}[a^2, m_f^0, \delta m, e_{v,s}]$$

$$= (A_0 + A_a a^2)(1 + B \Delta \hat{M}_{\pi_\chi}^2 + C \Delta \hat{M}_{ss}^2)$$

$$+ (D_0 + D_a a^2 + D_l \Delta \hat{M}_{\pi_\chi}^2 + D_s \Delta \hat{M}_{ss}^2) M^2 w_0^2$$

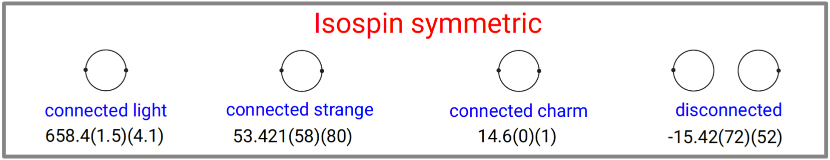
$$+ (E_0 + E_a a^2 + E_l \Delta \hat{M}_{\pi_\chi}^2 + E_s \Delta \hat{M}_{ss}^2) e_v^2$$

$$+ F \, e_v e_s$$

$$+ G \, e_s^2 \; .$$

- Correlations among observables are taken account in χ^2 defined with Covariance Matrix.

Isospin Symmetric Contributions



- Light quark contribution:

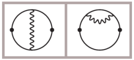
$$a_{\mu,ud}^{\text{iso-sym}} = A_{0,ud} + \Delta^{FV} a_{\mu,ud}$$
$$= 636.7(1.5)(3.1) + \frac{10}{9} \cdot 19.5(2.0)(1.4) = 658.4(1.5)(4.1) \ .$$

- Greatly suppressed uncertainties from PRL2018 (left) to Present (right),

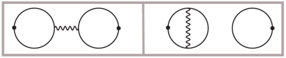
$$a_{\mu,\,ud}^{\text{LO-HVP}} : 647.6(7.5)(17.7)[3.0\%] \ \rightarrow 658.4(1.5)(4.1)[0.7\%] \ .$$

SIB/QED Corrections

QED isospin breaking, valence



connected
-1.20(32)(29)



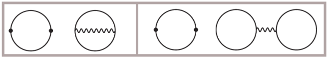
disconnected
-0.57(13)(11)

Etc.

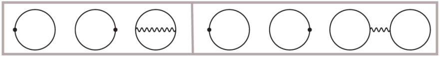
bottom; higher order; perturbative

0.11(4)

QED isospin breaking, sea

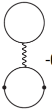


connected
0.51(13)(22)

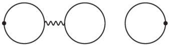


disconnected
-0.034(31)(19)

QED isospin breaking, mixed



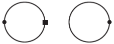
connected
-0.0065(60)(42)



disconnected
0.011(23)(12)

Strong isospin breaking

connected
6.32(55)(49)



disconnected
-3.57(42)(41)

BMW-2020 Summary

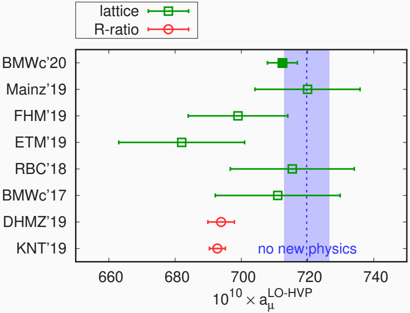


Figure: LO-HVP muon g-2 comparison.

BMW-2020

- $a_\mu^{\text{LO-HVP}} = 712.4(1.9)(4.0), \ 0.6\%$
- $w_{0,*} = 0.17180(18)(35)[\text{fm}], \ 0.2\%$
- LMA, Simulation-based SIB/QED/FV, full systematics of $O(10^5)$.
- Consistent with "no new physics".
- $(3.1/3.9)\sigma$ tension to DHMZ19/KNT19.

c.f. (no new phys.)
= (BNL-E821) − (SM wo. LO-HVP).

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LO-HVP Correction for Running $\alpha(Q^2)$

- Running Coupling: $\alpha(s) = \frac{\alpha(0)}{1-\Delta\alpha(s)}$, $\alpha(0) = \frac{1}{137.03\cdots}$.
- HVP Corrections with Data-Driven Dispersion:
  $\Delta^{had}\alpha(M_z^2) = 0.02761(11)$ [Keshavarzi et.al. PRD2019].
- Electroweak Global Fits [Keshavarzi et.al. 2006.12666]:
  $\Delta^{had}\alpha(M_z^2) = 0.2722(39)(12)$ and $M_{higgs} = 94^{+20}_{-18}$.
- Connection to LQCD [Jegerlehner hep-ph/0807.4206] (not yet in this talk):

$$\begin{aligned} \Delta^{had}\alpha(M_z^2) &= \Delta^{had}\alpha(-Q_0^2) \quad \longleftarrow 4\pi\hat{\Pi}_{lat}(Q_0^2) \\ &+ \left[\Delta^{had}\alpha(-M_z^2) - \Delta^{had}\alpha(-Q_0^2)\right]_{pqcd} \\ &+ \left[\Delta^{had}\alpha(M_z^2) - \Delta^{had}\alpha(-M_z^2)\right]_{pqcd} . \end{aligned} \quad (5)$$

- EW Physics with $\Delta^{had}\alpha(M_z^2)$ from LQCD estimate for $\Delta^{had}\alpha(-Q_0^2)$?

EW Global Fits

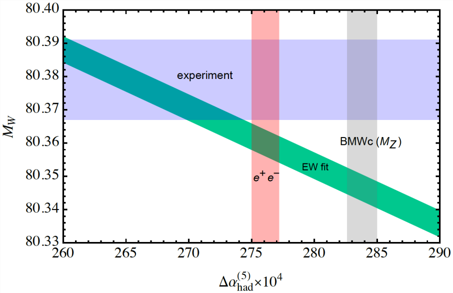


Figure:
Quoted from Crivellin et al, 2003.04886. Gray band is Project 1: $1.028 \cdot \Delta^{\text{had}}\alpha(M_Z^2)|_{pheno}$ is used as a prior in EW global fits.

- Pheno HVP:
  $$\Delta^{\text{had}}\alpha(s)|_{pheno} = \frac{-\alpha s}{3\pi}\int_0^\infty ds' \frac{R(s')}{s'(s'-s)} .$$
- Pheno Muon g-2:
  $$a_\mu^{\text{LO-HVP}}|_{pheno} = \left(\frac{\alpha}{\pi}\right)^2 \int ds' K(s', m_\mu^2) R(s') .$$
- Project 1:
  $R(s') \to 1.028 \cdot R(s')$ so that $a_\mu^{\text{LO-HVP}}|_{pheno} \to a_\mu^{\text{LO-HVP}}|_{BMW2020}$. Then, $\Delta^{\text{had}}\alpha(M_Z^2)|_{pheno} \to 1.028 \cdot \Delta^{\text{had}}\alpha(M_Z^2)|_{pheno}$.

BMW $\Delta^{had}\alpha(-Q^2)$

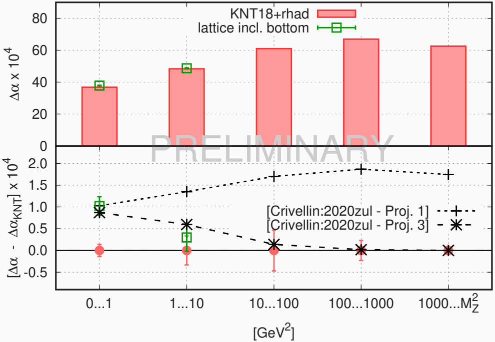


Figure: BMW2020 $\Delta^{had}\alpha(-Q^2)$ is compared with Data-Driven Pheno (KNT-18 + rhad).

- **Upper:** From the left,

  $[\Delta^{had}\alpha(\text{-}1) - \Delta^{had}\alpha(0)], [\Delta^{had}\alpha(\text{-}10) - \Delta^{had}\alpha(\text{-}1)], [\Delta^{had}\alpha(\text{-}100) - \Delta^{had}\alpha(\text{-}10)], \cdots$.

- **Lower:** KNT-Central Values (KNT-CV)are subtracted from the upper panel.
  $[+] = [\mathrm{KNT}(1.028)_{s\le M_Z^2}] - [\text{KNT-CV}]$, $[*] = [\mathrm{KNT}(1.028)_{s\le 1.94^2}] - [\text{KNT-CV}]$

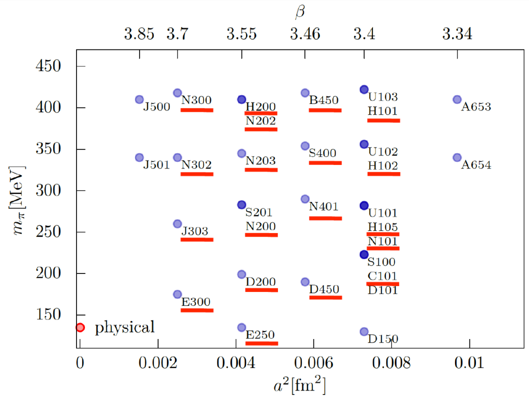
- Project 1 $(+)$ is shown to be too aggressive.

Mainz $\Delta^{\rm had}\alpha(Q^2)$ Collaboration

M. Cè, A. Géradin, H.B. Meyer, K. Miura, Teseo San José, and H. Wittig.

Reference: M. Cè et.al. PoS**Lattice2019** (2020), arXiv:1910.09525.

Mainz/CLS Ensembles



CLS Ensembles: [Bruno et al. JHEP2015].

- $N_f = (2{+}1)$ $\mathcal{O}(a)$ Improved Wilson-Clover Fermions.
- $\mathcal{O}(a^2)$ Improved Lüscher-Weisz Gauge Action.
- $M_\pi L = 4.1 - 6.4$.
- Mostly Open Boundary Conditions.
- $\beta(a) = \frac{6}{g^2(a)} \leftrightarrow a[fm]$ via $\frac{2}{3}(f_K + \frac{f_\pi}{2})$ [Bruno et.al. PRD2017].
- Low-Mode Deflation, Hierarchical Probe.

HVP Chiral/Continuum Extrap.

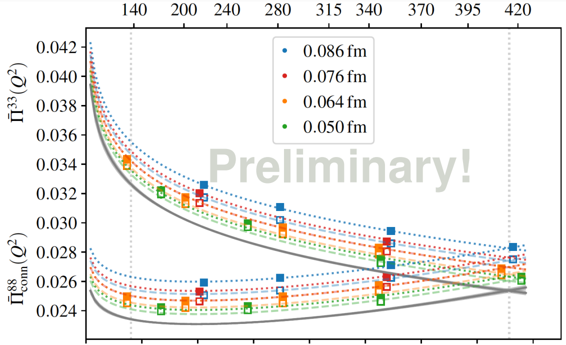
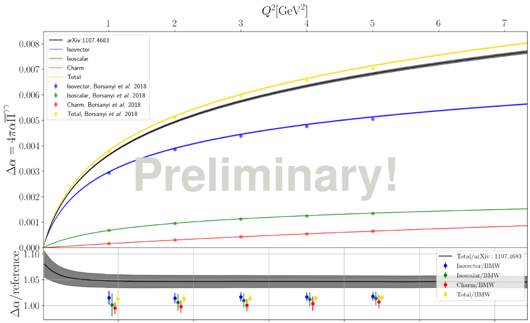


Fig: Chiral and Continuum Extrapolations at $Q^2 = 1$ $[GeV^2]$.
$\Pi^{33/88}$ = Isovector/Isoscalar plotted against $M_\pi [MeV]$.
Gray-bands shows continuum limits for a given M_π.

LQCD vs. Pheno.



- **Fig.:** $\Delta^{had}\alpha(Q^2)$ Comparison. Mainz/CLS vs. BMW [Borsarnyi et al. PRL2018] vs. Pheno [Jegerlehner, alphaQED19].
- Mainz/CLS total (yellow band) with no ISB corrections already larger than data-driven Pheno. (gray band).

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Summary

- BMWc has achieved per-mil level precision science in LQCD approach to LO-HVP muon g-2 with full systematics: $a_\mu^{\text{LO-HVP}} = 712.4(1.9)(4.0)\ , 0.6\%$.
- The BMW result is consistent with No New Physics, while it shows $(3.1/3.9)\sigma$ tension to data-driven pheno. DHMZ19/KNT19.
- LQCD-Pheno tension has led to new discussion in EW physics via $\Delta^{\text{had}}\alpha(Q^2)$.
- Both BMW and Mainz/CLS provide somewhat larger $\Delta^{\text{had}}\alpha(Q^2)$ than the data-driven method.
- Need to update LQCD consensus from whitepaper to per-mil presision.
- Need to specify a source of the above tensions. Some missing contributions in the integral of R-ratio? Problem in modeling the region $\sqrt{s} < 0.7 GeV$? [Keshavarzi et.al.(2006.12666)].
- Need to investigate connection between $\Delta^{\text{had}}\alpha(M_Z^2)$ and $\Delta^{\text{had}}\alpha(-Q^2)$ in detail, where the latter is accessible by LQCD.

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M_Ω

- **4-State Fit:**

$$h(t, A, M) = A_0 h_+(M_0, t) + A_1 h_-(M_1, t) + A_2 h_+(M_2, t) + A_3 h_-(M_3, t) ,$$
$$h_+(M, t) = e^{-Mt} + (-1)^{t-1} e^{-M(T-t)} , \qquad h_-(M, t) = -h_+(M, T-t) .$$

- **GEVP:** Construct

$$\mathcal{H}(t) = \begin{pmatrix} H_{t+0} & H_{t+1} & H_{t+2} & H_{t+3} \\ H_{t+1} & H_{t+2} & H_{t+3} & H_{t+4} \\ H_{t+2} & H_{t+3} & H_{t+4} & H_{t+5} \\ H_{t+3} & H_{t+4} & H_{t+5} & H_{t+6} \end{pmatrix} . \tag{6}$$

Solve $\mathcal{H}(t_a)v(t_a, t_b) = \lambda(t_a, t_b)\mathcal{H}(t_b)v(t_a, t_b)$.
Project out the ground state: $v^+(t_a, t_b)\mathcal{H}(t)v(t_a, t_b)$.
Fit the grand state to $\exp[-M_\Omega t]$.

Perturbative SIB/QED

- (QCD + QED) with strong isospin breaking:

$$Z = \int \mathcal{D}U \, e^{-S_g[U]} \int \mathcal{D}A \, e^{-S_\gamma[A]} \prod_{f=u,d,s,c} \mathrm{Det} M^{1/4}[U e^{ieq_f A}, m_f] \, . \qquad (7)$$

- *QED_L* in Coulomb gauge.
- Perturbative expansion w.r.t. $\alpha = e^2/(4\pi)$ and $\delta m = m_d - m_u$.
- Stochastic QED: N_{src} is optimised depending on valence $O[Ue^{ie_v q_f A}, m_f]$ or sea $R[Ue^{ie_s q_f A}, m_f] = \prod_f \mathrm{Det} M^{1/4}[Ue^{ie_s q_f A}, m_f] \, / \, \prod_f \mathrm{Det} M_0^{1/4}[U, m_f|_{\delta m \to 0}]$.
- $\langle O[Ue^{ie_v q_f A}, m_f] \rangle = \langle O_0 \rangle_U + \frac{\delta m}{m_l} \langle O \rangle'_m + e_v^2 \langle O \rangle''_{20} + e_v e_s \langle O \rangle''_{11} + e_s^2 \langle O \rangle''_{02}$,
  strong isospin: $\langle O \rangle'_m = m_l \left\langle \frac{\partial O}{\partial \delta m} \Big|_{\delta m \to 0} \right\rangle_U$,
  qed valence-valence: $\langle O \rangle''_{20} = \frac{1}{2} \left\langle \left\langle \frac{\partial^2 O}{\partial e_v^2} \right\rangle_A \Big|_{e_v \to 0} \right\rangle_U$,
  qed sea-valence: $\langle O \rangle''_{11} = \left\langle \left\langle \frac{\partial O}{\partial e_v} \frac{\partial R}{\partial e_s} \right\rangle_A \Big|_{e_v, e_s \to 0} \right\rangle_U$,
  qed sea-sea: $\langle O \rangle''_{02} = \left\langle O_0 \left\langle \cdot \frac{1}{2} \frac{\partial^2 R}{\partial e_s^2} \right\rangle_A \Big|_{e_s \to 0} \right\rangle_U - \langle O_0 \rangle_U \left\langle \left\langle \frac{1}{2} \frac{\partial^2 R}{\partial e_s^2} \right\rangle_A \Big|_{e_s \to 0} \right\rangle_U$.

SIB/QED in Various Observables

O	$\langle O\rangle'_m$	$\langle O\rangle''_{20}$	$\langle O\rangle''_{11}$	$\langle O\rangle''_{02}$
$M_\Omega, M_{\pi_\chi}, M_{K_\chi}$	—	★	★	★
$\Delta M_K^2, \Delta M^2$	★	★	★	—
w_0	—	—	—	★
$C_{l=ud}(t)$	★	★	★	★
$C_s(t)$	—	★	★	★
$D(t)$	★	★	★	★

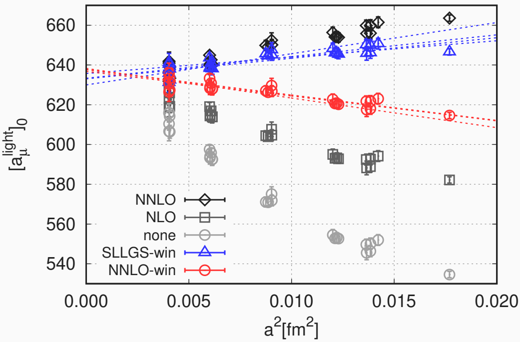
strong isospin: $\langle O\rangle'_m = m_l \left\langle \frac{\partial O}{\partial \delta m}\Big|_{\delta m \to 0} \right\rangle_U$,

qed valence-valence: $\langle O\rangle''_{20} = \frac{1}{2}\left\langle \left\langle \frac{\partial^2 O}{\partial e_v^2} \right\rangle_A \Big|_{e_v \to 0} \right\rangle_U$,

qed sea-valence: $\langle O\rangle''_{11} = \left\langle \left\langle \frac{\partial O}{\partial e_v} \frac{\partial R}{\partial e_s} \right\rangle_A \Big|_{e_v, e_s \to 0} \right\rangle_U$,

qed sea-sea: $\langle O\rangle''_{02} = \left\langle O_0 \left\langle \cdot \frac{1}{2} \frac{\partial^2 R}{\partial e_s^2} \right\rangle_A \Big|_{e_s \to 0} \right\rangle_U - \langle O_0\rangle_U \left\langle \left\langle \frac{1}{2} \frac{\partial^2 R}{\partial e_s^2} \right\rangle_A \Big|_{e_s \to 0} \right\rangle_U$.

Discretization Corrections



- Corrections depend on Windows: Win1: $t \in [0.5, 1.3] fm$, Win2: $t > 1.3 fm$.
- In advance to the continuum extrapolation, we correct data points as:

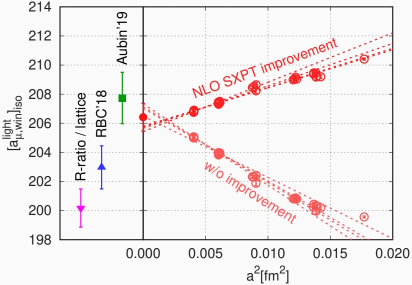
$$[a_\mu^{\text{light}}]_0(L,a) \to [a_\mu^{\text{light}}]_0(L,a) + (10/9)\left[a_{\mu,\text{win1}}^{\text{NLO-XPT}}(6.272 fm) - a_{\mu,\text{win1}}^{\text{NLO-SXPT}}(L,a)\right] + (10/9)\left[a_{\mu,\text{win2}}^{\text{NNLO-XPT}}(6.272 fm) - a_{\mu,\text{win2}}^{\text{NNLO-SXPT}}(L,a)\right] .$$

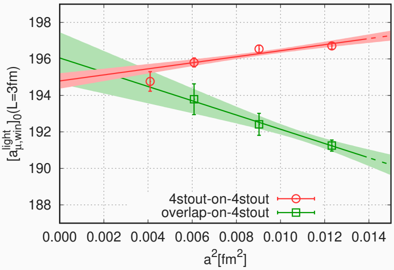
Finite Volume (FV) Effect for Isovector

- FV corrections for a continuum extrapolated iso-vector contribution $a_\mu^{\text{iso-v}}$.
- The average spatial extent of main ensembles (4stout): $L_{ref} = 6.274 fm$.
- 4HEX fermion ensembles: $L_{hex} = 10.752 fm$, $a = 0.112 fm$ with small UV artefact.
- FV via HEX and Models combined:

$$\begin{aligned}\Delta^{FV} a_\mu^{\text{iso-v}} &\equiv a_\mu^{\text{iso-v}}(\infty) - a_\mu^{\text{iso-v}}(6.274 fm) \,, \\ &= \left[a_\mu^{\text{iso-v}}(\infty) - a_\mu^{\text{iso-v}}(10.752 fm)\right]_{\text{NNLO XPT etc.}} \\ &\quad + \left[a_{\mu,4hex}^{\text{iso-v}}(10.752 fm) - a_{\mu,4stout}^{\text{iso-v}}(6.274 fm)\right]_{\text{LQCD}} \\ &= 1.4 + 18.1(2.0)(1.4) = 19.5(2.0)(1.4) \,.\end{aligned}$$

Window Method





Left: $[a^{\text{LO-HVP}}_{\mu,win,\,ud}]_{iso}$ from the window $t \in [0.4, 1.0]\,fm$.

Right: Comparison of $[a^{\text{LO-HVP}}_{\mu,win,\,ud}]_{iso}$ from 4stout and overlap valence quarks.