2020/5/20 @ R-CCS

Thermodynamics of 2+1 flavor QCD with the SF*t*X method based on the gradient flow

SF*t*X method : small flow-time expansion method

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Gradient Flow

Narayanan-Neuberger (2006), Lüscher (2010-)

Gradient Flow

(example) Yang-Mills theory in the continuum

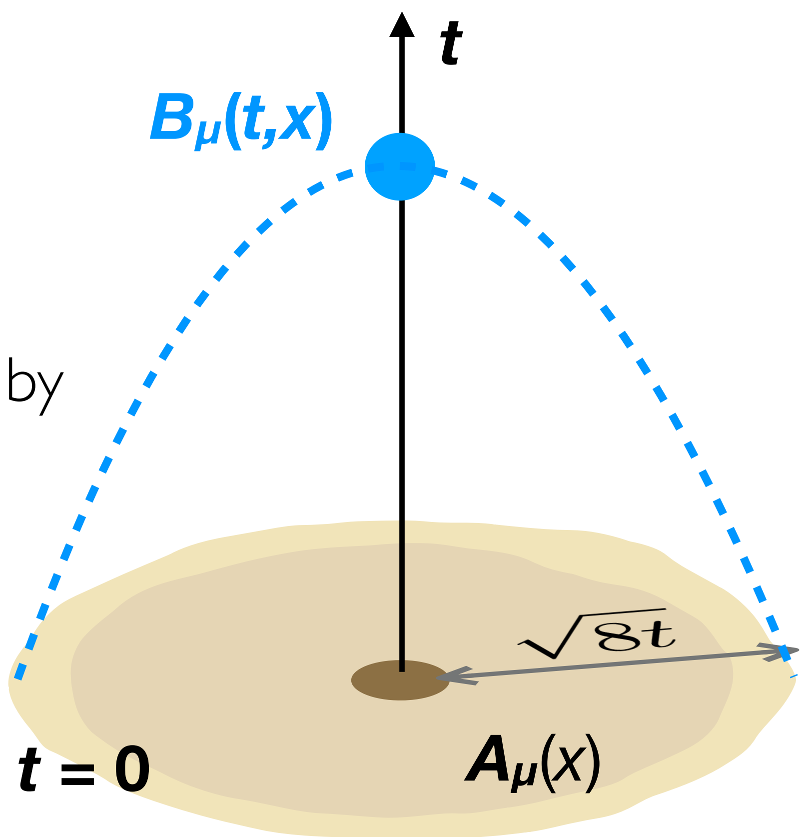
Original theory: gauge field $A_\mu(x)$ in D=4 dim. space-time,

$$S_{\rm YM}[A_\mu] = -\frac{1}{2g_0^2}\int d^Dx\,{\rm tr}[F_{\mu\nu}F_{\mu\nu}] = \frac{1}{2g_0^2}\int d^Dx\,F^a_{\mu\nu}F^a_{\mu\nu}$$

Introduce a fictitious "time" t, and evolve ("flow") the field A_μ by

$$\partial_t B_\mu(t,x) = -g_0^2\frac{\delta S_{\rm YM}[B_\mu]}{\delta B_\mu} = D_\nu G_{\nu\mu}(t,x)$$

with $B_\mu(t=0,x) = A_\mu(x)$. $\quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$



This is a kind of diffusion equation.

Its perturbative solution reads

B_μ ~ smeared A_μ over a physical range of $\sqrt{(8t)}$.

("8" = 2 x D with D=4)

Quantum expectation values $\stackrel{\text{def.}}{=}$ path-integration over the original fields A_μ

$$\langle B_\mu(t,x)B_\nu(s,y)\cdots\rangle \stackrel{\text{def.}}{=} \frac{1}{Z}\int\mathcal{D}A_\mu\,B_\mu(t,x)B_\nu(s,y)\cdots e^{-S[A_\mu]}$$

Flowed operators are free from UV divergences and short-distance singularities.

Lüscher-Weisz (2011)

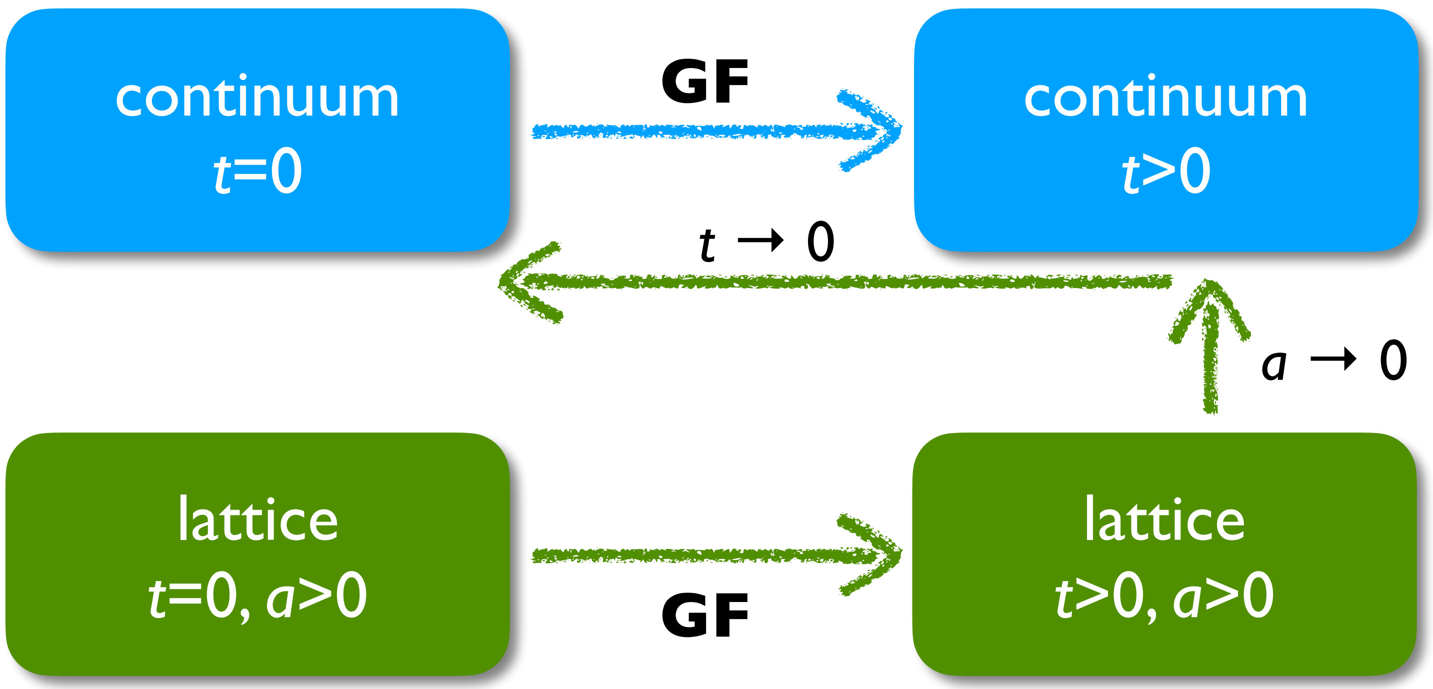
SFtX method based on GF

H. Suzuki, PTEP 2013, 083B03 (2013) [E: 2015, 079201]

Making use of the finiteness of the GF, H. Suzuki developed **a general method to correctly calculate any renormalized observables non-perturbatively on the lattice**.

Small Flow-*t*ime eXpansion (SF*t*X) method

physical obs.'s we want



finite,
physically well-defined

We can safely evaluate their expectation values non-perturbatively

by constructing corresponding operators on the lattice.

Because we can construct a lattice operator directly from the continuum operator,
this method is applicable also to observables whose base symmetry is broken on the lattice (**Poincaré inv.** etc.)

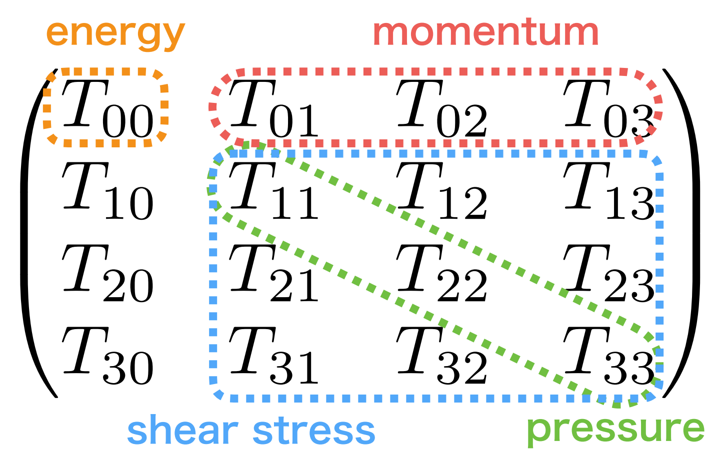
⇨ energy-momentum tensor

energy-momentum tensor

<u>In continuum</u>, EMT is defined as the generator of Poincaré transformation.

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu}=\delta_{\mu\nu}} = \frac{1}{g_0^2} \left[F^a_{\mu\rho} F^a_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} F^a_{\rho\sigma} F^a_{\rho\sigma} \right] \quad \text{for YM theory}$$

- source of the gravity
- conserved Noether current associated with the Poincaré inv.
- a fundamental observable of the theory to extract
  - EoS (energy, pressute), momentum, shear stress, ...
  - fluctuation/correlation functions => specific heat, viscosity, ...



<u>On the lattice</u>, the Poincaré invariance is explicitly broken.

We have to

- fine-tune the **renormalization** and **mixing coefficients** of many operators

  to make the current conserved and to get the correct values of en. density etc. in the continuum limit.

Caracciolo et al., NP B309, 612 (1988); Ann.Phys. 197, 119 (1990)

$$\{T_{\mu\nu}\}_R(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_\rho F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar\psi(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar\psi(x) \overleftrightarrow{\not{D}} \psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar\psi(x) \psi(x),$$

allowed by the lattice rotation symmetry =>

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_\rho F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu} \bar\psi(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

YM EMT with the SF*t*X method

Small-*t* expansion

Lüscher-Weisz, JHEP1102.051(2011)
Suzuki, PTEP 2013, 083B03 [E: 2015, 079201]

At small *t*, flowed operators can be expanded in terms of un-flowed operators.

In QCD, the coefficients at small *t* can be calculated by perturbation theory thanks to AF.

For YM EMT,

$$U_{\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x) = \alpha_U(t)\left[T_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu}T_{\rho\rho}(x)\right] + O(t),$$

$$E(t,x) \equiv \frac{1}{4}G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x) = \langle E(t,x)\rangle_0 + \alpha_E(t)T_{\rho\rho}(x) + O(t).$$

with

$$\alpha_U(t) = \bar{g}(1/\sqrt{8t})^2[1 + 2b_0\bar{s}_1\bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4)],$$

$$\alpha_E(t) = \frac{1}{2b_0}[1 + 2b_0\bar{s}_2\bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4)],$$

$$\bar{s}_1 = \frac{7}{22} + \frac{1}{2}\gamma_E - \ln 2 \simeq -0.08635752993,$$

$$\bar{s}_2 = \frac{21}{44} - \frac{b_1}{2b_0^2} = \frac{27}{484} \simeq 0.05578512397,$$

$$b_0 = \frac{1}{(4\pi)^2}\frac{11}{3}N_c,\ b_1 = \frac{1}{(4\pi)^4}\frac{34}{3}N_c^2 \text{ with } N_c = 3$$

Inverting these, **the correctly normalized EMT is given by**

$$T_{\mu\nu}(x) = \lim_{t\to 0}\left[\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}(E(t,x) - \langle E(t,x)\rangle_0)\right]$$

matching coefficients

- to make $t \to 0$ smoother by removing known small-*t* mixings & *t*-dep. in the continuum
- to match the renormalization schemes, when the observable is scheme-dependent

They are finite => safe to evaluate on the lattice.

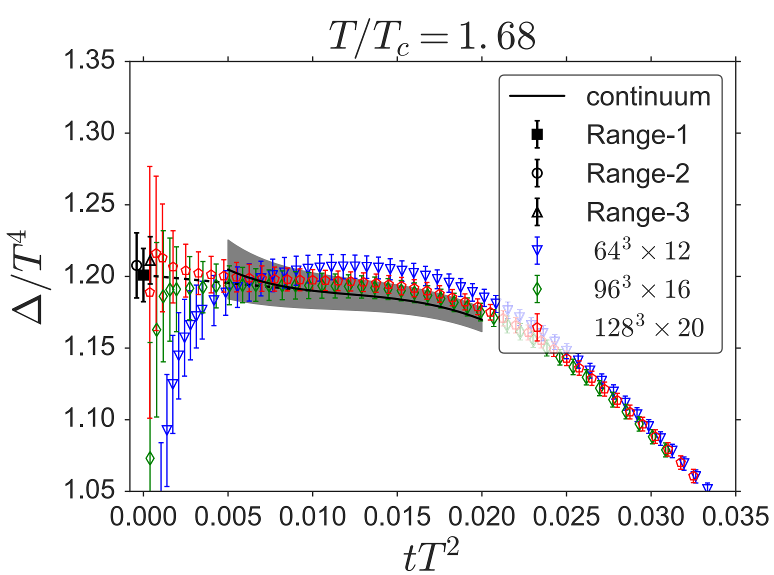
A test in quenched QCD (FlowQCD Collab.)

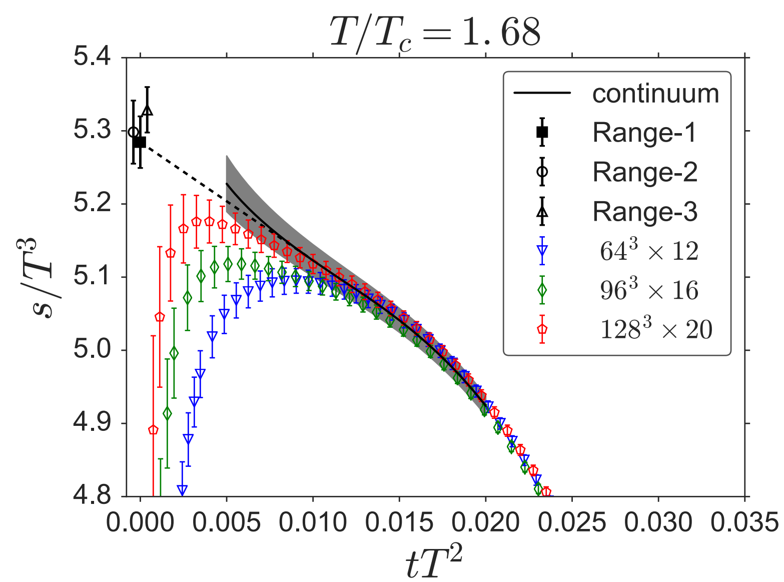
Kitazawa-Iritani-Asakawa-Hatsuda-Suzuki, Phys.Rev. D 94, 114512 (2016)

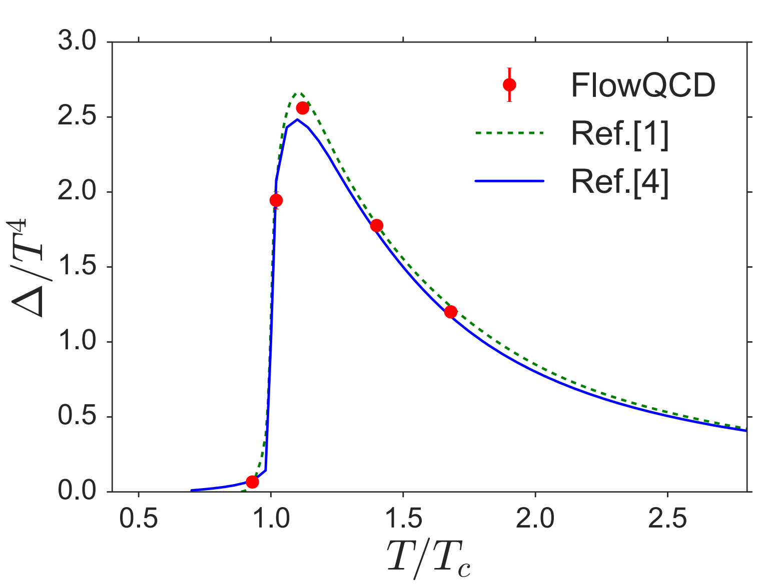
- Wilson plaquette gauge action both for $S_{YM}[A_\mu]$ and $S_{YM}[B_\mu]$.
- Clover definition for $G_{\mu\nu}$.
- Improved operator for E : $O(a^2)$ removed in the tree-level.

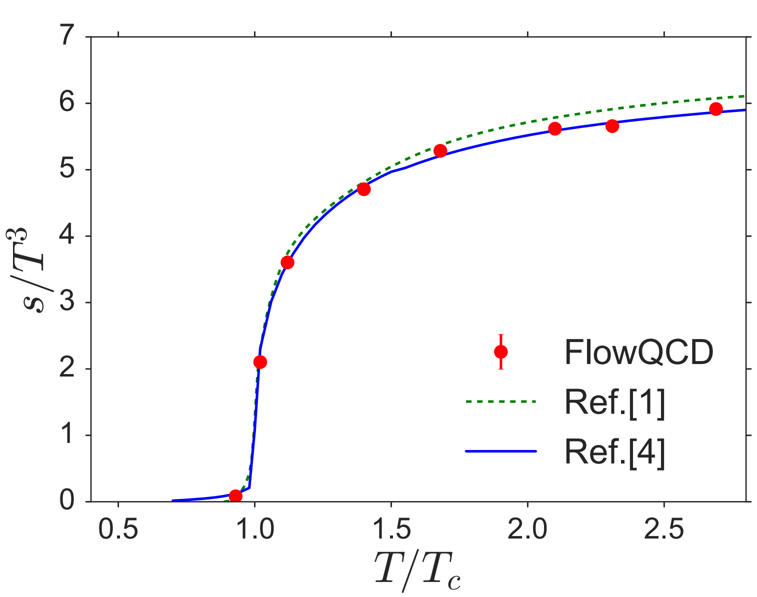
***a* → 0 & *t* → 0** $\Delta = \varepsilon - 3p = -\langle T_{\mu\mu}(x)\rangle$ $sT = \varepsilon + p = -\langle T_{44}(x)\rangle + \frac{1}{3}\sum_{i=1}^{3}\langle T_{ii}(x)\rangle$

T/T_c	β	N_s	N_τ
0.93	6.287	64	12
	6.495	96	16
	6.800	128	24
1.02	6.349	64	12
	6.559	96	16
	6.800	128	22
1.12	6.418	64	12
	6.631	96	16
	6.800	128	20
1.40	6.582	64	12
	6.800	128	16
	7.117	128	24
1.68	6.719	64	12
	6.941	96	16
	7.117	128	20
2.10	6.891	64	12
	7.117	128	16
	7.296	128	20
2.31	7.200	96	16
	7.376	128	20
	7.519	128	24
2.69	7.086	64	12
	7.317	96	16
	7.500	128	20









Lattice error expected at $\sqrt{(8t)} \leq a$
[$tT^2 \leq 1/(8N_t^2)$ ~ 0.0009, 0.0003 for N_t=12, 20]

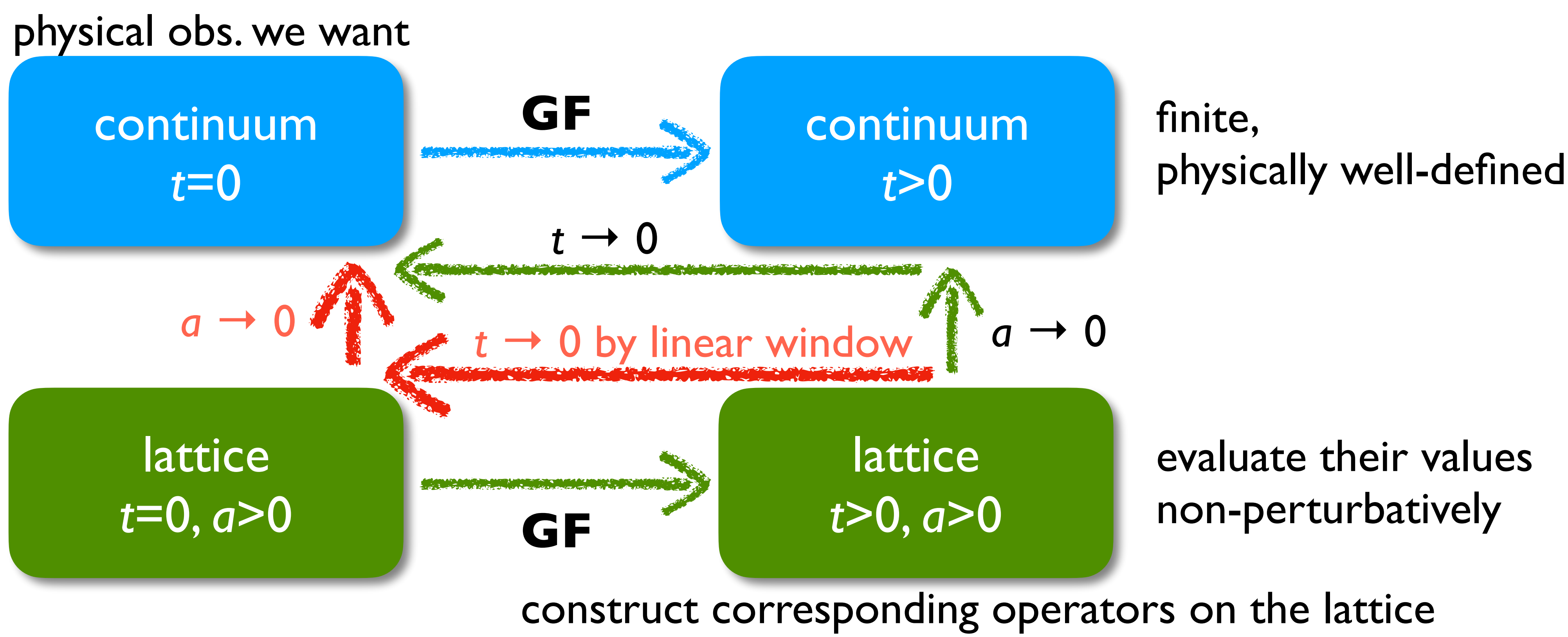
=> SF*t*X well reproduces the results of conventional integral method.

SF*t*X method based on GF

H. Suzuki, PTEP 2013, 083B03 (2013) [E: 2015, 079201]

Making use of the finiteness of the GF, H. Suzuki developed **a general method to correctly calculate any renormalized observables non-perturbatively on the lattice**.

Small Flow-*t*ime eXpansion (SF*t*X) method



Because we can construct a lattice operator directly from the continuum operator,
this method is applicable also to observables whose base symmetry is broken on the lattice
(Poincaré inv., **chiral sym.**, etc.)

⇨ energy-momentum tensor

⇨ **QCD with Wilson-type quarks,** to cope with the problems due to chiral violation.

When we can identify a proper window, we may exchange the order of two extrapolations.

[0] Introduction

[1] N_F = 2+1 QCD with slightly heavy u,d and ≈physical s quarks

[1A] Issue of renormalization-scale in N_F = 2+1 QCD with slightly heavy u,d
--- an improvement of the SF*t*X method ---

[1B] 2-loop matching coefficients in N_F = 2+1 QCD with slightly heavy u,d

[2] N_F = 2+1 QCD with physical u,d,s quarks
--- a status report ---

[3] Summary

[1]
N_F = 2+1 QCD
with slightly heavy u,d
and ≈physical s quarks

Taniguchi-Ejiri-Iwami-KK-Kitazawa-Suzuki-Umeda-Wakabayashi, Phys.Rev. D 96, 014509 (2017)

Taniguchi-KK-Suzuki-Umeda, Phys.Rev. D 95, 054502 (2017)

test of SF*t*X with dynamical quarks

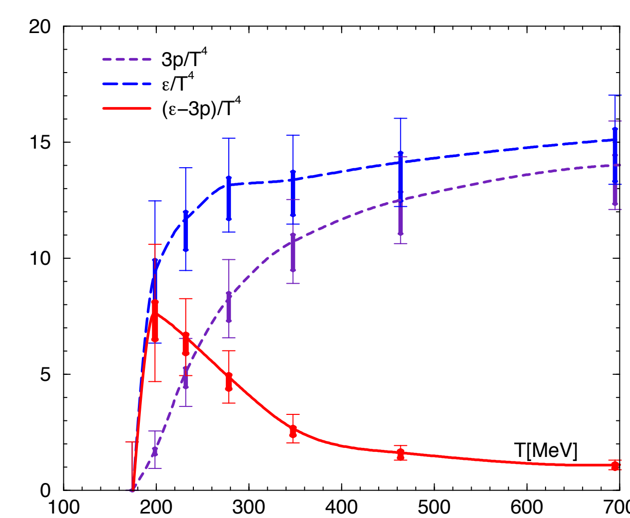
Taniguchi-Ejiri-Iwami-KK-Kitazawa-Suzuki-Umeda-Wakabayashi, Phys.Rev. D 96, 014509 (2017)

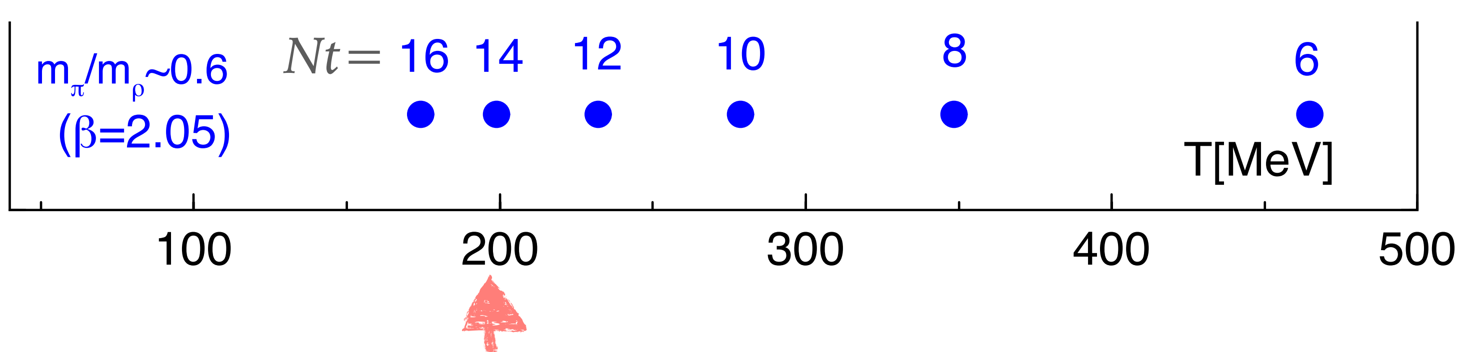
As the 1st step with dynamical quarks:

- Heavy ud quarks ($m_\pi/m_\rho \approx 0.63$) with ≈physical s quark ($m_{\eta ss}/m_\phi \approx 0.74$).
- Fine lattice ($a \approx 0.07$fm with improved action) using the fixed-scale approach.
- Compare with EoS by the conventional *T*-integration method.

WHOT-QCD Collab., Phys.Rev. D 85, 094508 (2012)

- ☑ N_f=2+1 QCD, RG-improved Iwasaki gauge + NP O(a)-improved Wilson quarks
- ☑ CP-PACS+JLQCD's $T=0$ config. ($\beta = 2.05$, 28^3x56, $a \approx 0.07$fm, $m_\pi/m_\rho \approx 0.63$):
  the lightest and the finest among the 3β x $5m_{ud}$ x $2m_s$ data points available.
- ☑ $T > 0$ by fixed-scale approach, WHOT-QCD config.(32^3xNt, Nt = 4, 6, 8, 10, 12, 14, 16)
- ☑ gauge measurements at every config.
- ☑ quark measurements every 10 config's, using a noisy estimator method.
- ☐ continuum extrapolation => to do





$T_{pc} \approx 190$ MeV

T (MeV)	T/T_{pc}	N_t	$t_{1/2}$	gauge confs.
0	0	56	24.5	650
174	0.92	16	8	1440
199	1.05	14	6.125	1270
232	1.22	12	4.5	1290
279	1.47	10	3.125	780
348	1.83	8	2	510
464	2.44	6	1.125	500
697	3.67	4	0.5	700

GF with dynamical quarks

Lüscher, JHEP1304.123(2013)

For the finiteness, the flow action can be different from the original action as far as the gauge-covariance is preserved. To include quarks (matter fields), Lüscher proposed a simple method, in which the gauge flow is the same as the pure gauge case.

gauge flow

the same as the pure YM case

original gauge field at $t = 0$

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x), \qquad B_\mu(t=0,x) = A_\mu(x)$$

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)],$$

$$D_\nu G_{\nu\mu}(t,x) = \partial_\nu G_{\nu\mu}(t,x) + [B_\nu(t,x), G_{\nu\mu}(t,x)],$$

quark flow

original quark field at $t = 0$

$$\partial_t \chi_f(t,x) = \Delta \chi_f(t,x), \qquad \chi_f(t=0,x) = \psi_f(x),$$

$$\partial_t \bar\chi_f(t,x) = \bar\chi_f(t,x)\overleftarrow{\Delta}, \qquad \bar\chi_f(t=0,x) = \bar\psi_f(x),$$

$$\Delta\chi_f(t,x) \equiv D_\mu D_\mu \chi_f(t,x), \qquad D_\mu\chi_f(t,x) \equiv [\partial_\mu + B_\mu(t,x)]\,\chi_f(t,x),$$

$$\bar\chi_f(t,x)\overleftarrow{\Delta} \equiv \bar\chi_f(t,x)\overleftarrow{D}_\mu\overleftarrow{D}_\mu, \qquad \bar\chi_f(t,x)\overleftarrow{D}_\mu \equiv \bar\chi_f(t,x)\left[\overleftarrow{\partial}_\mu - B_\mu(t,x)\right]$$

only gauge fields involved

1) quark flow preserves the gauge and chiral symmetries.

   χ_f has the same gauge and chiral transformation properties as ψ_f.

2) quark flow is independent of spinor and flavor indices.

3) quark fields need renormalization <= can be handled numerically *a la* Makino-Suzuki

GF with dynamical quarks

quark field renormalization

It turned out that wave function renormalization is required for quarks.

Lüscher, JHEP1304.123(2013)

$$\chi = Z_\chi^{-1/2}\chi_R \qquad Z_\chi = 1 + \frac{g^2}{(4\pi)^2}C_2(R)3\frac{1}{\epsilon} + O(g^4)$$

for the MS scheme.

But this is all.
All other UV divergences as well as the short-distance singularities are absent.

Perturbative Z_χ is not quite useful in MC simulations.
<= need additional matching to lattice scheme, non-perturbative effects, ...

<u>Makino and Suzuki</u>

Makino-Suzuki, PTEP 2014, 063B02 [Erratum: 2015, 079202]

$$\mathring{\chi}_f(t,x) = \sqrt{\frac{-2\dim(R)}{(4\pi)^2 t^2 \left\langle \bar{\chi}_f(t,x) \overleftrightarrow{\not{D}} \chi_f(t,x) \right\rangle_0}}\,\chi_f(t,x),$$

$$\mathring{\bar{\chi}}_f(t,x) = \sqrt{\frac{-2\dim(R)}{(4\pi)^2 t^2 \left\langle \bar{\chi}_f(t,x) \overleftrightarrow{\not{D}} \chi_f(t,x) \right\rangle_0}}\,\bar{\chi}_f(t,x),$$

R: gauge representation of quarks
[dim(R)=N_c=3 for fund.repr. quarks]
f : flavor index (no summation over f)

$$\overleftrightarrow{D}_\mu \equiv D_\mu - \overleftarrow{D}_\mu$$

evaluated at $T = 0$

The divergences in χ are correctly cancelled by the denominator.

full QCD EMT by SFtX

Makino-Suzuki, PTEP 2014, 063B02 [E: 2015. 079202]

Measure flowed operators at $t \neq 0$:

$$\tilde{O}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x) G^a_{\nu\rho}(t,x),$$
$$\tilde{O}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu} G^a_{\rho\sigma}(t,x) G^a_{\rho\sigma}(t,x),$$

$$\tilde{O}^f_{3\mu\nu}(t,x) \equiv \varphi_f(t) \bar{\chi}_f(t,x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right) \chi_f(t,x),$$
$$\tilde{O}^f_{4\mu\nu}(t,x) \equiv \varphi_f(t) \delta_{\mu\nu} \bar{\chi}_f(t,x) \overleftrightarrow{\not{D}} \chi_f(t,x),$$
$$\tilde{O}^f_{5\mu\nu}(t,x) \equiv \varphi_f(t) \delta_{\mu\nu} \bar{\chi}_f(t,x) \chi_f(t,x), \qquad \varphi_f(t) \equiv \frac{-6}{(4\pi)^2 t^2 \left\langle \bar{\chi}_f(t,x) \overleftrightarrow{\not{D}} \chi_f(t,x) \right\rangle_0}.$$

and combine them as
$$T_{\mu\nu}(x) = \lim_{t\to 0} \Big\{ c_1(t) \left[\tilde{O}_{1\mu\nu}(t,x) - \frac{1}{4} \tilde{O}_{2\mu\nu}(t,x) \right]$$
$$+ c_2(t) \left[\tilde{O}_{2\mu\nu}(t,x) - \left\langle \tilde{O}_{2\mu\nu}(t,x) \right\rangle_0 \right]$$
$$+ c_3(t) \sum_{f=u,d,s} \left[\tilde{O}^f_{3\mu\nu}(t,x) - 2\tilde{O}^f_{4\mu\nu}(t,x) - \left\langle \tilde{O}^f_{3\mu\nu}(t,x) - 2\tilde{O}^f_{4\mu\nu}(t,x) \right\rangle_0 \right]$$
$$+ c_4(t) \sum_{f=u,d,s} \left[\tilde{O}^f_{4\mu\nu}(t,x) - \left\langle \tilde{O}^f_{4\mu\nu}(t,x) \right\rangle_0 \right]$$
$$+ \sum_{f=u,d,s} c^f_5(t) \left[\tilde{O}^f_{5\mu\nu}(t,x) - \left\langle \tilde{O}^f_{5\mu\nu}(t,x) \right\rangle_0 \right] \Big\},$$

Physical EMT extracted by $t\to 0$ extrapolation.

c_i : matching coefficients

- to make $t\to 0$ smoother by removing known small-t mixings & t-dep. in the continuum
- to match the renormalization schemes when the observable is scheme-dependent
- perturbation theory applicable to calculate c_i in AF theories

In this study, we mainly use 1-loop c_i by Makino-Suzuki. We revisit the issue with 2-loop c_i later.

an issue of $a \neq 0$

We have configurations at $a \approx 0.07$fm only. This lattice is father fine but not in the continuum limit!

=> Exchange the order of $a \to 0$ and $t \to 0$ extrapolations.

In the continuum

combination of dim=6 operators

$$T_{\mu\nu}(t,x) = T_{\mu\nu}(x) + tS_{\mu\nu}(x) + O(t^2)$$

conserved EMT we want

At $a \neq 0$

additional mixing with unwanted operators

Note: lattice artifacts of NP-clover is $O(a^2)$.

$$T_{\mu\nu}(t,x,a) = T_{\mu\nu}(t,x) + A_{\mu\nu}\frac{a^2}{t} + \sum_f B_{f\mu\nu}(am_f)^2 + C_{\mu\nu}(aT)^2 + D_{\mu\nu}\left(a\Lambda_{\mathrm{QCD}}\right)^2 + a^2 S'_{\mu\nu}(x) + \mathcal{O}(a^4),$$

dim=4 operators

dim=6 operators

Singular terms at $t \approx 0$

- Stronger singularities such as a^4/t^2 can appear from higher orders in a^2.
- When we take $a \to 0$ first, the singular terms are removed and we can take $t \to 0$ safely.

an issue of $a \neq 0$

At $a \neq 0$

$$T_{\mu\nu}(t,x,a) = T_{\mu\nu}(t,x) + A_{\mu\nu}\frac{a^2}{t} + \sum_f B_{f\mu\nu}(am_f)^2 + C_{\mu\nu}(aT)^2 + D_{\mu\nu}\,(a\Lambda_{\rm QCD})^2$$
$$+ a^2 S'_{\mu\nu}(x) + \mathcal{O}(a^4),$$

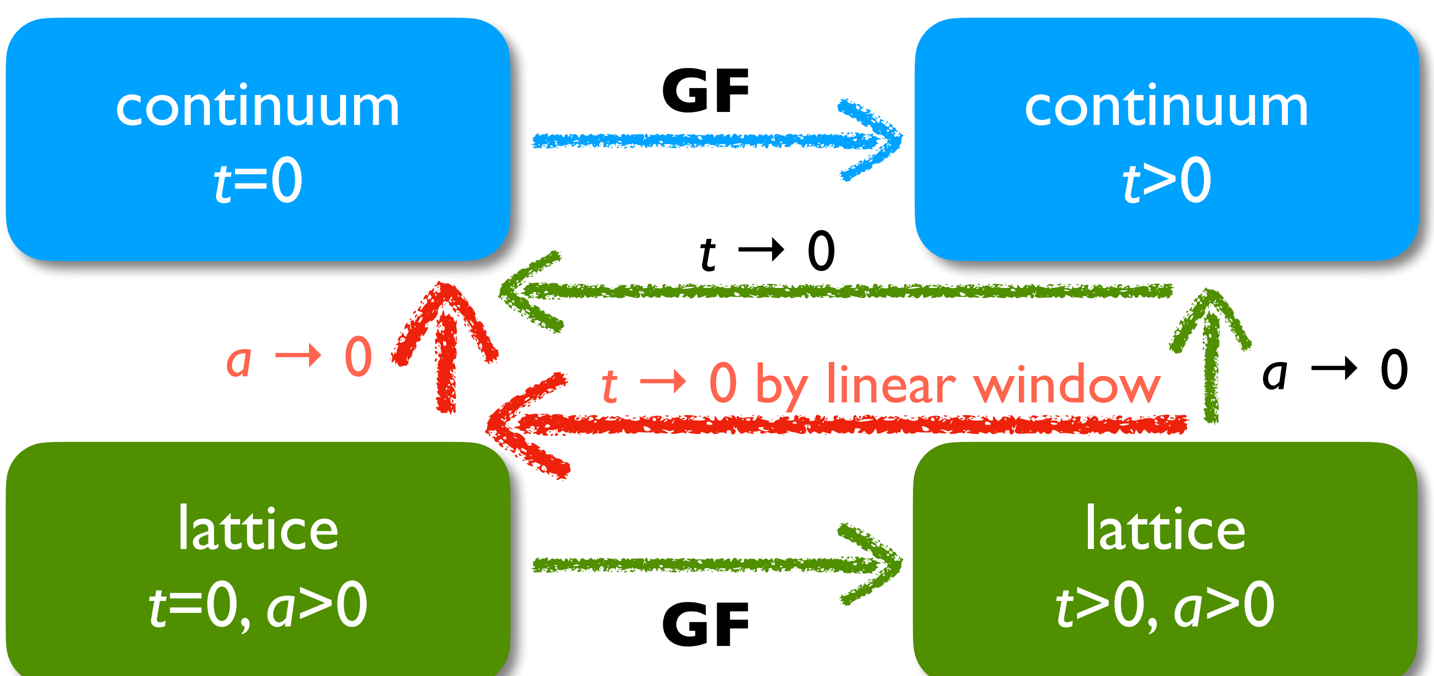
dim=4 operators

dim=6 operators

Singular terms at $t \approx 0$

=> should be disregarded in the $t \to 0$ extrapolation at $a \neq 0$.

This is possible when we have a "*linear window*" where const. + linear terms are dominating.



Notes:

1. Contamination of B, C, D, S', ... remains.

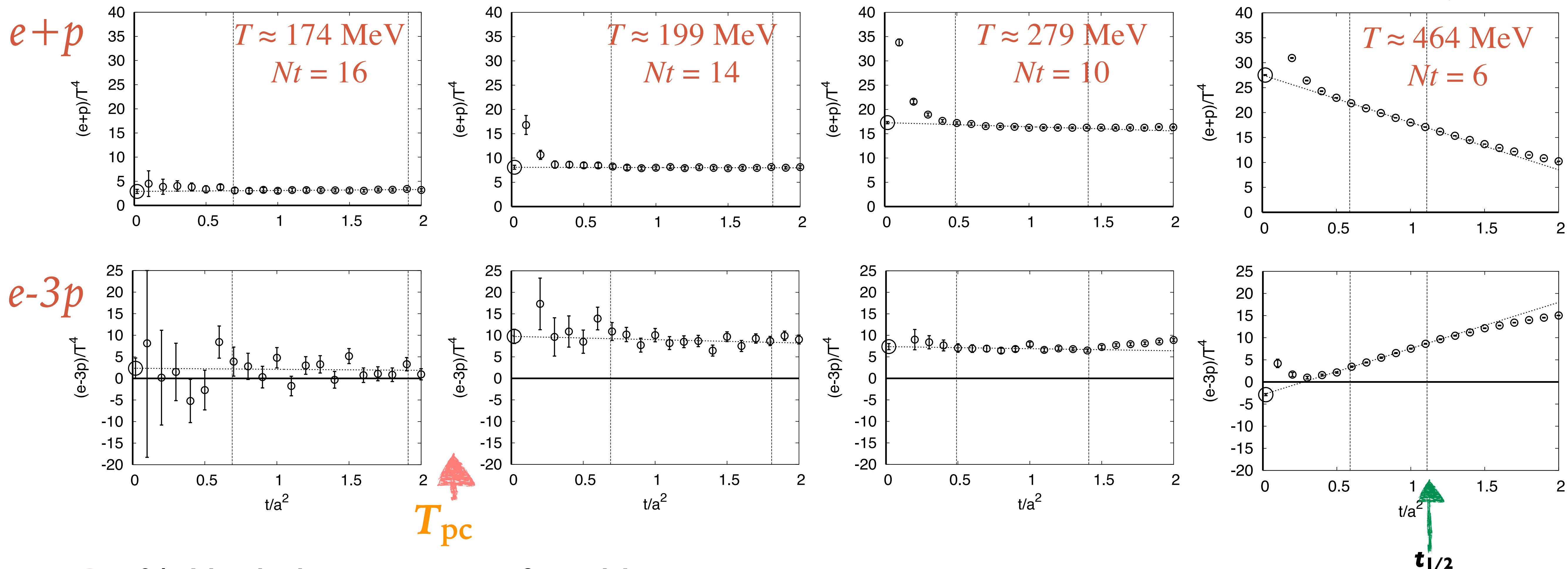
   => $a \to 0$ mandatory at end.

2. Small-t data had to be removed also in the qQCD study in which $a \to 0$ was done before $t \to 0$, i.e., $a \to 0$ not possible when singular terms are dominating.

EMT with dynamical quarks

N_f=2+1 EMT with heavy u,d

$$\epsilon = -\langle T_{00} \rangle, \ p = \frac{1}{3}\sum_i \langle T_{ii} \rangle$$

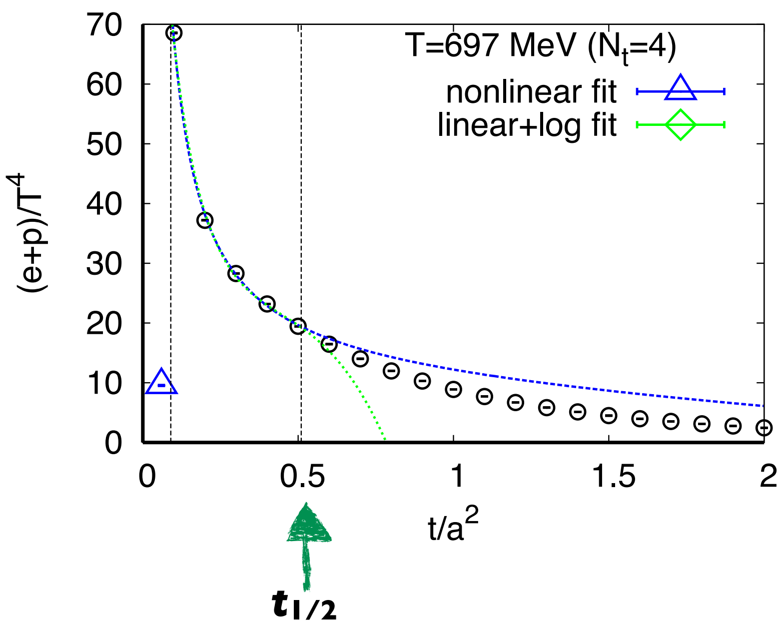


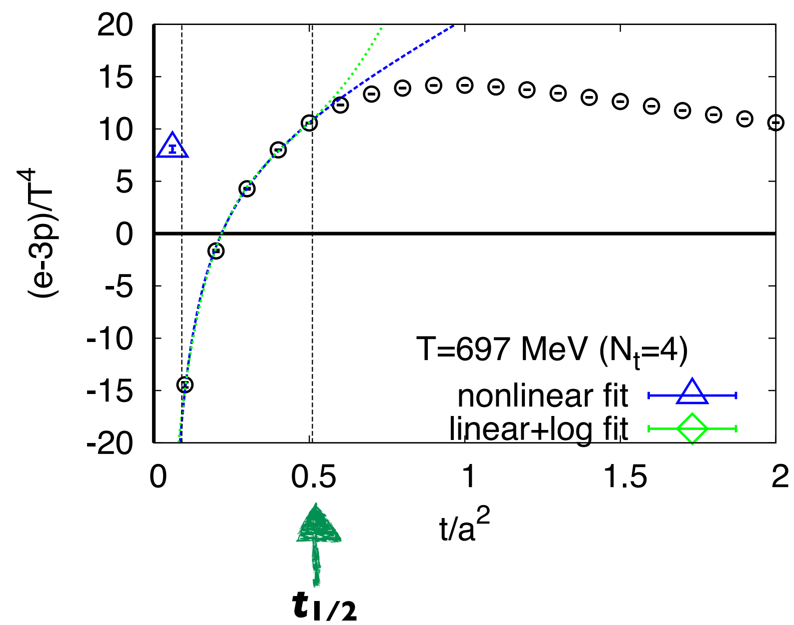
- a^2/t-like behavior at $t \approx 0$ visible.
- Linear behavior visible below $t_{1/2}$. (Nt=6 may be marginal.)
- a^2/t term looks negligible in the "linear windows" => Linear fit using the windows.
- At $T \approx 697$ MeV (Nt=4), no linear windows found.
- Smaller errors for e+p <= no T=0 subtraction required

EMT with dynamical quarks

N_f=2+1 EMT with heavy u,d

- At $T \approx$ 697 MeV (Nt=4), no linear windows found.





Though we may try non-linear fits, unphysical contributions are dominating in the data.

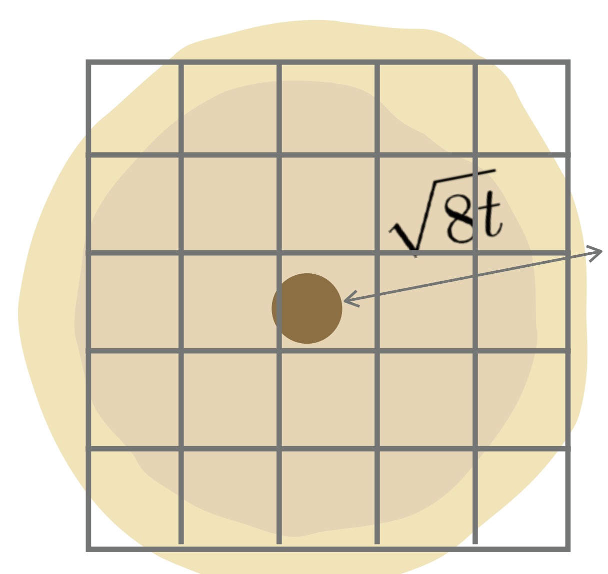
=> We can not extrapolate reliably at this T.

$t_{1/2}$ **To avoid oversmearing** wrapping around the lattice,

$\sqrt{(8t/a^2)} \leq \min(Ns/2, Nt/2)$

i.e., $\boldsymbol{t/a^2 \leq t_{1/2}} = [\min(Ns/2, Nt/2)]^2 / 8$

besides $(t/a^2)_{max}$ in the simulation.



T (MeV)	T/T_{pc}	N_t	$t_{1/2}$
0	0	56	24.5
174	0.92	16	8
199	1.05	14	6.125
232	1.22	12	4.5
279	1.47	10	3.125
348	1.83	8	2
464	2.44	6	1.125
697	3.67	4	0.5

EMT with dynamical quarks

N_f=2+1 EMT with heavy u,d

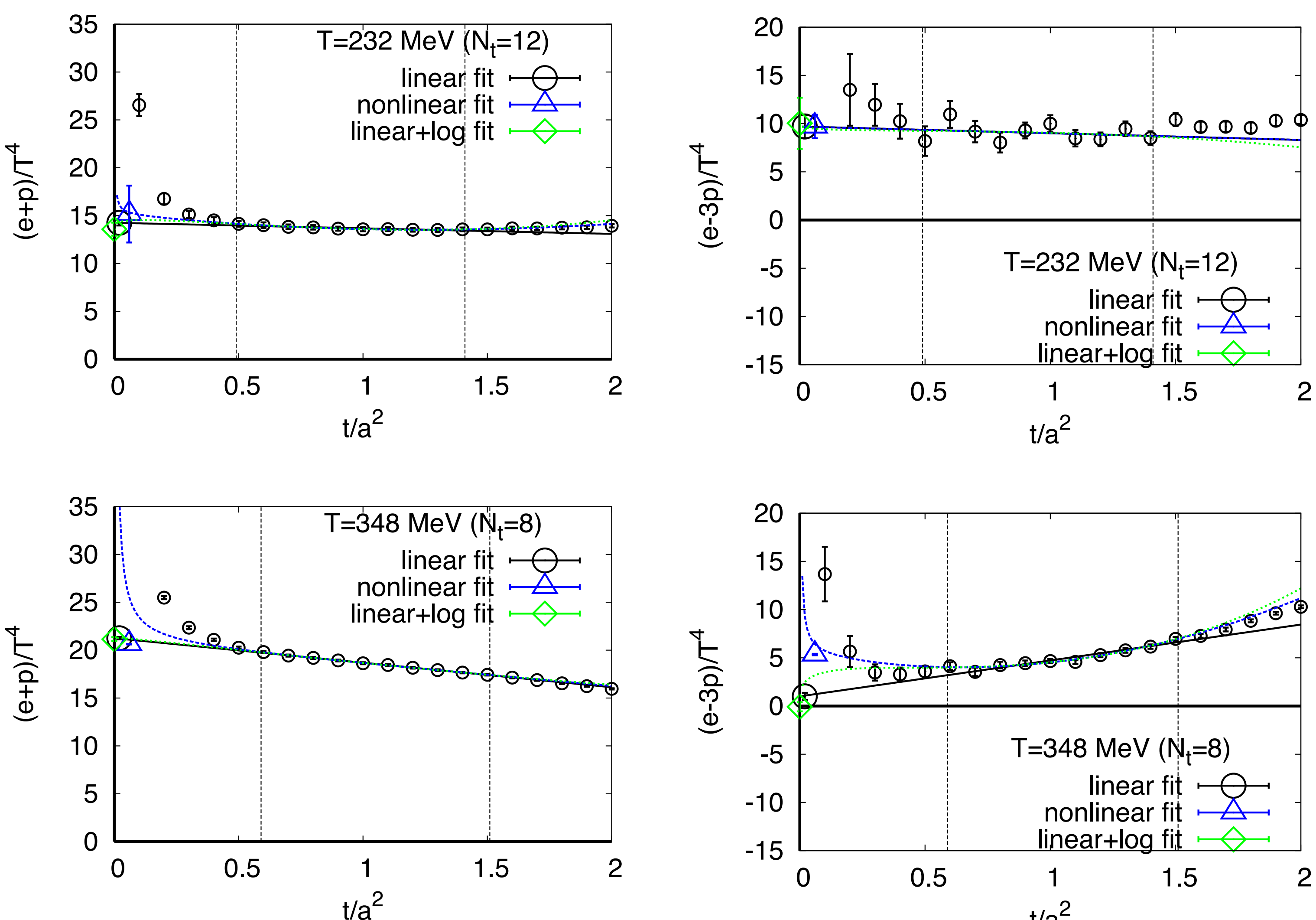
- A series of additional analyses
  - to confirm the linear extrapolation procedure at a>0
  - to estimate systematic error due to the fit ansatz

$$\langle T_{\mu\nu}(t,a)\rangle = \langle T_{\mu\nu}\rangle + A_{\mu\nu}\frac{a^2}{t} + t\,S_{\mu\nu} + t^2 R_{\mu\nu}$$

- nonlinear fit, inspired from a^2/t as well as next-leading t corrections.

$$\langle T_{\mu\nu}(t,a)\rangle = \langle T_{\mu\nu}\rangle + t\,S_{\mu\nu} + \frac{Q_{\mu\nu}}{\log^2(\sqrt{8t}/a)}$$

- linear+log fit, inspired from higher order PT corrections in the one-loop Suzuki coeff's. c_i.

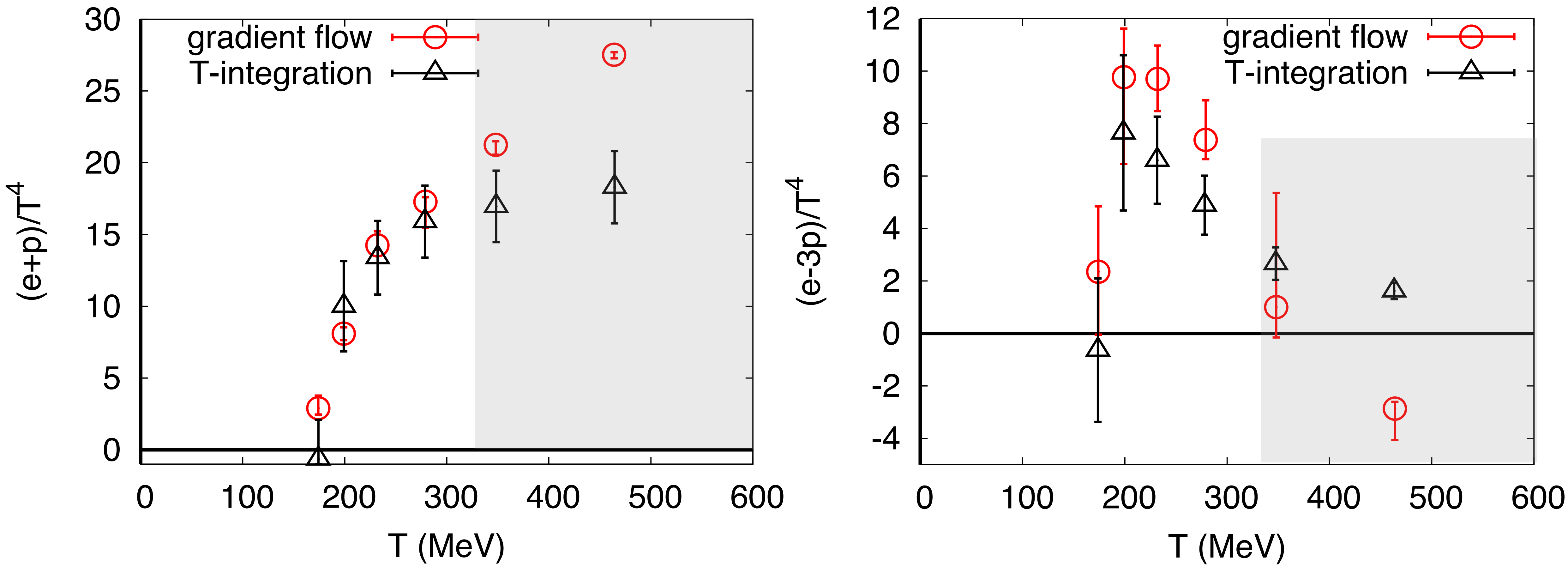


- In most cases, all the fits are consistent with each other using the same window.
- Take the deviations as an estimate of systematic error due to the fit ansatz.

EMT with dynamical quarks

Taniguchi-Ejiri-Iwami-KK-Kitazawa-Suzuki-Umeda-Wakabayashi, Phys.Rev. D 96, 014509 (2017)

N_f=2+1 EoS with heavy u,d

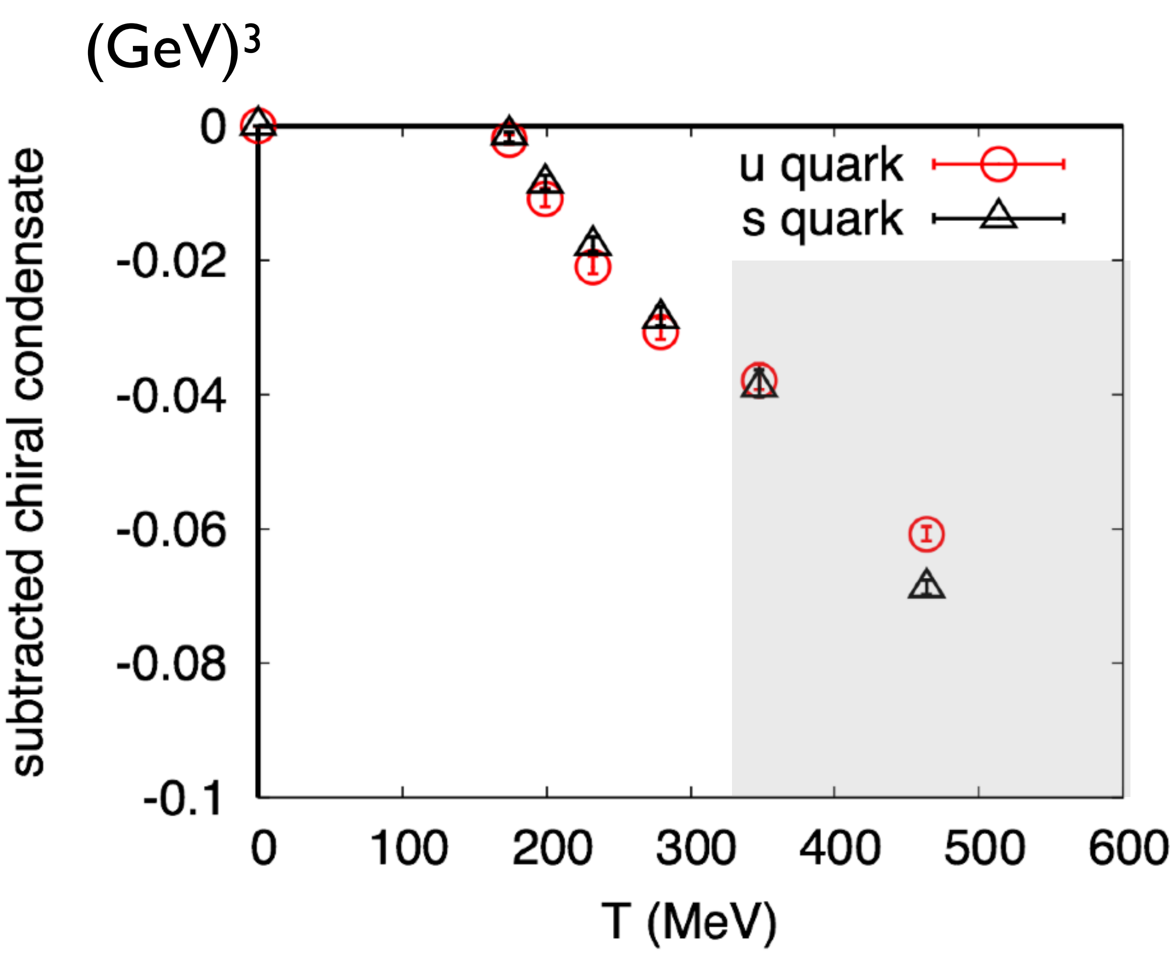


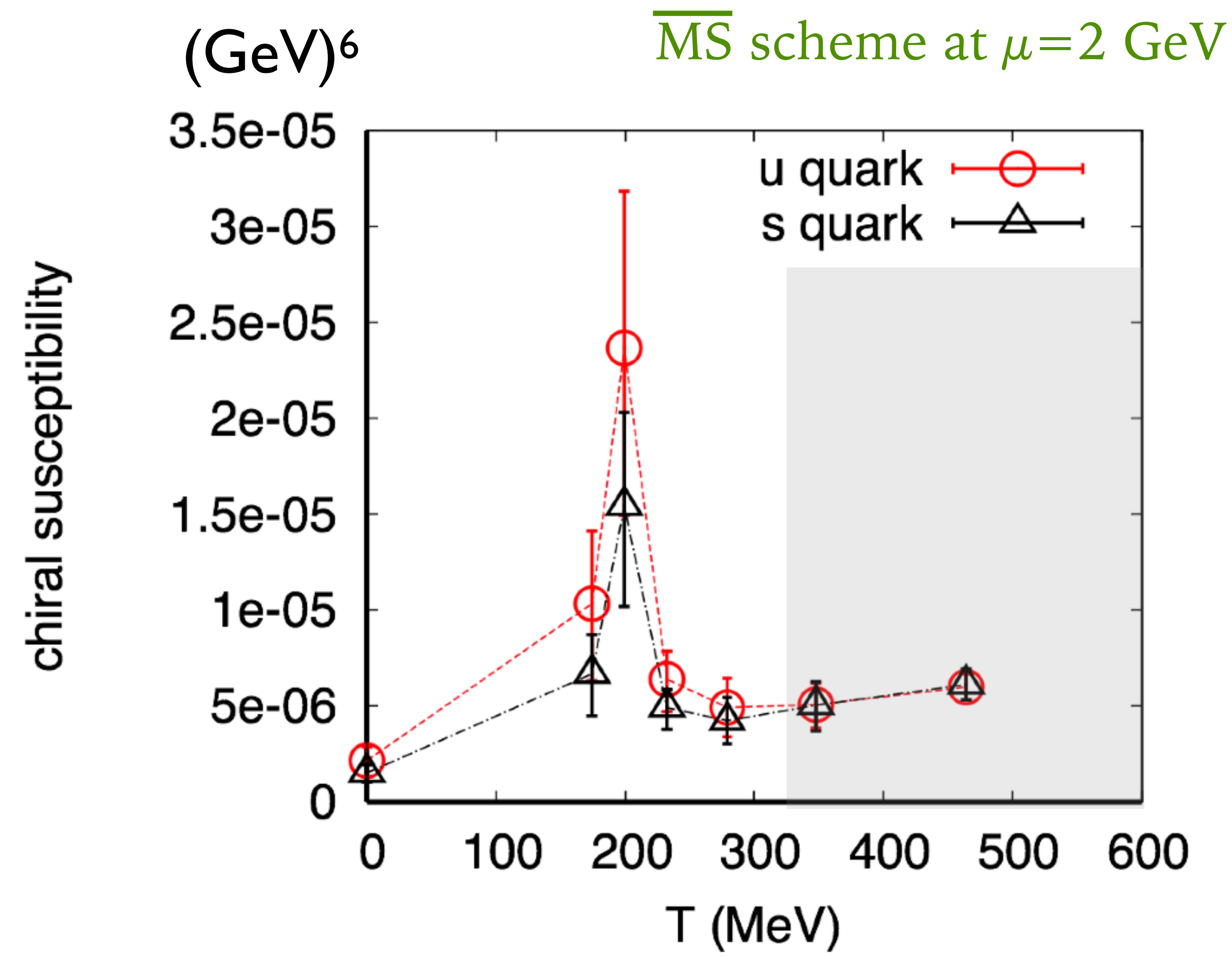
- ☑ EoS by SF*t*X agrees with conventional method at $T \leq 300$ MeV ($Nt \geq 10$). Suggest $a \approx 0.07$fm close to the cont. limit.
- ☑ Disagreement at $T \geq 350$ MeV due to $O((aT)^2 = 1/Nt^2)$ lattice artifact at $Nt \lesssim 8$. [Note that this lattice artifact is independent of *a*.]

chiral condensate / susceptibility

Taniguchi-Ejiri-Iwami-KK-Kitazawa-Suzuki-Umeda-Wakabayashi, Phys.Rev. D 96, 014509 (2017)

N_f=2+1 chiral cond. / disconnected susceptibility



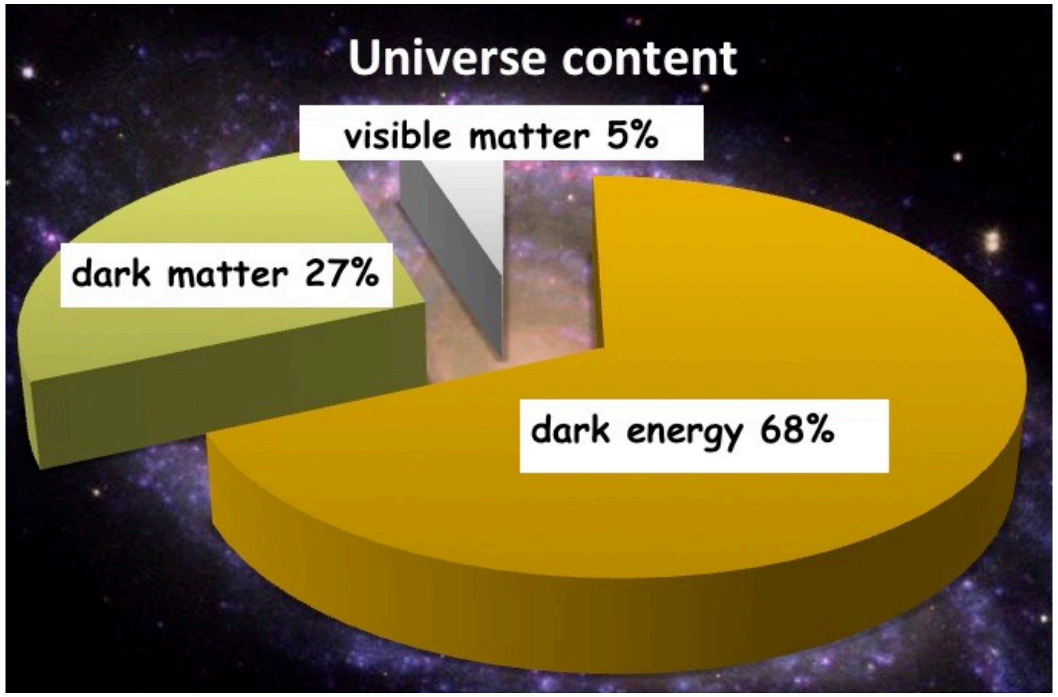


- ☑ Crossover suggested around $T_{pc}\approx$190 MeV, consistent with previous study.
- ☑ Peak higher with decreasing m_q, as expected.

=> Physically expected results even with Wilson-type quarks!

SF*t*X powerful to extract physical properties.

topological charge / susceptibility



Axion is a candidate of the CDM.

T-dependence of the axion mass is important in judging its cosmic abundance.

According to invisible axion models,

$$m_a^2(T) = \frac{1}{f_a^2}\,\chi_{\rm t}(T)$$

gluonic definition

topological susceptibility

$$\chi_{\rm t} = \int d^4x \langle q(x)q(0)\rangle = \frac{1}{V_4}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$

($\langle Q\rangle \to 0$)

topological charge

$$Q = \int d^4x\, q(x), \quad q = \frac{1}{64\pi^2}\epsilon_{\mu\nu\sigma\rho}F^a_{\mu\nu}F^a_{\sigma\rho}$$

Use GF as a cooling procedure.

The resulting Q is correctly normalized (satisfy the chiral WT).

Hieda-Suzuki, Mod.Phys.Lett.A 31, 1650214 (2016)
Cé-Consonni-Engel-Giusti, PR D 92, 074502 (2015)

topological charge / susceptibility

$$\chi_t = \int d^4x \langle q(x)q(0)\rangle = \frac{1}{V_4}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$

($\langle Q\rangle \to 0$)

fermionic definition

Giusti-Rossi-Testa, PL B 587, 157 (2004)
Bochicchio-Rossi-Tessa-Yoshida, PL B 149, 487 (1998)

chiral Ward-Takahashi identities

$$\langle \partial_\mu A^a_\mu(x)\mathcal{O}\rangle - 2m\langle \pi^a(x)\mathcal{O}\rangle + 2n_f\delta^{a0}\langle q(x)\mathcal{O}\rangle = i\langle \delta^a\mathcal{O}\rangle$$

$$A^a_\mu = \bar\psi T^a\gamma_\mu\gamma_5\psi,\ \pi^a = \bar\psi T^a\gamma_5\psi$$

n_f = # of degenerate flavors with mass m (n_f=2, m=m_{ud} in our case)
T^a = generator in the degenerate flavor space (T^0=1)

$$P^a = \int d^4x\bar\psi(x)T^a\gamma_5\psi(x) \qquad S^a = \int d^4x\,\bar\psi(x)T^a\psi(x)$$

$\mathcal{O} = Q$, $\mathcal{O} = P^0$ → **Abelian** →

$$-m\langle P^0Q\rangle + n_f\langle Q^2\rangle = 0,$$

$$-m\langle P^0P^0\rangle + n_f\langle QP^0\rangle = -\langle S^0\rangle$$

(← singlet scalar)

→ **Abelian** →

$$n_f^2\langle Q^2\rangle = m^2\langle P^0P^0\rangle - m\langle S^0\rangle$$

$\mathcal{O} = P^b$ → **non-A** →

$$-2m\langle P^aP^b\rangle = -\left(\delta^{ab}\frac{2}{n_f}\langle S^0\rangle + d_{abc}\langle S^c\rangle\right)$$

(non-singlet: $\langle S^c\rangle \to 0$)

Combining them, we obtain

$$\chi_t = \frac{1}{V_4}\langle Q^2\rangle = \frac{m^2}{V_4 n_f^2}\langle P^0P^0\rangle_{\rm disc}$$

(disconnected part)

We evaluate $P^0 = \int d^4x\bar\psi(x)\gamma_5\psi(x)$ **by the SFtX method.**

topological charge / susceptibility

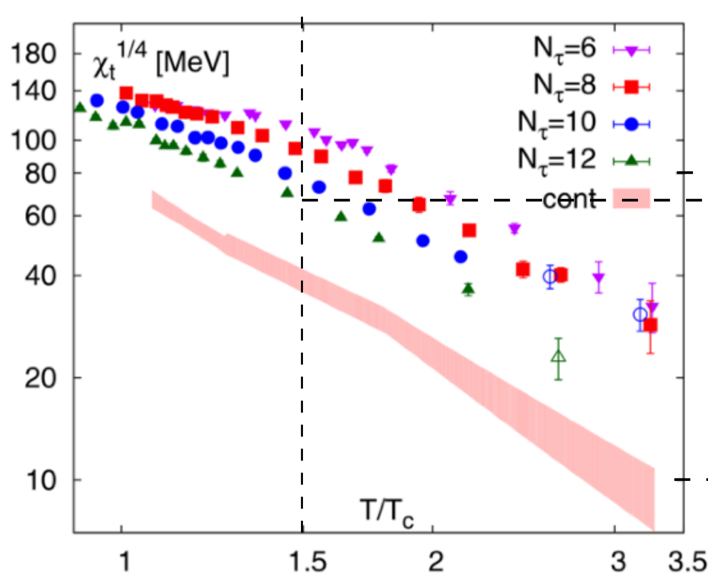
The two definitions should give identical results.

On the lattice, <= violation of chiral W-T identities by lattice quarks

gluonic and fermionic susceptibilities largely discrepant at $a \neq 0$ with the conventional method.

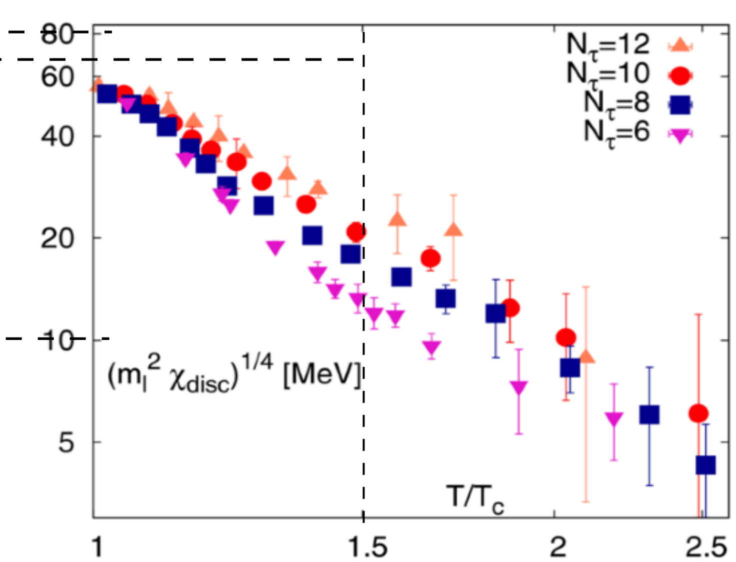
Petreczky et al, PL B 762, 498 (2016): Nf=2+1 HISQ

$\chi_t^{1/4}$

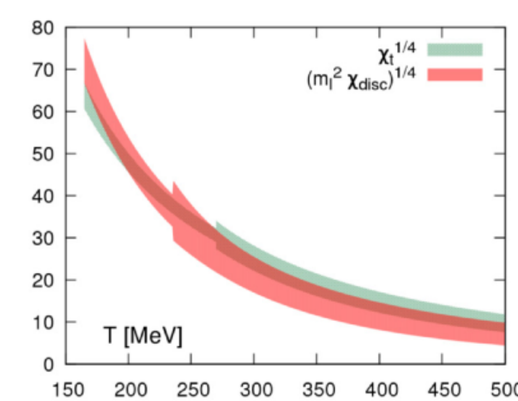


gluonic definition

fermionic definition



Their continuum extrapolations suggest that the two definitions may be consistent in the continuum limit.



But the extrapolations are quite long and not fully unambiguous.

$\approx$2 orders of magnitude different χ_t even at N_t=12

topological charge

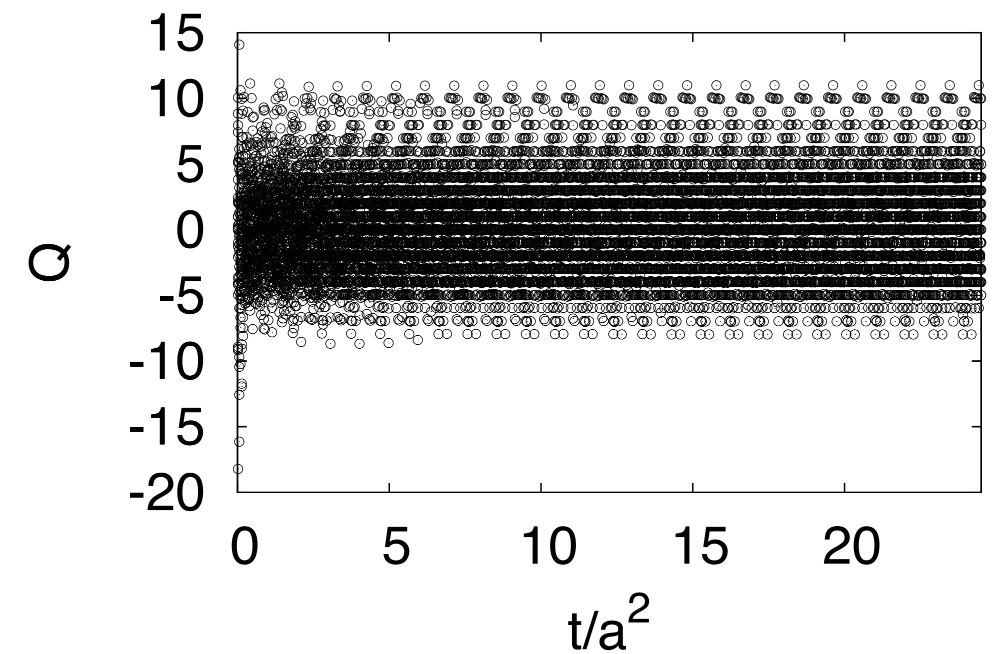
Taniguchi-KK-Suzuki-Umeda, Phys.Rev. D 95, 054502 (2017)

gluonic definition

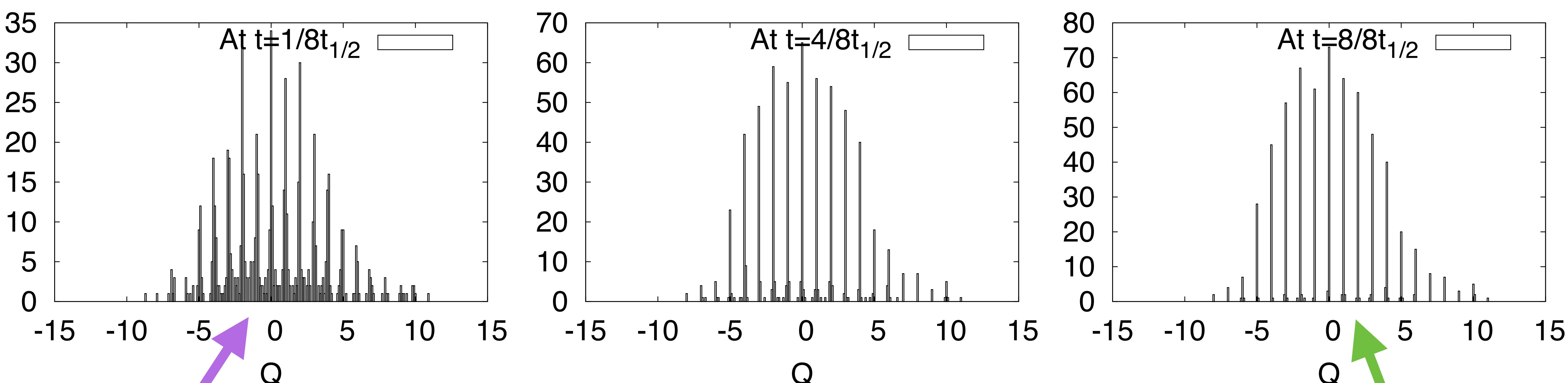
Use GF as a cooling.

$T = 0$ 28^3x56

Q-distribution as a function of t.



$t_{1/2} = 24.5$



Non-integer Q's

Accumulate to integer Q's

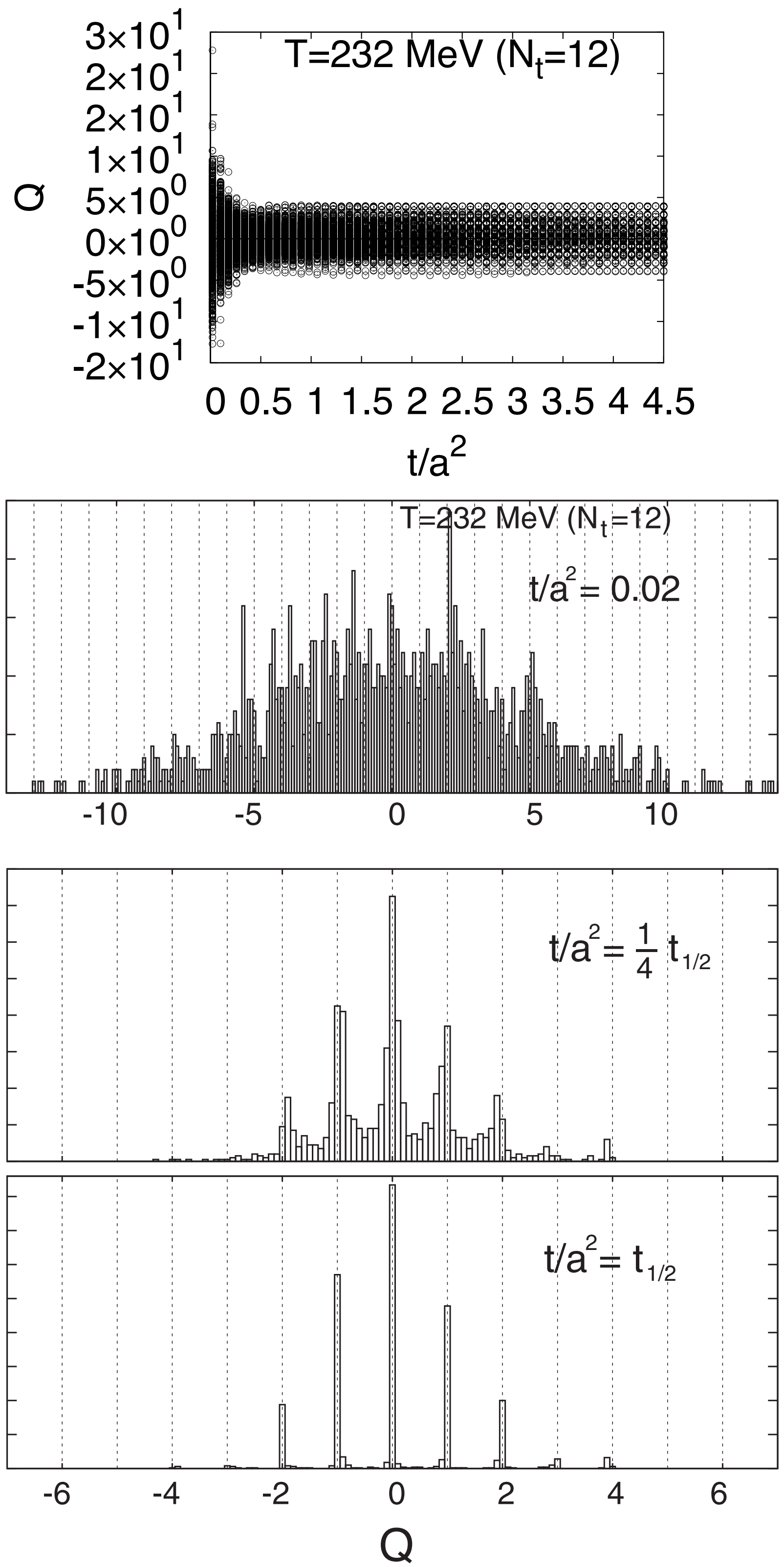
=> GF works well as a cooling.

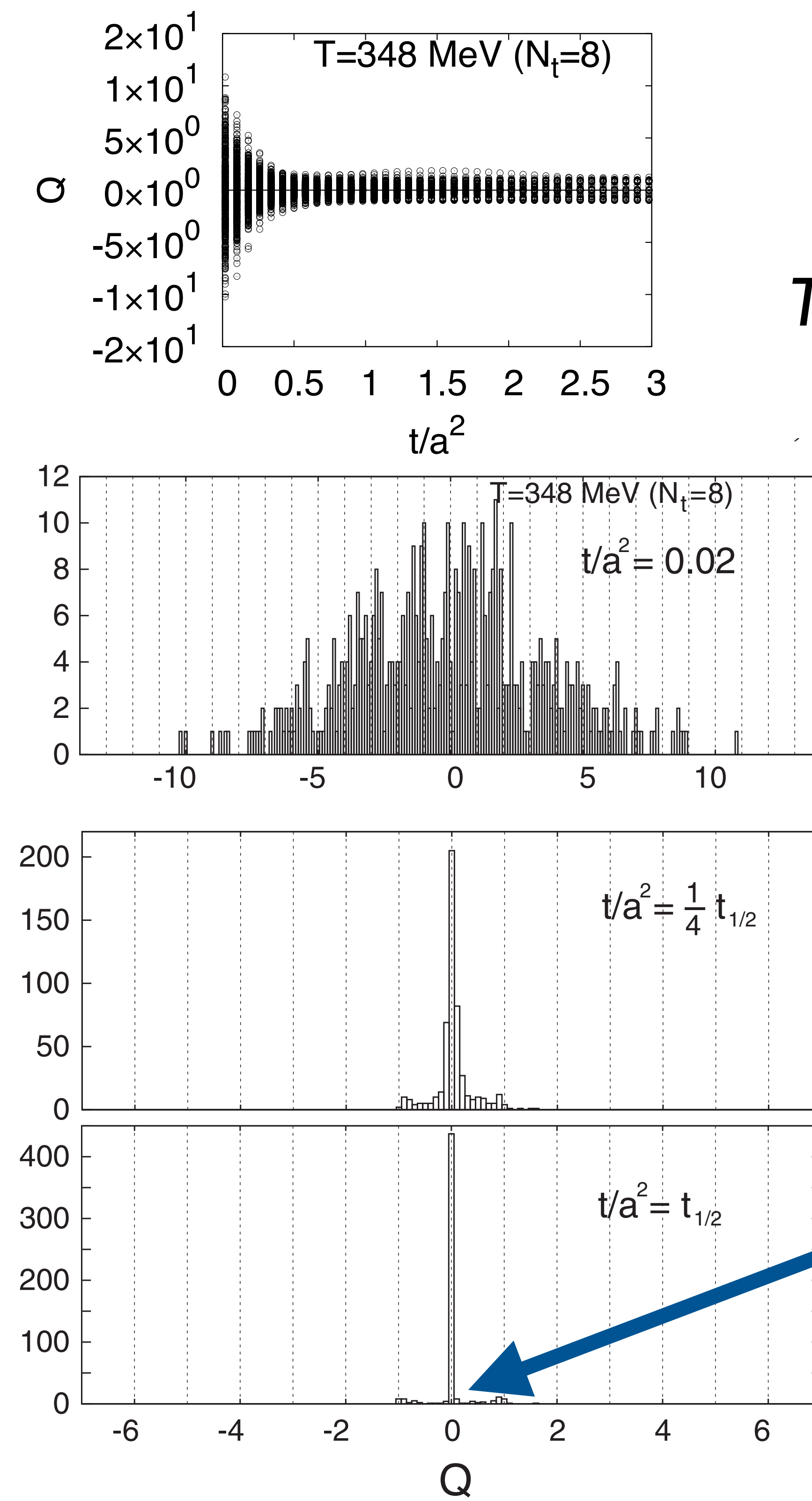
topological charge

Taniguchi-KK-Suzuki-Umeda, Phys.Rev. D 95, 054502 (2017)

gluonic definition

$T/T_c = 1.22$





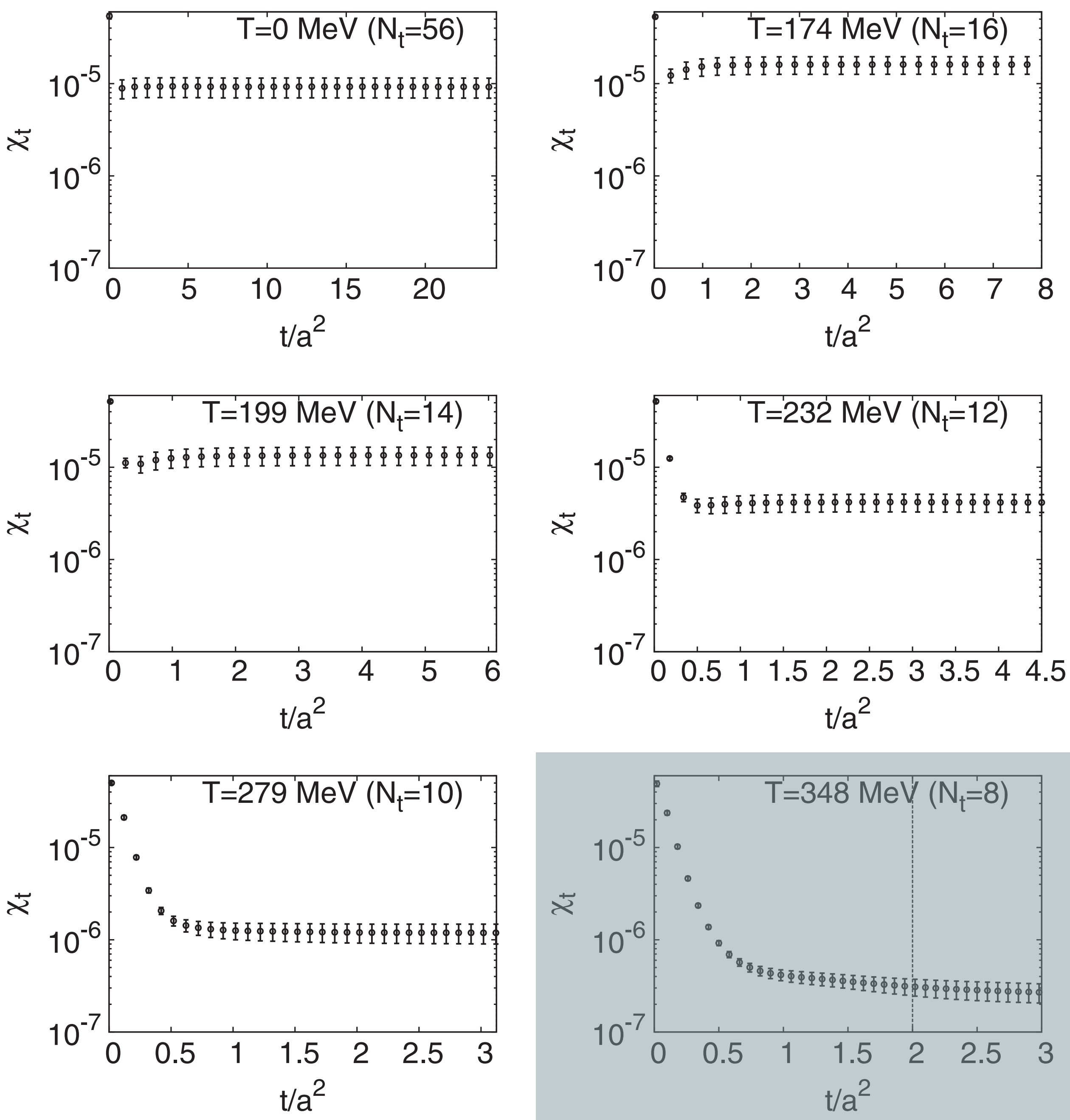
$T/T_c = 1.83$

Starting to freeze to $Q = 0$.

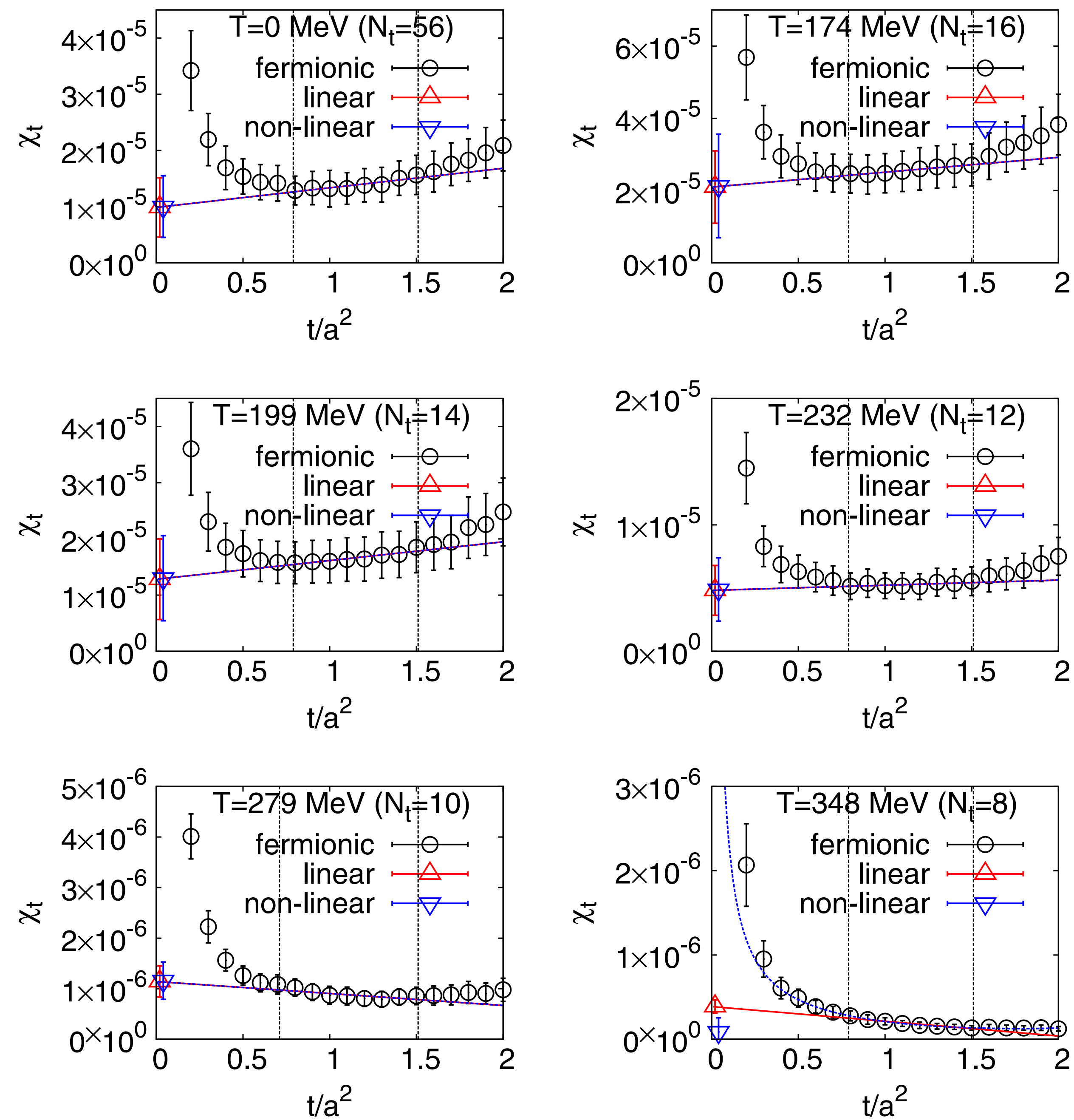
topological susceptibility

Taniguchi-KK-Suzuki-Umeda, Phys.Rev. D 95, 054502 (2017)

gluonic definition (cooling)



fermionic definition (SF*t*X)

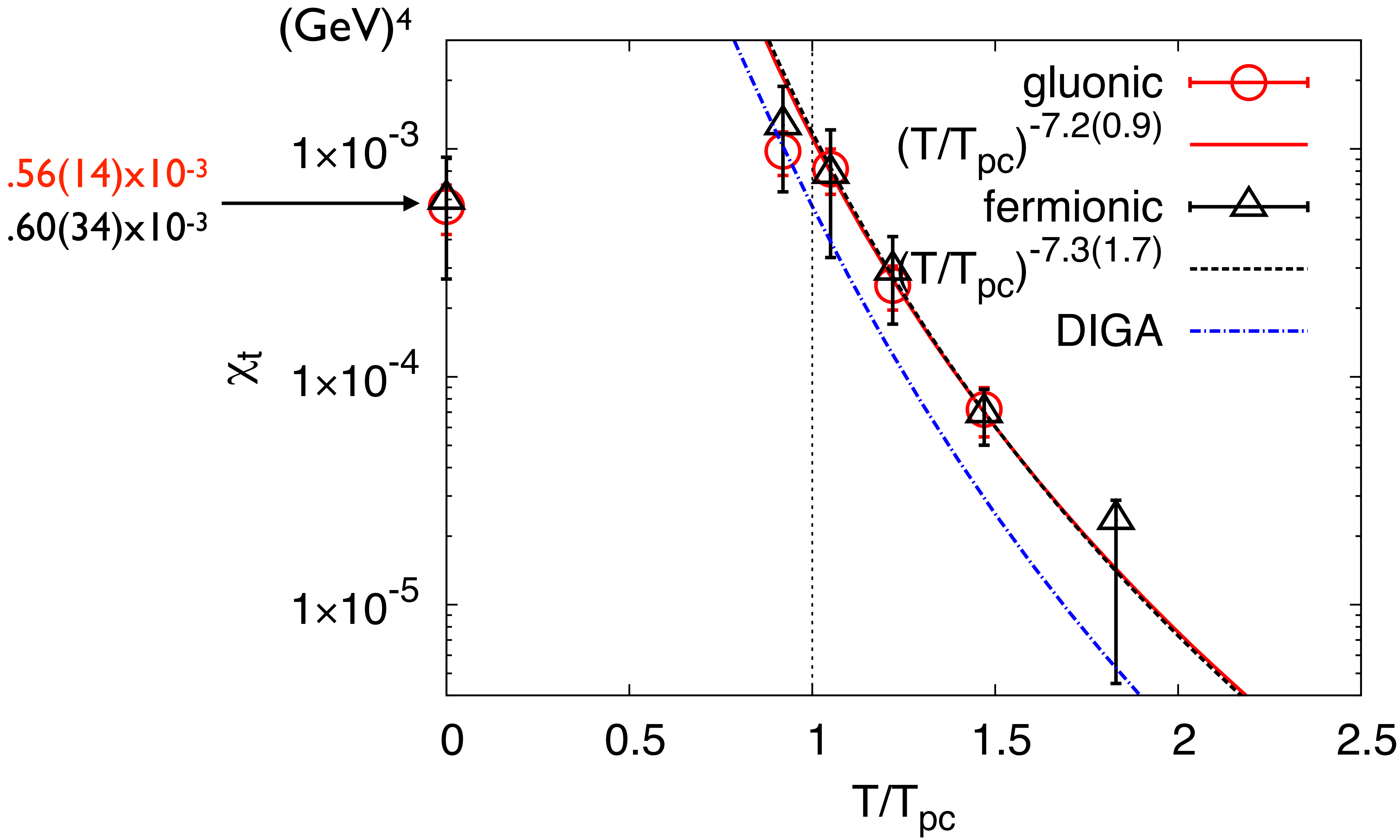


plateau at large *t*

linear extrapolation *t*→0

topological susceptibility

Taniguchi-KK-Suzuki-Umeda, Phys.Rev. D 95, 054502 (2017)



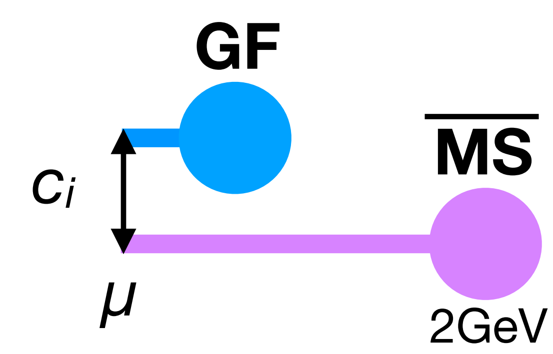
- Two definitions agree well => SF*t*X enables us reliable predictions.
- Power low consistent with a prediction of Dilute Instanton Gas model.

[1A]
Issue of renormalization-scale in $N_F = 2+1$ QCD with slightly heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

renormalization scale μ

- matching coefficients of the SFtX method



$$
\begin{aligned}
c_1(t) &= \frac{1}{g^2}\Bigg(1+\frac{g^2}{(4\pi)^2}\left[-\beta_0 L(\mu,t)-\frac{7}{3}C_A+\frac{3}{2}T_F\right] \\
&\quad +\frac{g^4}{(4\pi)^4}\Bigg\{-\beta_1 L(\mu,t)+C_A^2\left(-\frac{14482}{405}-\frac{16546}{135}\ln 2+\frac{1187}{10}\ln 3\right) \\
&\quad +C_A T_F\left[\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right)+\frac{10873}{810}+\frac{73}{54}\pi^2-\frac{2773}{135}\ln 2+\frac{302}{45}\ln 3\right] \\
&\quad +C_F T_F\left[-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right)+\frac{2587}{108}-\frac{7}{9}\pi^2-\frac{106}{9}\ln 2-\frac{161}{18}\ln 3\right]\Bigg\}\Bigg)
\end{aligned}
$$

etc. with $L(\mu,t)\equiv \ln\left(2\mu^2 t\right)+\gamma_E$

Harlander-Kluth-Lange, EPJC 78:944 (2018)

c_i at small t are calculated in terms of the MS-bar running coupling $g(\mu)$ and mass $m(\mu)$.
The MS-bar renorm. scale μ is free to choose, as far as the perturbative expansions are OK.
Final results should be indep. of μ.

A conventional choice is $\mu(t)=\mu_d(t)\equiv\dfrac{1}{\sqrt{8t}}$, a natural scale of flowed operators.

HKL suggested $\mu_0(t)\equiv\dfrac{1}{\sqrt{2e^{\gamma_E}t}}$ which makes $L(\mu,t)=0$ and suppresses NNLO in a similar level as μ_d.

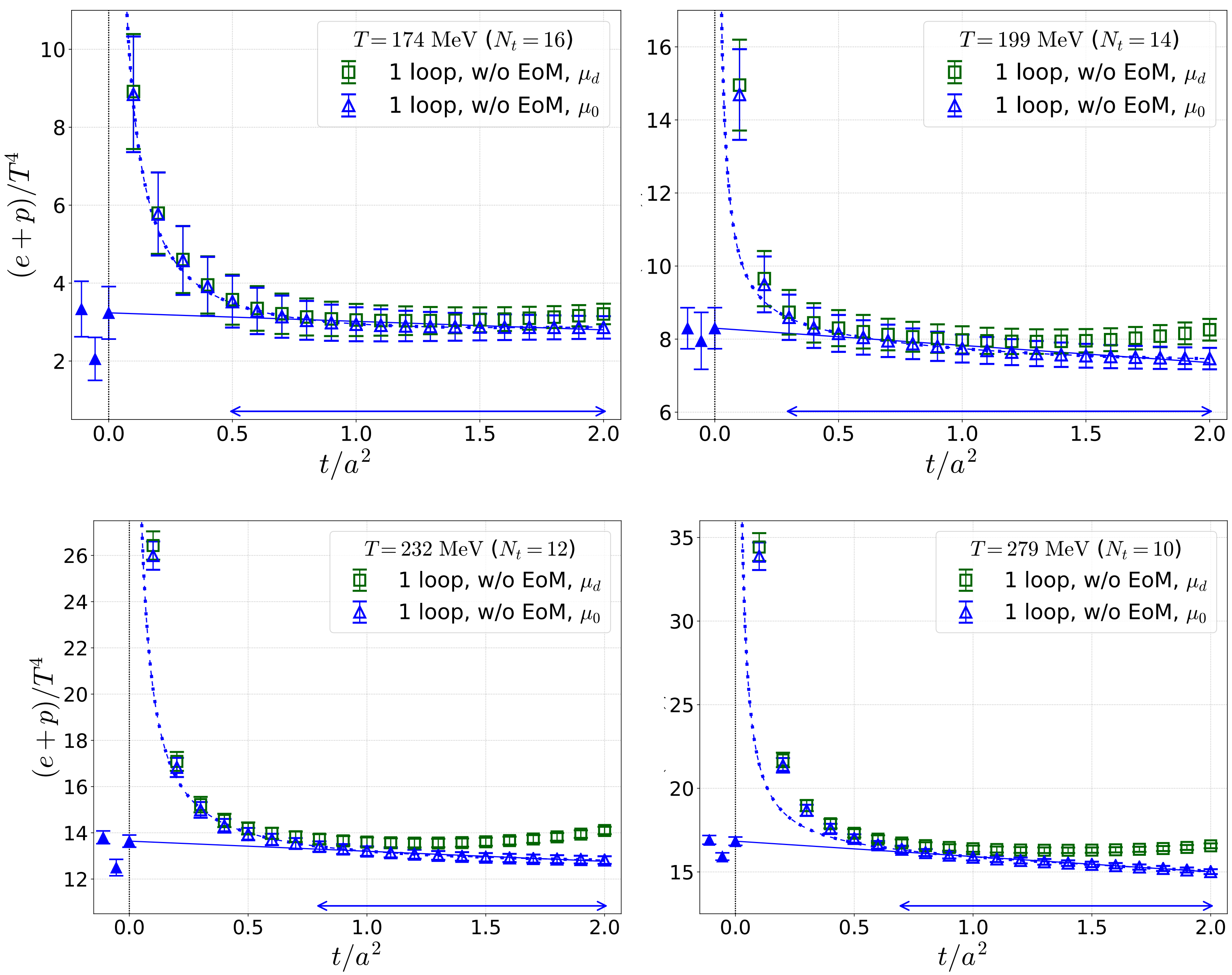
Practically $\mu_0(t)\approx 1.5\,\mu_d(t)$ => μ_0 more perturbative
extends the perturbative region towards larger t

[A larger $\mu(t)$ is even more perturbative, but a huge $L(\mu,t)$ breaks the perturbative expansion.]

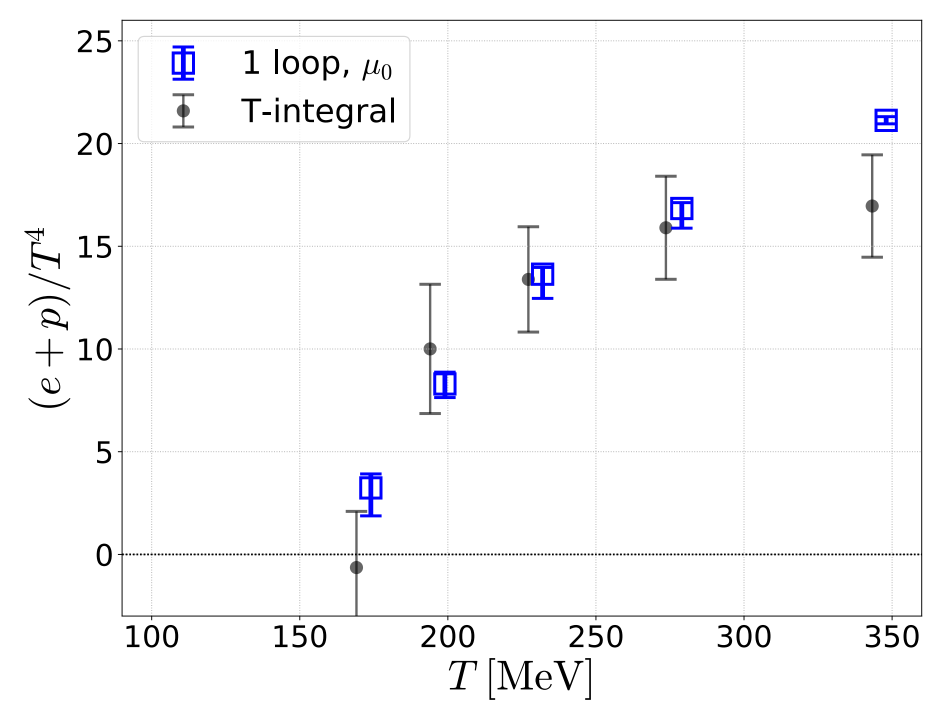
EoS with heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

entropy density $(e+p)/T^4$



Results with the μ_0-scale

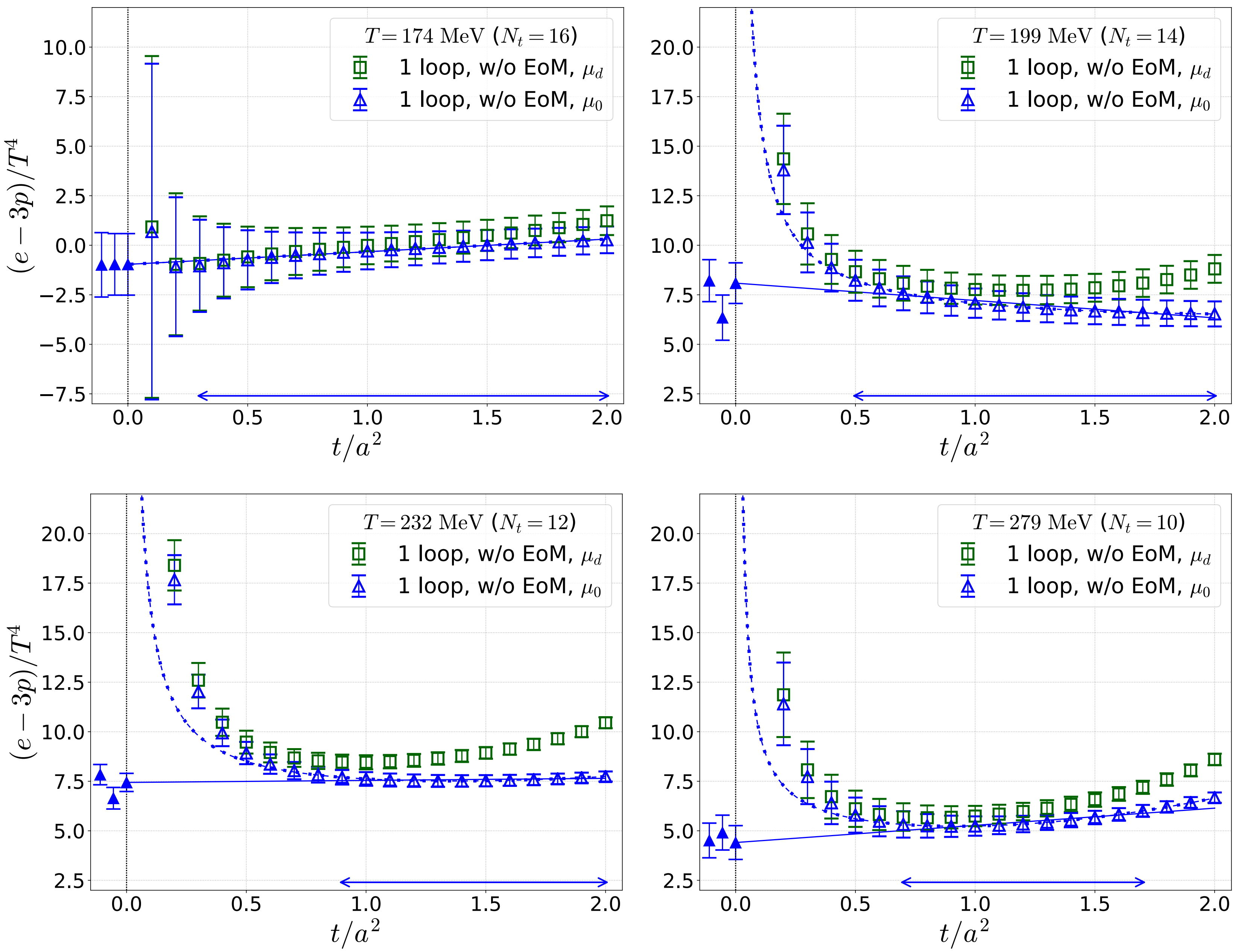


☑ μ_0 and μ_d results consistent with each other

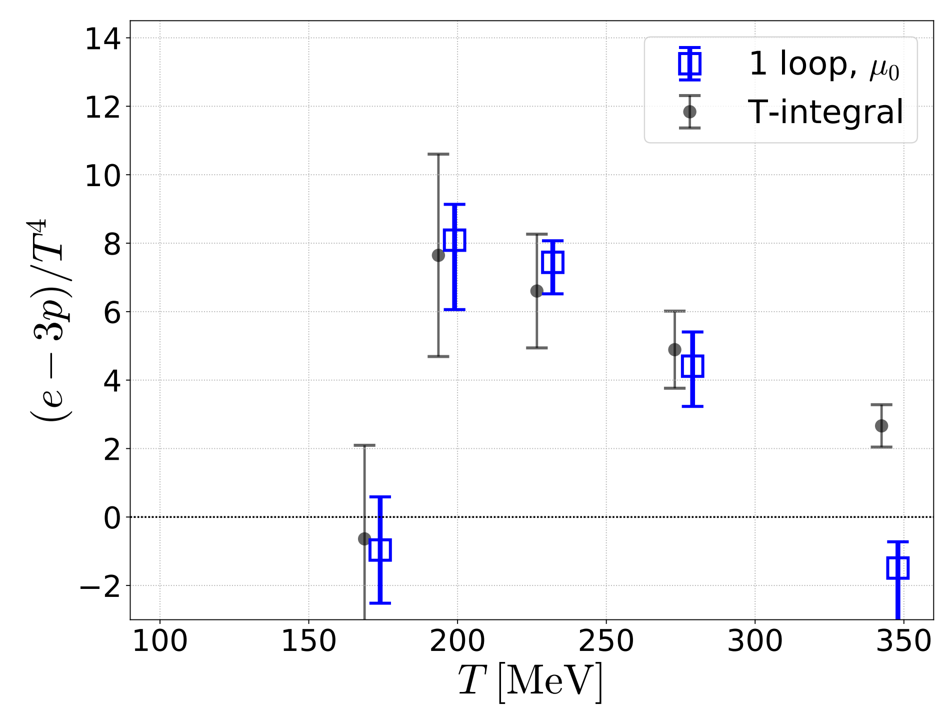
EoS with heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

trace anomaly $(e\text{-}3p)/T^4$



Results with the μ_0-scale



☑ μ_0 and μ_d results consistent with each other

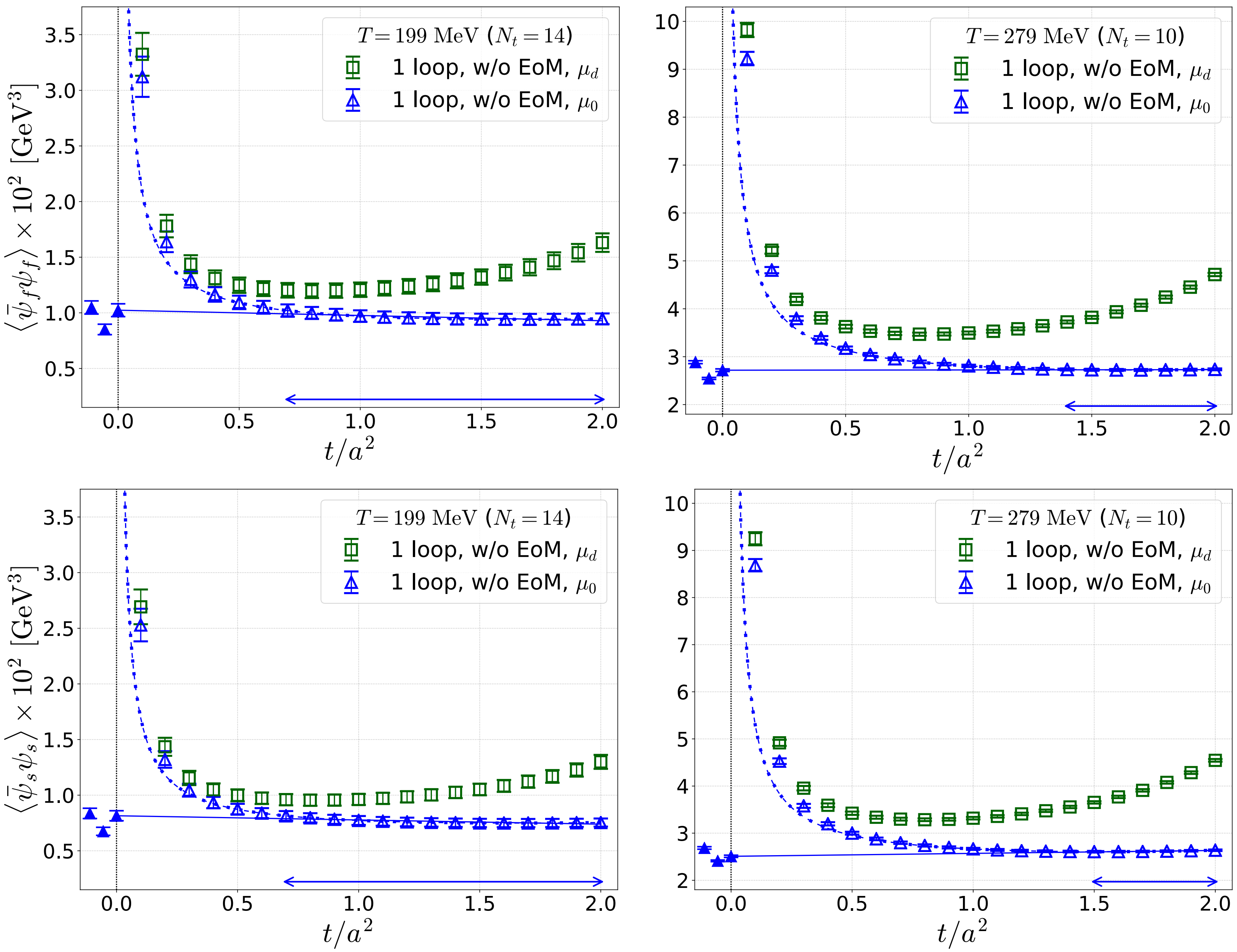
chiral condensate with heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

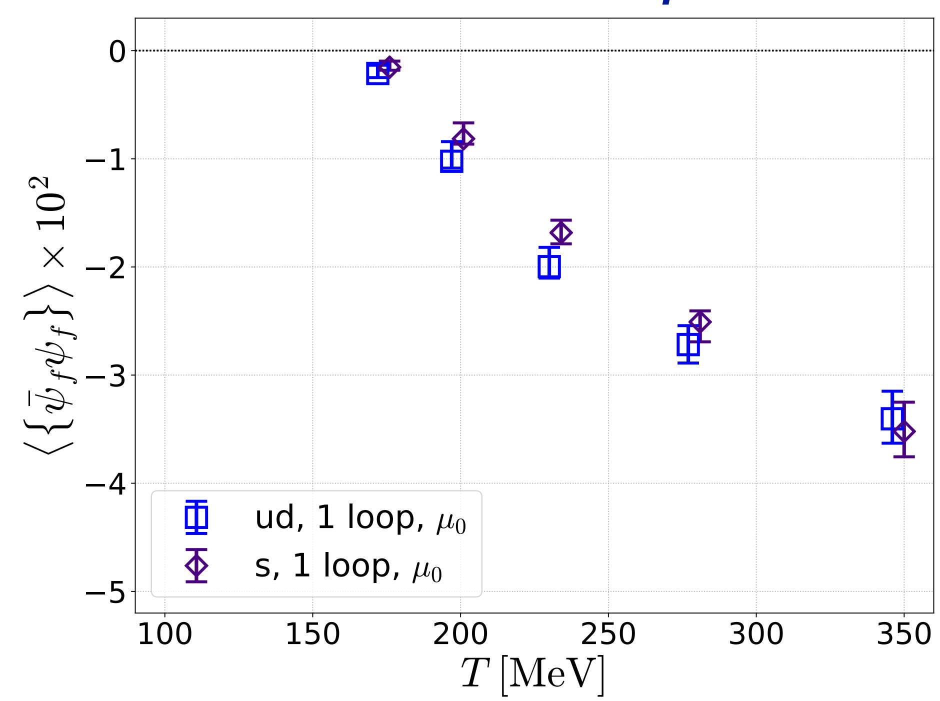
ud- and s-chiral cond. (VEV-subtracted)

$-\langle\{\bar{\psi}_u\psi_u\}\rangle_{\overline{\mathrm{MS}}}(\mu=2\,\mathrm{GeV})$

in unit of GeV^3



Results with the μ_0-scale

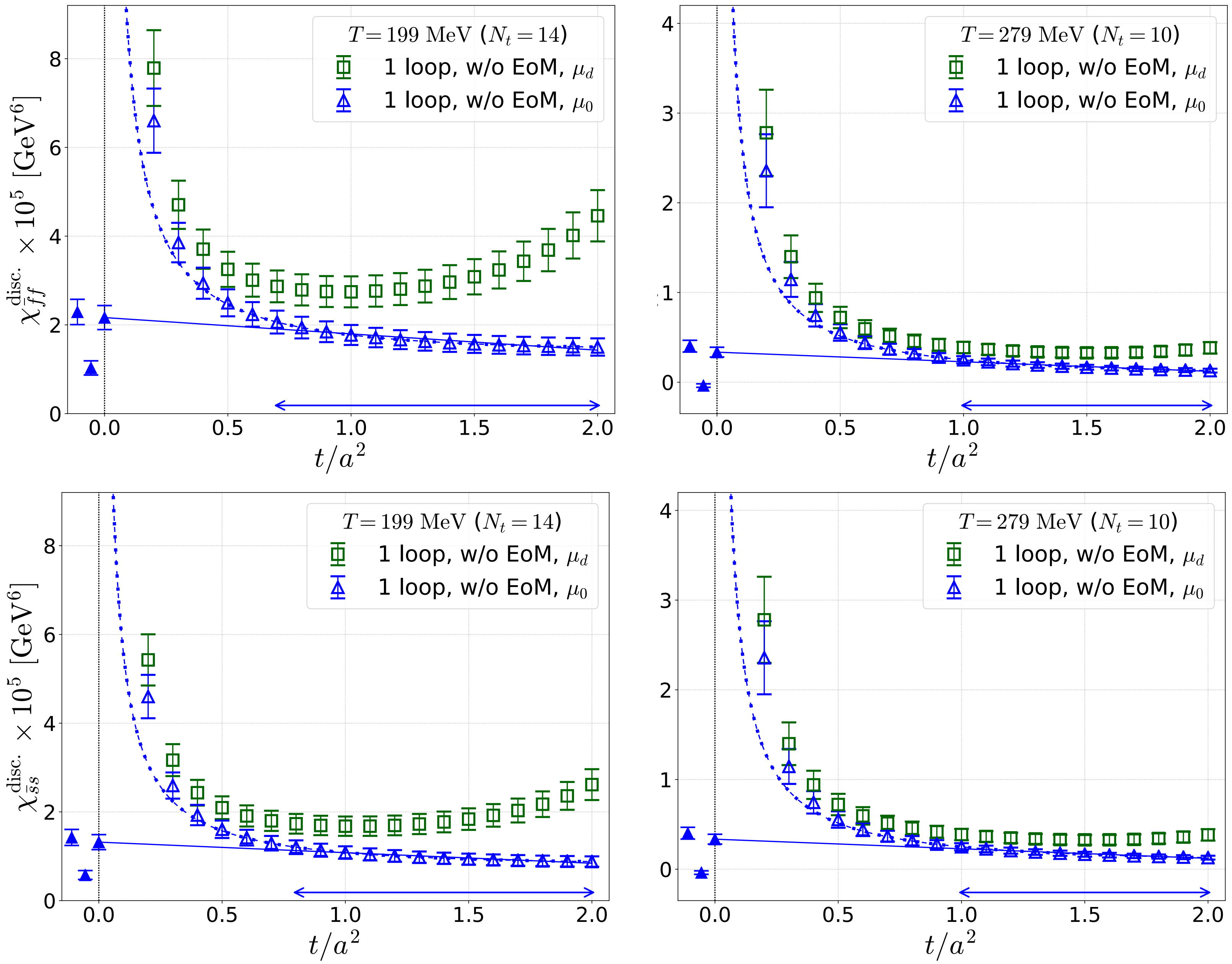


- ☑ μ_0 and μ_d results consistent with each other
- ☑ μ_0 improves linear behavior at large t => more reliable linear extrapolations

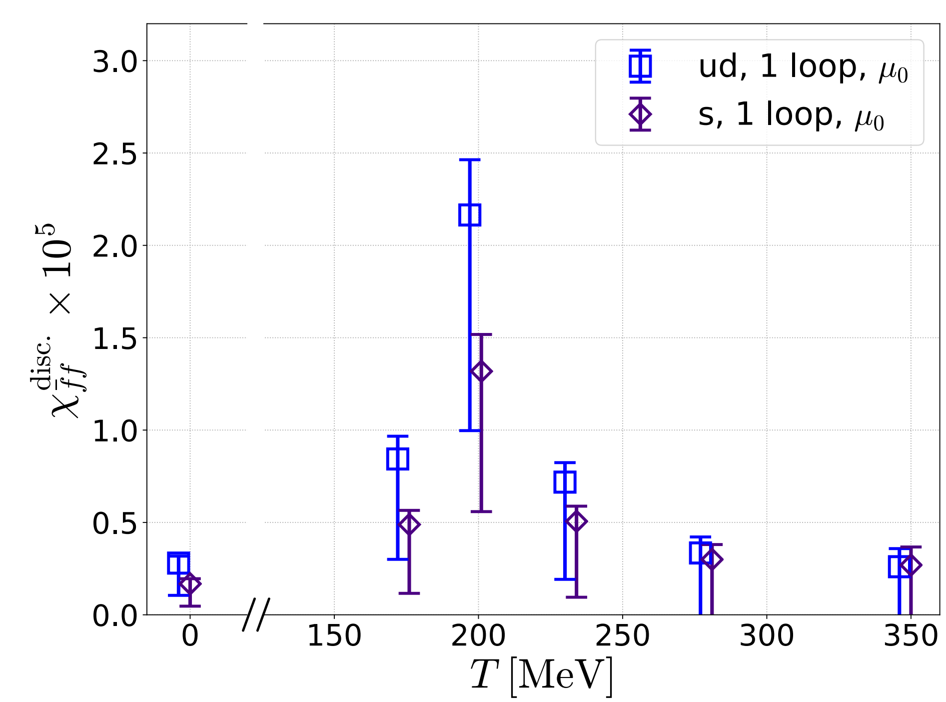
chiral susceptibility with heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

ud- and s-chiral suscept. (disconnected)



Results with the μ_0-scale



- μ_0 and μ_d results consistent with each other
- μ_0 improves linear behavior at large t

=> μ_0 extend the reliability/applicability of the SF*t*X method => helps the phys. pt. study

[1B]
2-loop matching coefficients in N_F = 2+1 QCD with slightly heavy u,d

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

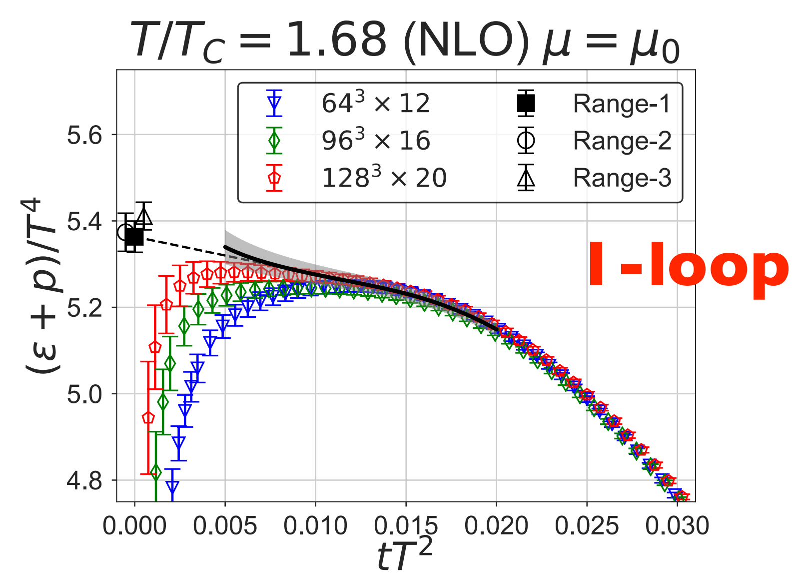
2-loop matching coefficients for EMT

Harlander-Kluth-Lange, EPJC 78:944 (2018)

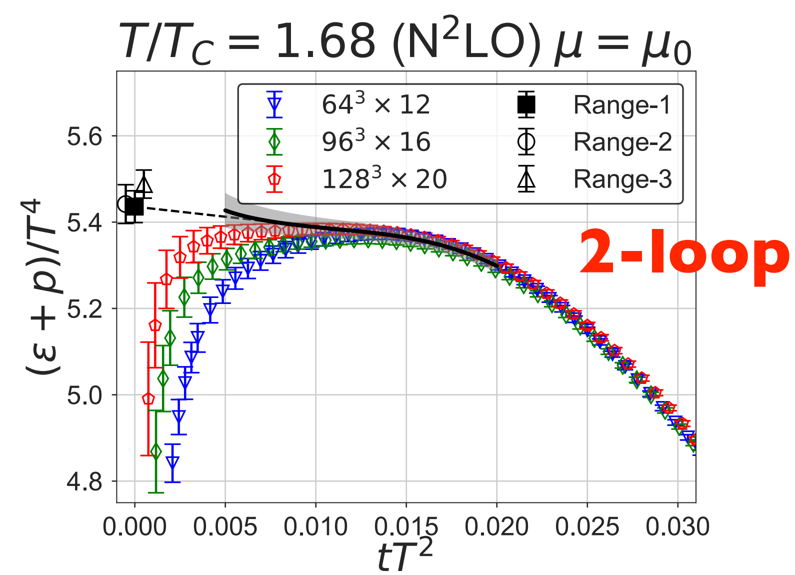
$$c_1(t) = \frac{1}{g^2}\left\{1 + \frac{g^2}{(4\pi)^2}\left[-\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0 L(\mu,t)\right]\right.$$
$$+\frac{g^4}{(4\pi)^4}\left[-\beta_1 L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2\right.\right.$$
$$\left. + \frac{1187}{10}\ln 3\right) + C_A T_F\left(\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2\right.$$
$$\left. - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right) + C_F T_F$$
$$\left(-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{2587}{108} - \frac{7}{9}\pi^2\right.$$
$$\left.\left.\left. - \frac{106}{9}\ln 2 - \frac{161}{18}\ln 3\right)\right] + \mathcal{O}(g^6)\right\},$$

etc. with $L(\mu,t) \equiv \ln\left(2\mu^2 t\right) + \gamma_E$

Removing more known small-t properties, we may expect a milder t-dep. at small t.



(a)



- ➤ first test in quenched QCD

Iritani-Kitazawa-Suzuki-Takaura, PTEP 2019, 023B02 (2019)

- ☑ Results of EoS with 1- and 2-loop coefficients are consistent with each other.
- ☑ With 2-loop coefficients, t-dep. is milder.
- ☑ Thus, 2-loop coefficients reduce systematic errors from the $t\rightarrow 0$ extrapolation.

2-loop matching coefficients for EMT

- **matching coefficients for full QCD EMT**

Harlander *et al.* used the **equation of motion (EoM) for quarks**

$$\bar{\psi}_f(x)\left(\frac{1}{2}\overleftrightarrow{\not{D}}+m_{0,f}\right)\psi_f(x)=0$$

to reduce the number of independent operators/coefficients for EMT.

This should be OK when we take the continuum limit.

However, **EoM gets corrections** at ***a* ≠ 0** on the lattice.

=> May introduce another source of lattice errors.

(Note 1) EoM not used in the quenched coefficients.

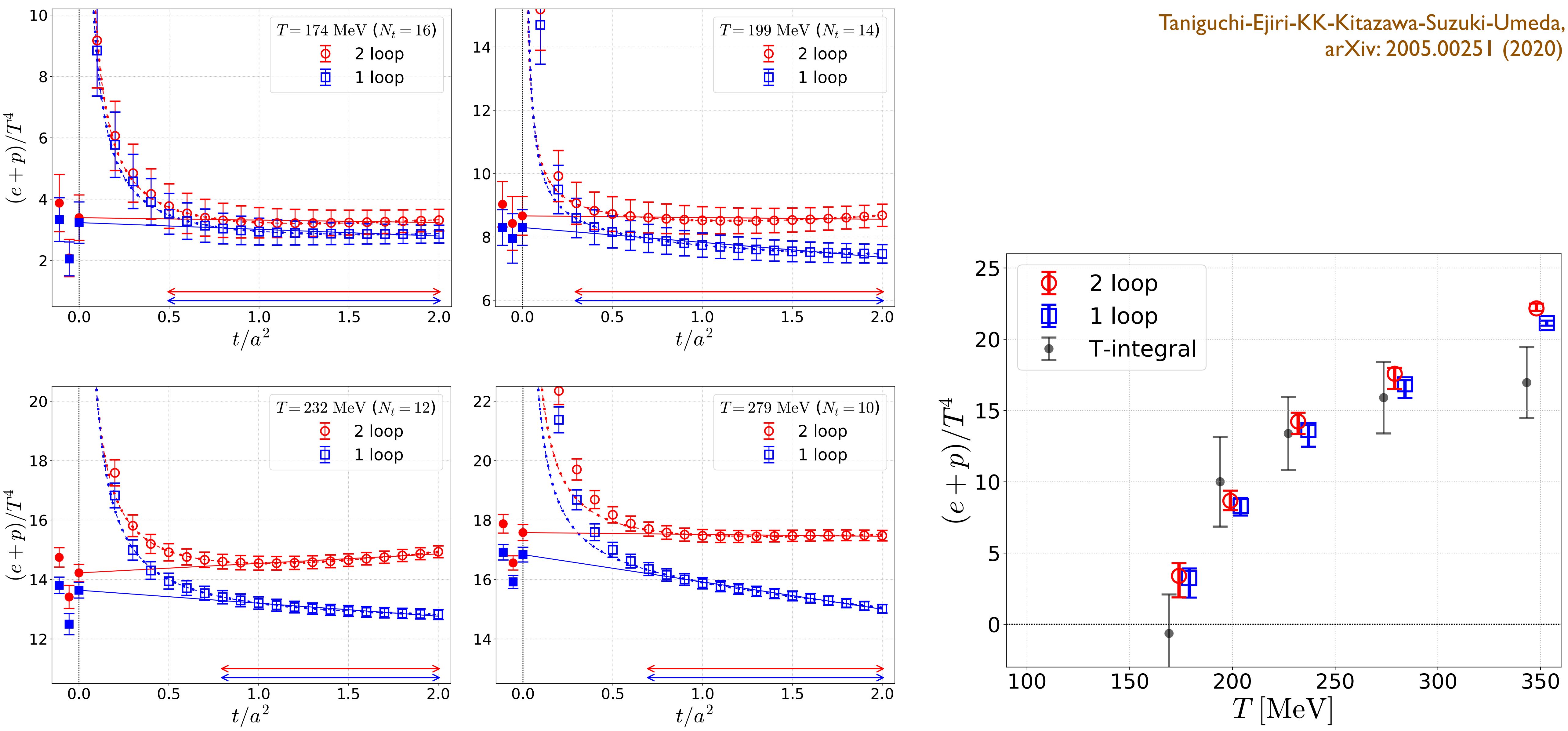
(Note 2) EoM affects the trace-part of EMT only.

(2+1)-flavor heavy u,d QCD w/ 2-loop coefficients

μ_0-scale

entropy density $(e+p)/T^4$ in which EoM not used. <= trace-less combination of EMT

Taniguchi-Ejiri-KK-Kitazawa-Suzuki-Umeda, arXiv: 2005.00251 (2020)

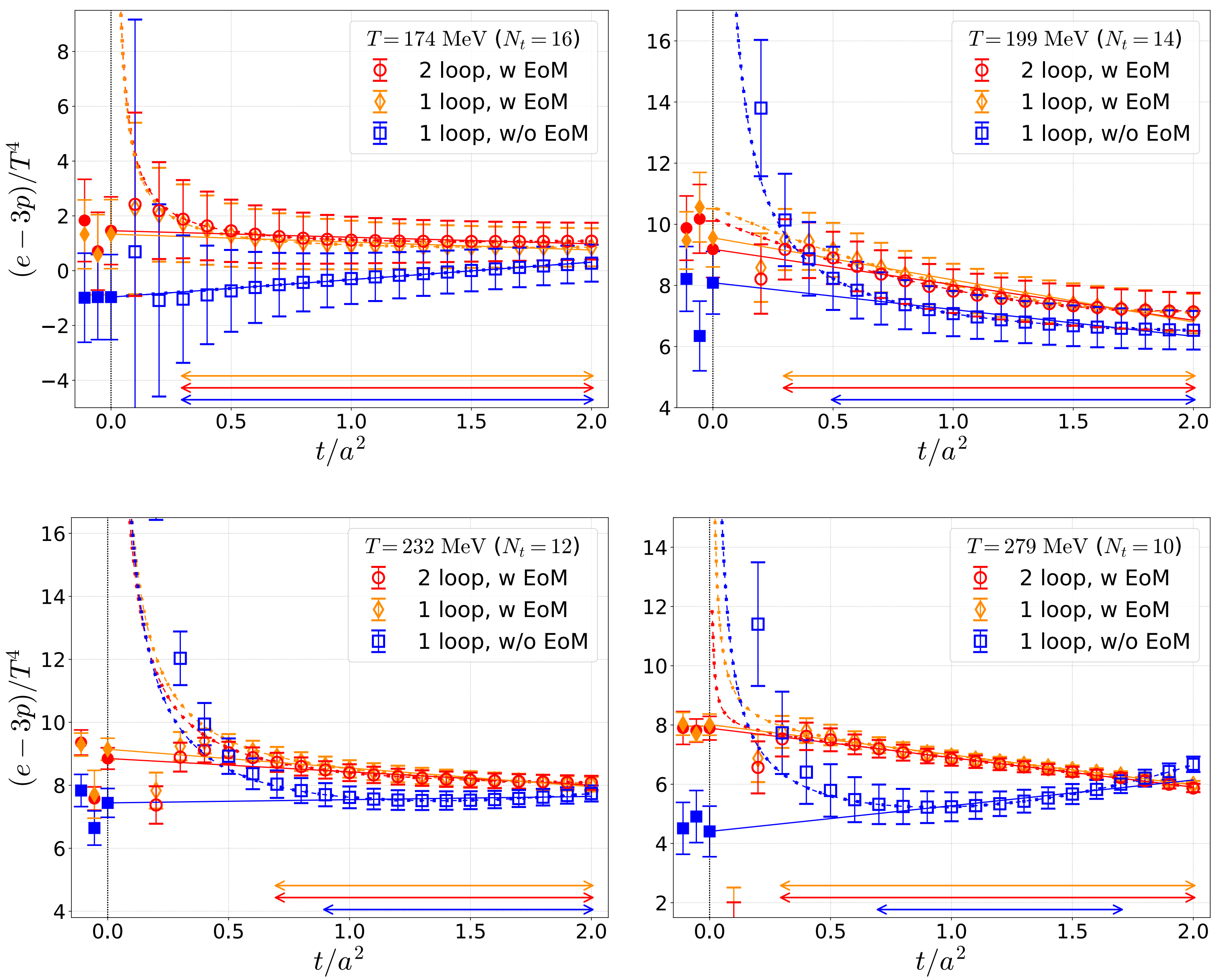


☑ 1- and 2-loop results consistent with each other. No apparent improvements with 2-loop.

(2+1)-flavor heavy u,d QCD w/ 2-loop coefficients

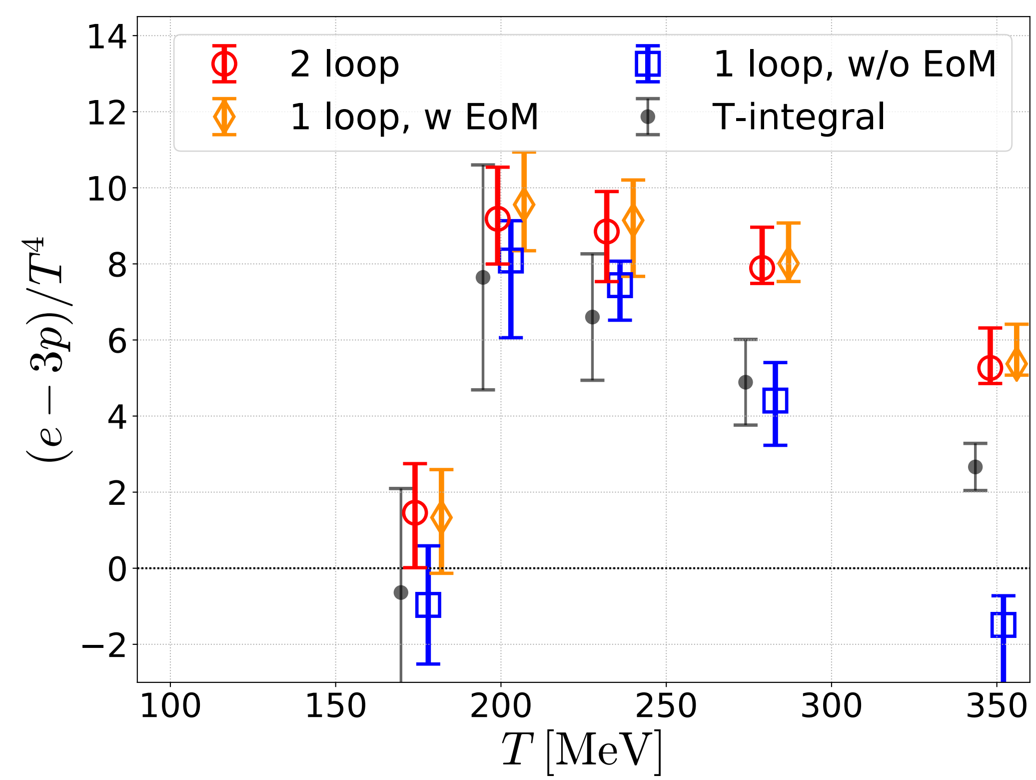
μ_0-scale

trace anomaly $(e\text{-}3p)/T^4$ in which EoM is used in the 2-loop HKL coefficients.



To identify the effects of EoM, we compare

- 1-loop Makino-Suzuki w/o EoM
- 2-loop HKL coefficients w/ EoM
- 1-loop HKL coefficients w/ EoM



☑ 1-loop (w/ EoM) and 2-loop (w/ EoM) well consistent at all T. No apparent improvements with 2-loop.

☑ 1-loop (w/o EoM) and 2-loop (w/ EoM) disagree at $N_t \leq 10$.

=> EoM gets $O((aT)^2) = O(1/N_t^2)$ lattice artifacts at $N_t \leq 10$.

[2]
N_F = 2+1 QCD
with **physical** u,d,s quarks

KK-Baba-SuzukiA-Ejiri-Kitazawa-SuzukiH-Taniguchi-Umeda, PoS Lattice2019, 088 (2020)

(2+1)-flavor **phys.pt.** QCD

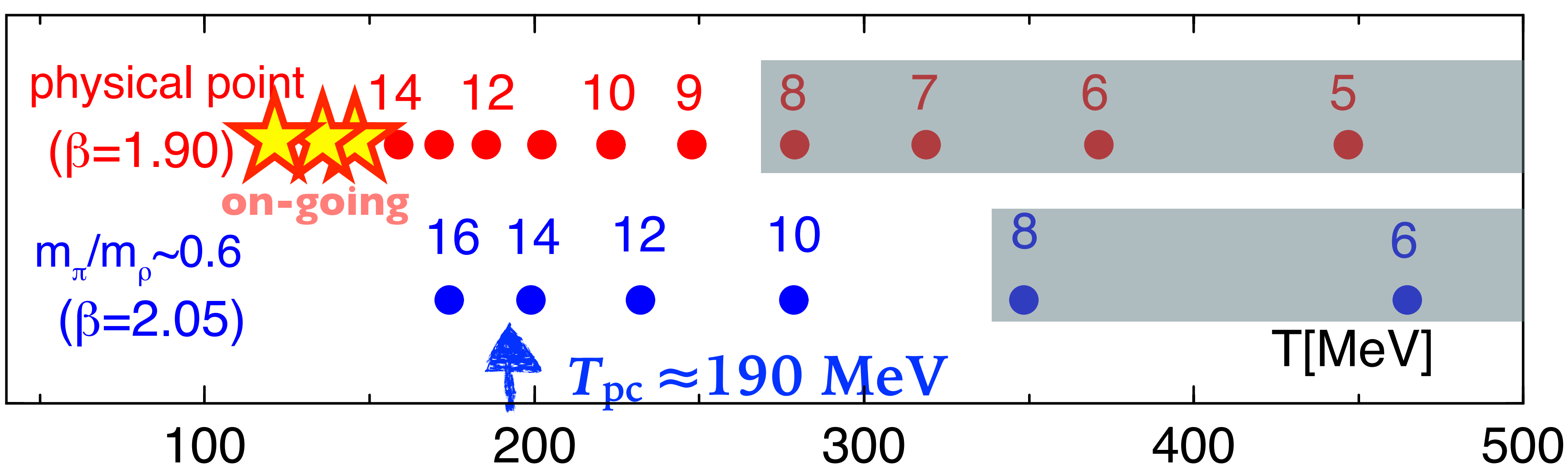
WHOT-QCD, EPJ Conf. 175, 07023 (2018)

+ New data at $T \approx 122$–146 MeV (prelim.)

- RG-improved Iwasaki gauge + NP O(a)-improved Wilson quarks
- T=0 configs. of PACS-CS (β=1.9, $32^3 \times 64$, ***$a \approx 0.09$fm***) [Phys.Rev.D79, 034503 (2009)] 80 configs.
- All quarks fine-tuned to the **phys.pt.** by reweighting [Phys.Rev.D81, 074503 (2010)] using m_π, m_K, m_Ω inputs.
- T>0 by fixed-scale approach, ($32^3 \times Nt$, Nt = 4, 5, ... , 18): **$T \approx 122$ – 549 MeV.**

  Odd Nt too, to have a finer T-resolution.

  Generated directly at the phys.pt. w/o reweighting [β=1.9, Kud=0.13779625, Ks=0.13663377].



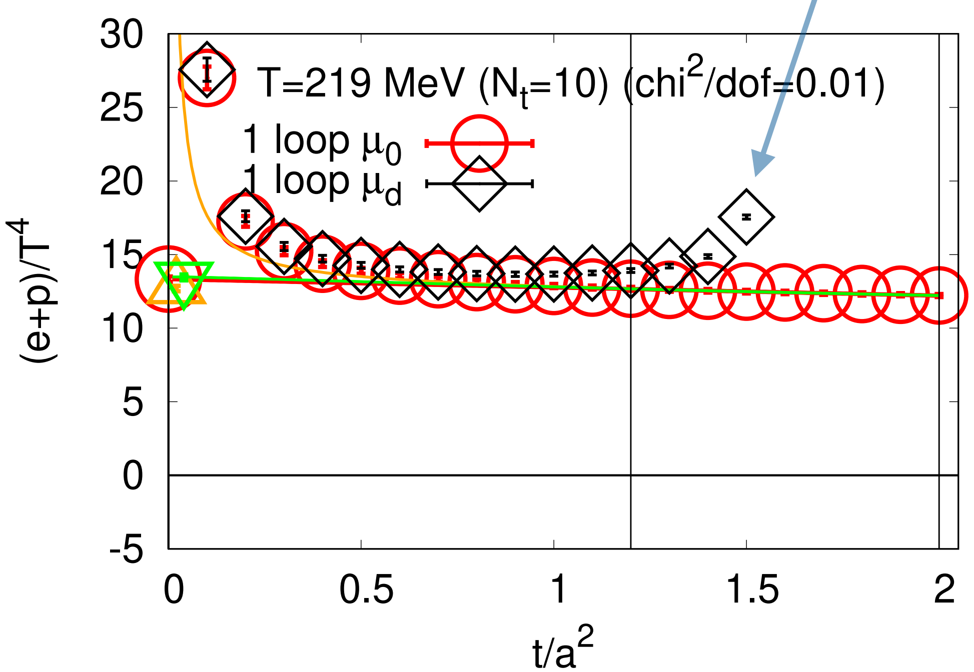
T[MeV]	T/T_{pc}	N_t	$t_{1/2}$	gauge confs.	fermion confs.
0	0	64	32	80	80
122		18	10.125	308	308
129		17	9.03125		
137		16	8	239	239
146		15	7.03125	143	143
157		14	6.125	650	65
169		13	5.28125	550	55
183		12	4.5	610	61
199		11	3.78125	890	89
219		10	3.125	690	69
244		9	2.53125	780	78
274		8	2	680	68
313		7	1.53125	220	22
366		6	1.125	280	280
439		5	0.78125	130	130
548		4	0.5	70	70

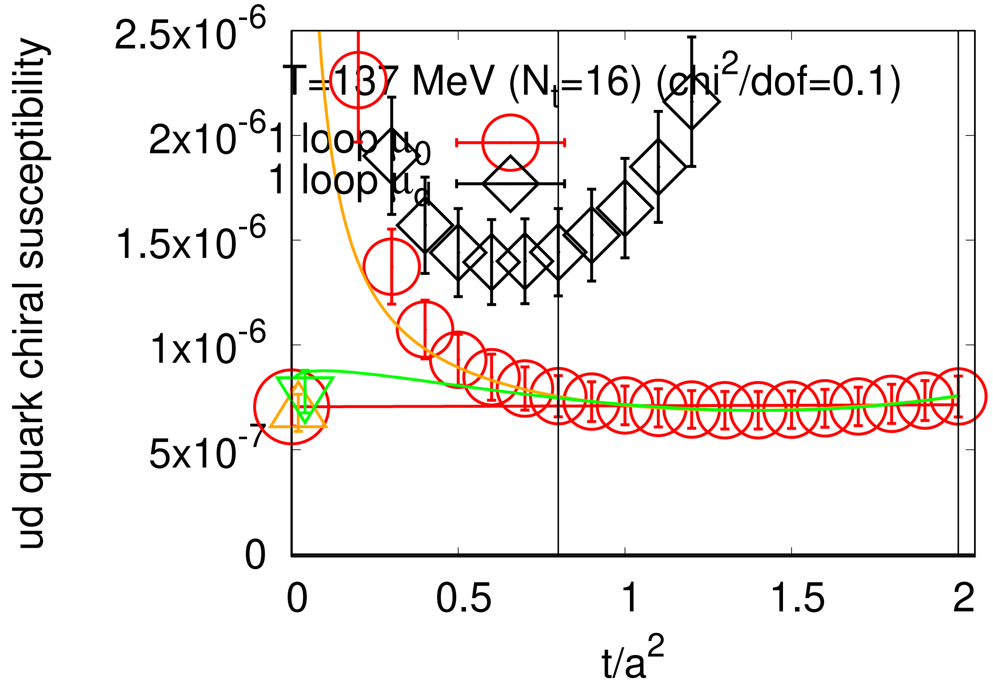
on-going

- ☐ Where is T_{pc} for physical m_q? Expect T_{pc}^{phys} < 190 MeV.
- ☐ Lattice is slightly coarser than the heavy QCD case ($a \approx 0.07$fm).
- ☐ Expect a-indep. lattice artifacts of O($(aT)^2 = 1/N_t^2$) at $N_t \leq 8$ ($T \geq 274$ MeV)

renormalization scale μ

- Lattice at $a \approx 0.09$fm is slightly coarser than the heavy QCD case ($a \approx 0.07$fm).
  => Perturbative behavior worse --- μ_0 may help.
  $g(\mu(t))$ becomes large at $t/a^2 \approx 1.5$ with $\mu_d(t)$, but remains small up to ≈ 3 with $\mu_0(t)$.





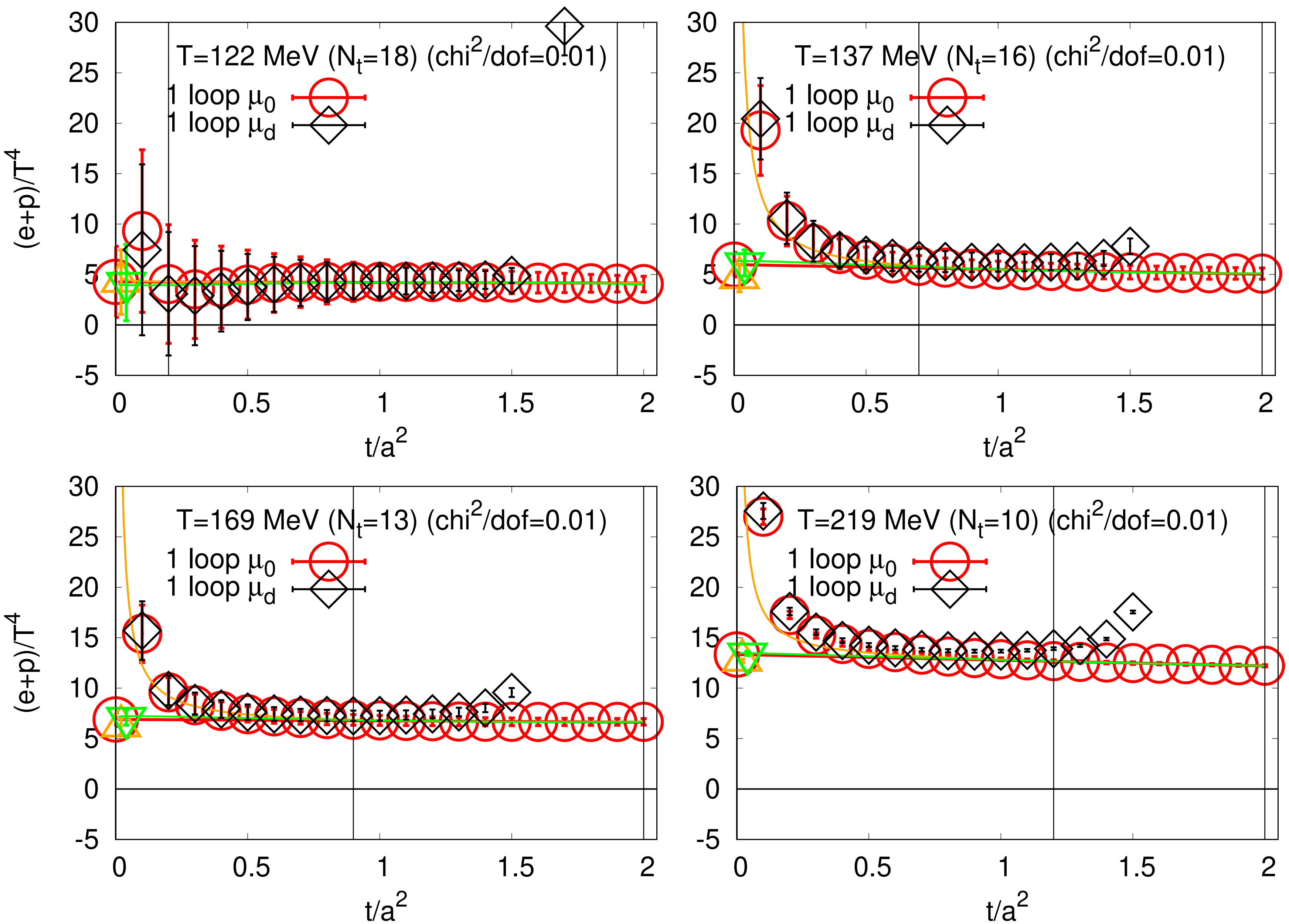
- ☑ μ_0 and μ_d results consistent with each other
- ☑ μ_0 improves linear behavior at large t
  => μ_0 extend the reliability/applicability of the SFtX method

EoS at the physical point

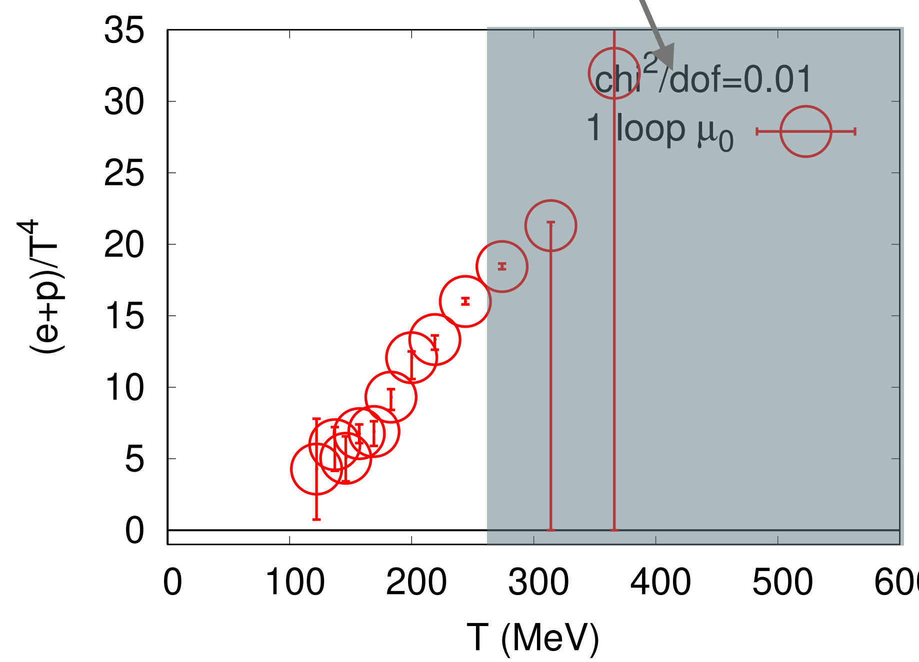
entropy density $(e+p)/T^4$

1-loop
μ_0-scale

Preliminary



O$((aT)^2)$ lattice artifacts at $Nt \leq 8$



- ☑ μ_0 and μ_d results consistent with each other
- ☑ μ_0 improves linear behavior at large t

=> μ_0 extend the reliability/applicability of the SFtX method

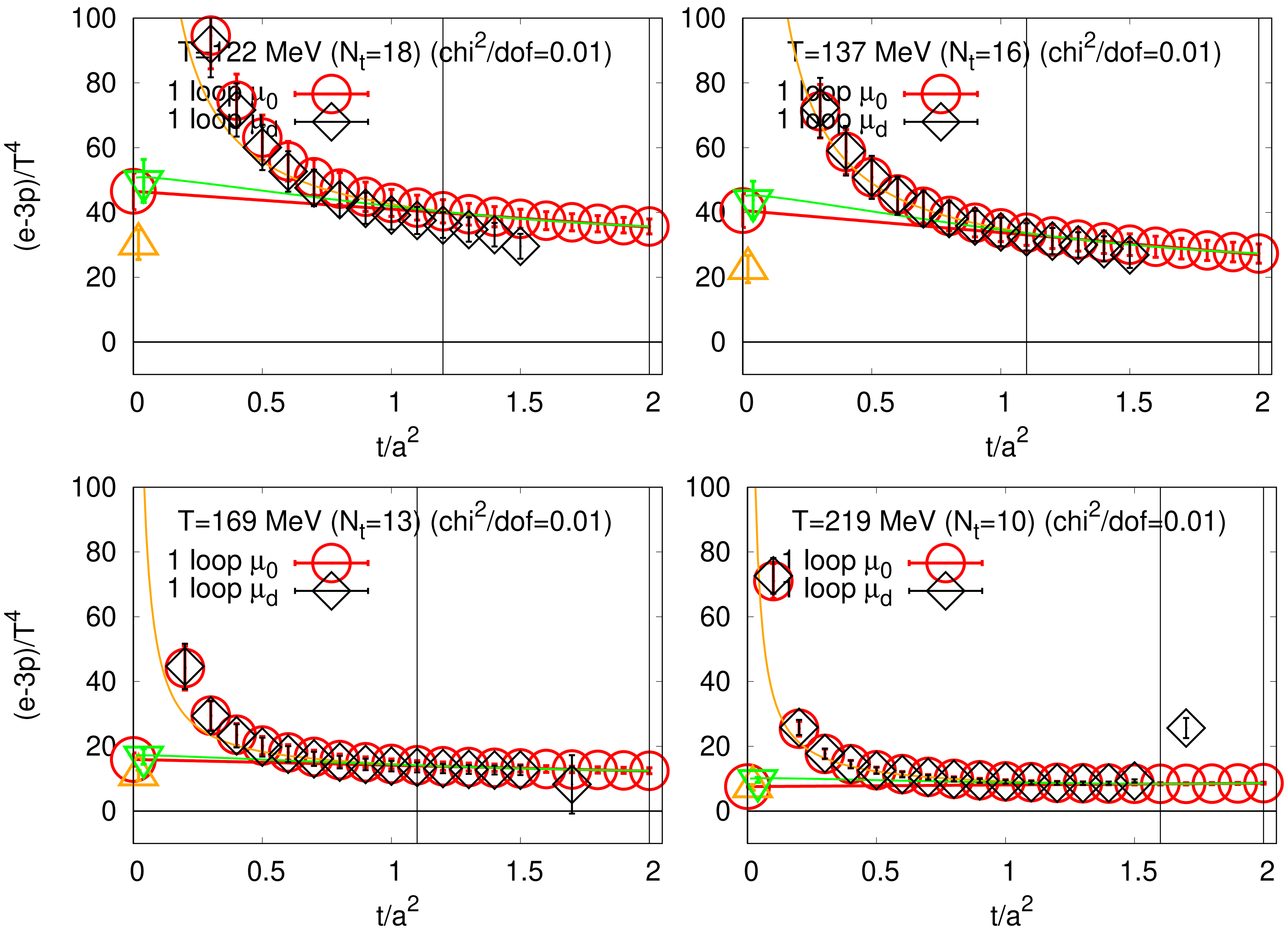
Results with the μ_0-scale

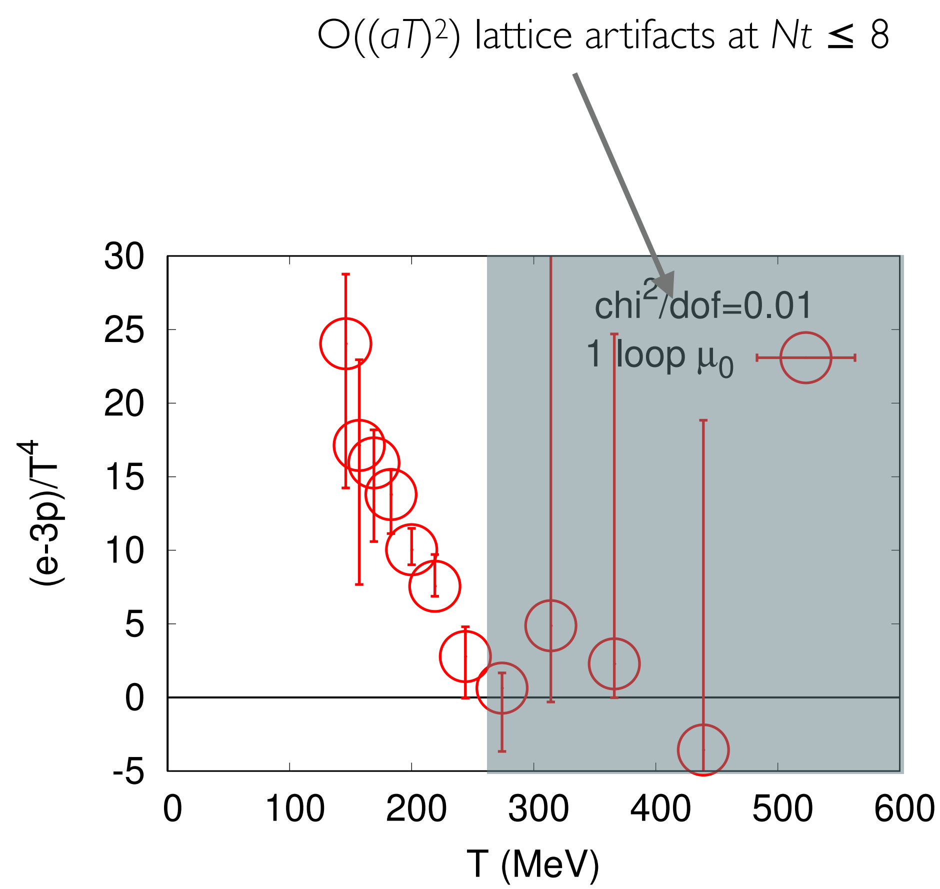
EoS at the physical point

1-loop
μ_0-scale

trace anomaly $(e-3p)/T^4$

Preliminary





- ☑ μ_0 and μ_d results consistent with each other
- ☑ μ_0 improves linear behavior at large t

=> μ_0 extend the reliability/applicability of the SFtX method

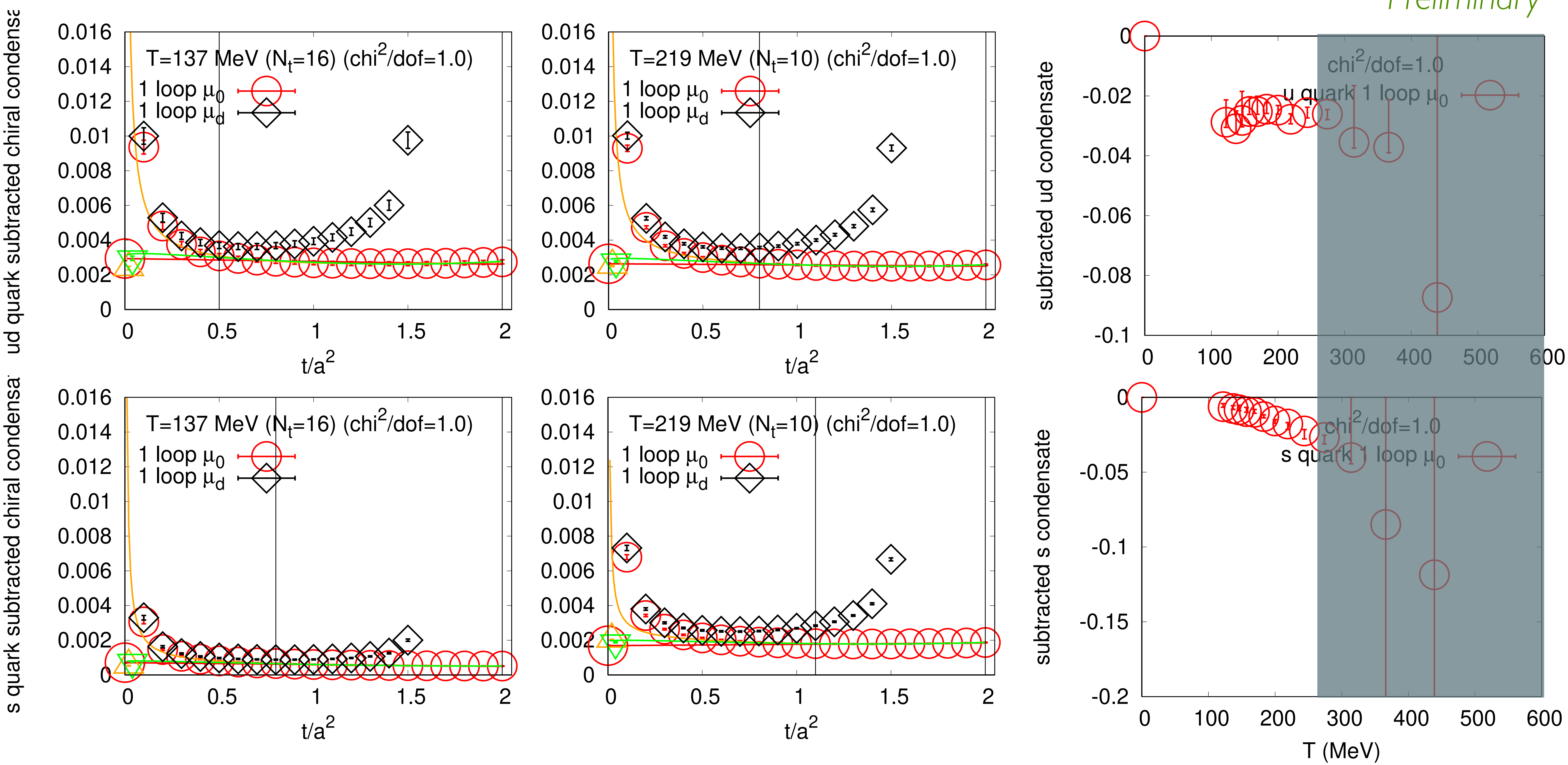
Results with the μ_0-scale

chiral condensate at the physical point

ud- and s-quark chiral cond. (VEV-subtracted)

1-loop
μ_0-scale

Preliminary



Results with the μ_0-scale

☑ μ_0 improves linear behavior at large *t*

=> μ_0 extend the reliability/applicability of the SF*t*X method

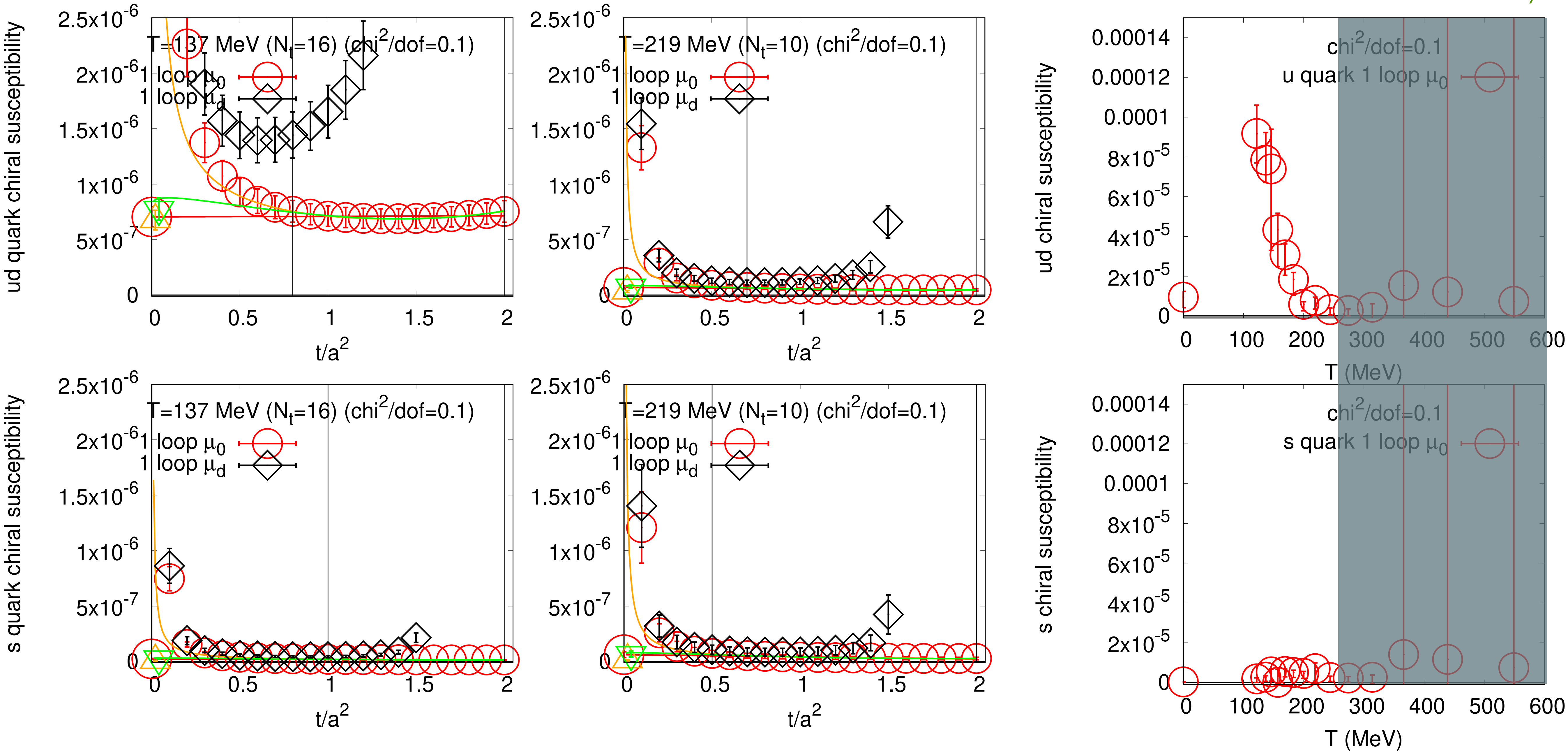
$-\langle\{\bar{\psi}_u\psi_u\}\rangle_{\overline{\rm MS}}(\mu=2\,{\rm GeV})$
in unit of GeV^3

chiral susceptibility at the physical point

ud- and s-quark chiral suscept. (disconnected)

1-loop
μ_0-scale

Preliminary

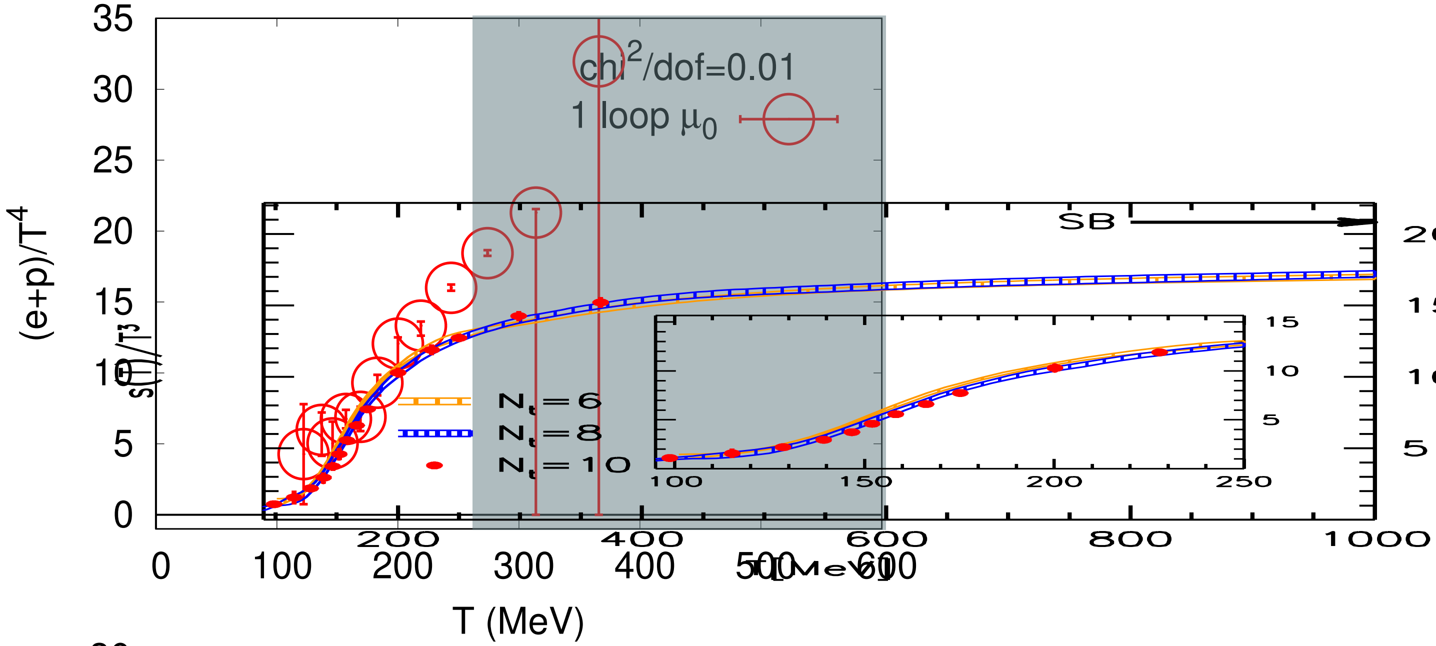


Results with the μ_0-scale

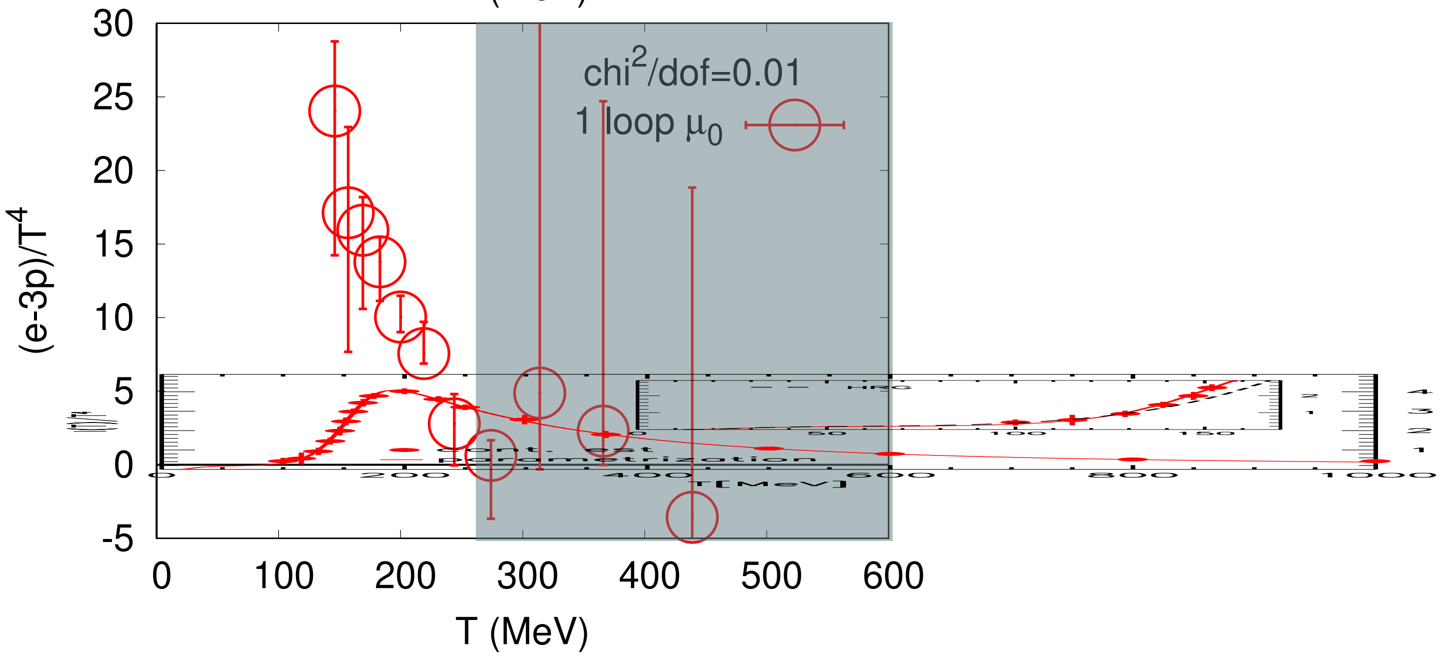
☑ μ_0 improves linear behavior at large t

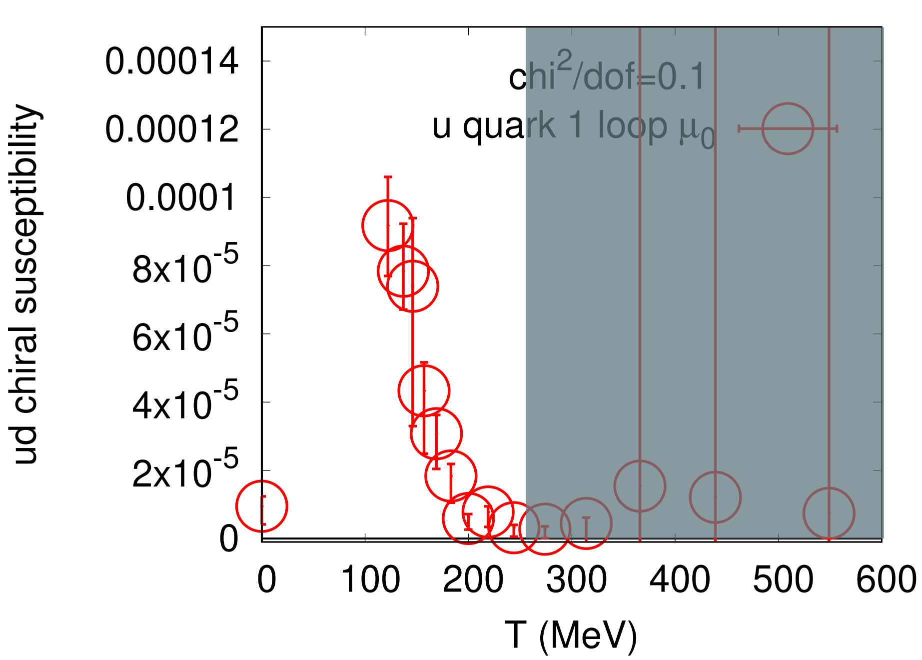
=> μ_0 extend the reliability/applicability of the SFtX method

in unit of GeV^6



Borsany et al., JHEP 1011, 077 (2010), with KS(stout).





☑ **T_{pc}^{phys} < 157 MeV (T≈122-146MeV critical ??)**

(cf.) Result with 2+1 staggered quarks
156.5 ± 1.5 MeV
Bazavov et al. PLB795, 15 (2019), HISQ

- ☐ Need more statistics / more data points at low T's. (*on-going*)
- ☐ A definite conclusion possible only after continuum extrapolation.


summary

summary: SF*t*X method in 2+1 flavor QCD

1. 2+1 flavor QCD with slightly heavy u,d and ≈physical s quarks

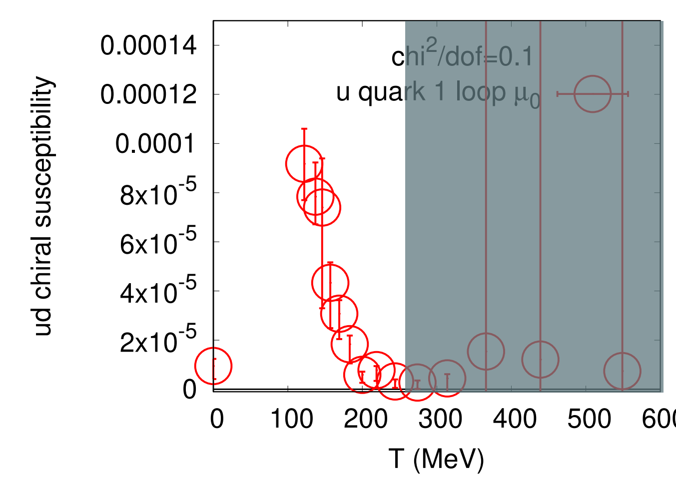
- ➤ fine $a \approx 0.07$fm lattice with improved Wilson quarks, $32^3 \times N_t$ (N_t=4-16): $T \approx$ 174-697MeV
- ☑ EoS agrees well with conventional integral method at $T \leq 300$ MeV ($N_t \geq 10$), while $O((aT)^2 = 1/N_t^2)$ lattice artifacts suggested at $N_t \leq 8$.
- ☑ Chiral suscept. show clear peak at $T_{pc} \approx 190$ MeV expected from Polyakov loop etc.
- ☑ Topological suscepts. by gluonic and fermionic definitions agree well.
- ☑ μ_0-scale extends the reliability/applicability of the SF*t*X method.
- ☑ 1- and 2-loop matching coefficients lead to consistent results, while EoM gets $O((aT)^2 = 1/N_t^2)$ lattice artifacts at $N_t \leq 10$.

=> SF*t*X powerful in evaluating physical observables.

- ☐ A definite conclusion possible only after continuum extrapolation, though our results suggest that $a \approx 0.07$fm is fine enough.

2. 2+1 flavor QCD with physical u,d,s quarks

- ➤ less fine $a \approx 0.09$fm lattice, $32^3 \times N_t$ (N_t=4-18): $T \approx$ 122-549MeV
- ☑ The μ_0-scale helps much.
- ☑ **$T_{pc}^{phys} < 157$ MeV ($T \approx$ 122-146MeV critical ??)**
- ☐ Need more statistics / more data points at low T's. => *on-going.*
- ☐ Data at larger t/a^2 may help. => *on-going.*
- ☐ Need continuum extrapolation too. => *being started.*

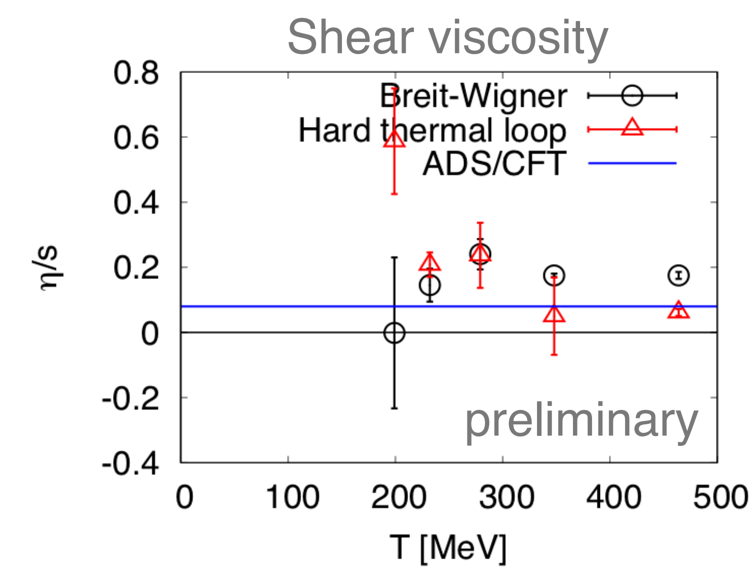


preliminary

prospects / to do

☆ other observables

- EMT correlation functions
  - transport coefficients of QGP: shear/bulk viscosity, etc.
  - test: thermodynamic relations vs. linear response relations
- chiral observables
  - matrix elements: B_K, etc.
- topological observables at the physical point



☆ continuum extrapolation

- Slightly heavy ud + ≈phys. s on a **less fine lattice** (a≈0.097fm), $24^3 \times N_t$ (N_t=8-12): T≈170-254MeV
  - ☑ Look similar to the fine lattice case
  - ☑ Linear windows narrower than the fine lattice case. => μ_0 will help
  - ☑ a-dep. looks small up to this a
  - ☐ need more statistics + a finer point
- PACS10 configurations (T=0) at the physical point
  - ☐ generation of finite temperature configurations


thank you!