Zero-temperature phase structure of the 1+1 dimensional Thirring model from matrix product states

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Outline

- Preliminaries: motivation and introduction
- Lattice formulation and the MPS
- Simulations and numerical results:
  Phase structure of the Thirring model
- Remarks and outlook (spectrum, real-time dynamics)

Preliminaries

Logic flow

Hamiltonian formalism for QFT

↓

Quantum spin model

↓

MPS & variational method for obtaining the ground state

↓

Compute correlators and excited state spectrum

Motivation

- New formulation for lattice field theory
- No sign problem
- Real-time dynamics
- Future quantum computers?

In this talk: BKT phase transition

The 1+1 dimensional Thirring model

$$S_{\mathrm{Th}}[\psi,\bar{\psi}] = \int d^2x \left[\bar{\psi} i\gamma^\mu \partial_\mu \psi - m\,\bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma_\mu\psi\right)\left(\bar{\psi}\gamma^\mu\psi\right)\right]$$

- ★ Conformality of the massless theory
- ★ Duality with the sine-Gordon theory

Bosonisation and duality

- Basic ingredients from free field theories

$$\left\langle \prod_{i=1}^{n} \mathrm{e}^{i\kappa_i \phi(x)} \right\rangle_{\text{ren.}} = \prod_{i<j} \left(\mu \left|x_i - x_j\right|\right)^{\kappa_i \kappa_j / 2\pi}, \text{ where } \left[\mathrm{e}^{i\kappa_i \phi(x)}\right]_{\text{bare}} = (\Lambda/\mu)^{-\kappa_i^2/4\pi} \left[\mathrm{e}^{i\kappa_i \phi(x)}\right]_{\text{ren.}}$$

And similar power law for $\bar{\psi}\psi$ correlators.

Works in the zero-charge sector

- The dictionary (zero total fermion number)

$$S_{\text{Th}}\left[\psi, \bar{\psi}\right] = \int d^2x \left[\bar{\psi} i \gamma^\mu \partial_\mu \psi - m_0 \bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma_\mu \psi\right)^2\right]$$

field redifinition, anomaly

$$S_{\text{SG}}\left[\phi\right] = \frac{1}{t} \int d^2x \left[\frac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) + \alpha_0 \cos\left(\phi(x)\right)\right]$$

$$\bar{\psi}\gamma_\mu \psi \leftrightarrow \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi,$$

$$\bar{\psi}\psi \leftrightarrow \frac{\Lambda}{\pi} cos\phi,$$

$$\frac{4\pi}{t} = 1 + \frac{g}{\pi}.$$

$$\frac{\alpha_0}{t} = \frac{m_0 \Lambda}{\pi}.$$

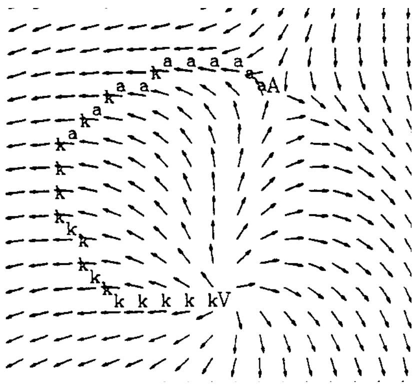
$$m_0 = m\left(\mu/\Lambda\right)^{g/(g+\pi)},$$

$$\alpha_0 = \alpha\left(\mu/\Lambda\right)^{-t/4\pi}.$$

★ **Coleman: Unstable vacuum at** $g \sim -\pi/2$

Dualities and phase structure

Thirring	sine-Gordon	XY
g	$\frac{4\pi^2}{t} - \pi$	$\frac{T}{K} - \pi$



Picture from: K. Huang and J. Polonyi, 1991

★ The K-T phase transition at $T \sim K\pi/2$ in the XY model.

$g \sim -\pi/2$, Coleman's instability point

★ The phase boundary at $t \sim 8\pi$ in the sine-Gordon theory.

→ The cosine term becomes relevant or irrelevant.

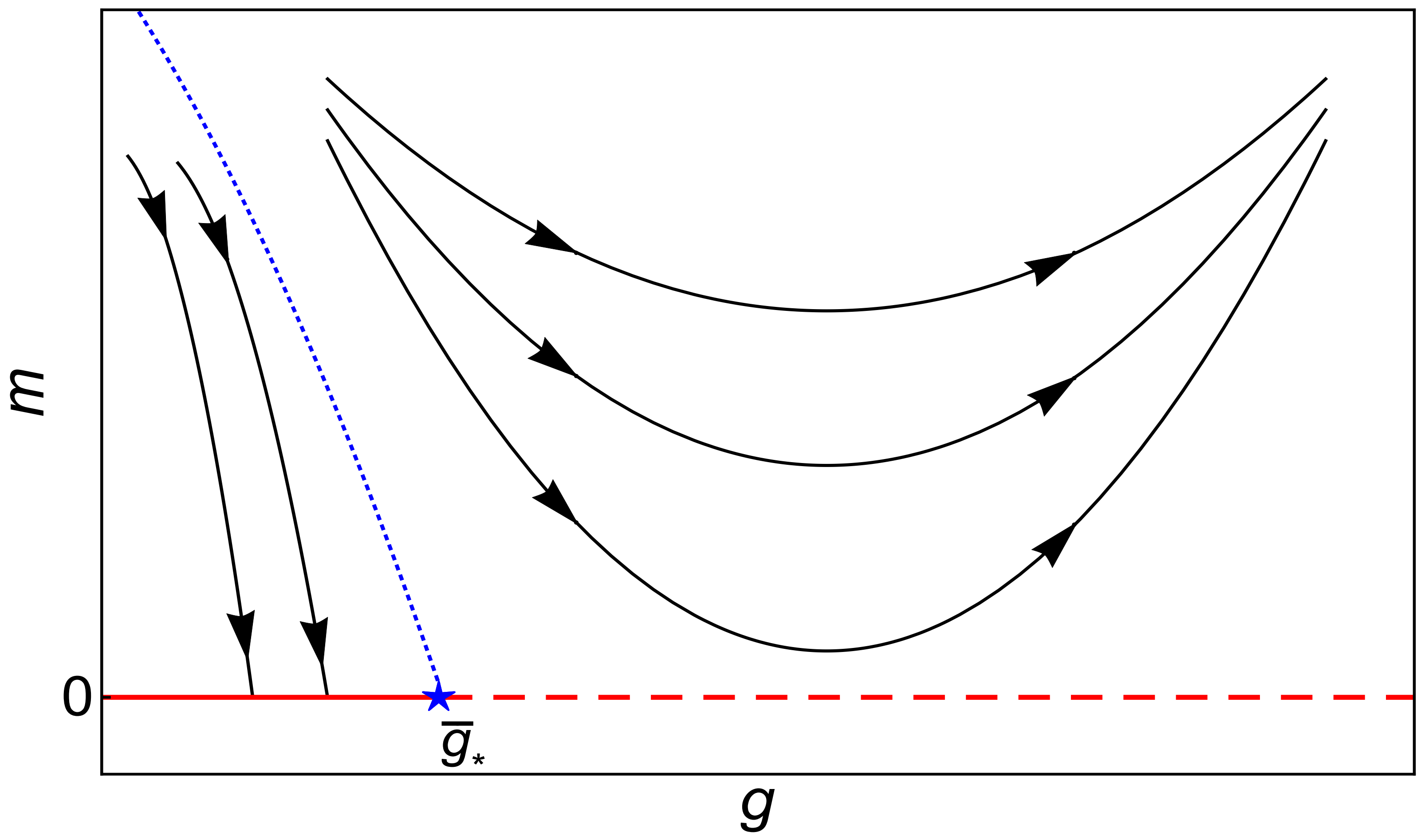
Thirring	sine-Gordon
$\bar{\psi}\gamma_\mu\psi$	$\frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi$
$\bar{\psi}\psi$	$\frac{\Lambda}{\pi}cos\phi$

RG flows of the Thirring model

Perturbative expansion in mass

$$\beta_g \equiv \mu \frac{dg}{d\mu} = -64\pi \left(\frac{m}{\Lambda}\right)^2 ,$$

$$\beta_m \equiv \mu \frac{dm}{d\mu} = m \left[\frac{-2(g+\frac{\pi}{2})}{g+\pi} - \frac{256\pi^3}{(g+\pi)^2} \left(\frac{m}{\Lambda}\right)^2 \right]$$



Lattice formulation and the MPS

Operator formalism and the Hamiltonian

- Operator formaliam of the Thirring model Hamiltonian

C.R. Hagen, 1967

$$H_{\mathrm{Th}} = \int dx \left[-i\bar{\psi}\gamma^1\partial_1\psi + m_0\bar{\psi}\psi + \frac{g}{4}\left(\bar{\psi}\gamma^0\psi\right)^2 - \frac{g}{4}\left(1+\frac{2g}{\pi}\right)^{-1}\left(\bar{\psi}\gamma^1\psi\right)^2 \right]$$

- Staggering, J-W transformation ($S_j^{\pm} = S_j^x \pm iS_j^y$):

J. Kogut and L. Susskind, 1975; A. Luther, 1976

$$\bar{H}_{XXZ} = \nu(g)\left[-\frac{1}{2}\sum_n^{N-2}\left(S_n^+S_{n+1}^- + S_{n+1}^+S_n^-\right) + a\tilde{m}_0\sum_n^{N-1}(-1)^n\left(S_n^z+\frac{1}{2}\right) + \Delta(g)\sum_n^{N-1}\left(S_n^z+\frac{1}{2}\right)\left(S_{n+1}^z+\frac{1}{2}\right) \right]$$

$$\nu(g) = \frac{2\gamma}{\pi\sin(\gamma)},\ \tilde{m}_0 = \frac{m_0}{\nu(g)},\ \Delta(g) = \cos(\gamma),\ \text{with } \gamma = \frac{\pi - g}{2}$$

$$\bar{H}_{XXZ}^{(\mathrm{penalty})} = \bar{H}_{XXZ} + \lambda\left(\sum_{n=0}^{N-1} S_n^z - S_{\mathrm{target}}\right)^2$$

projected to a sector of total spin

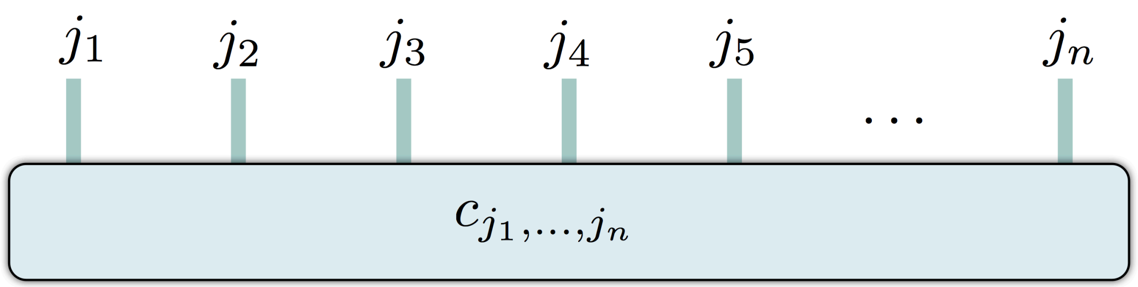
JW-trans of the total fermion number,
Bosonise to topological index in the SG theory.

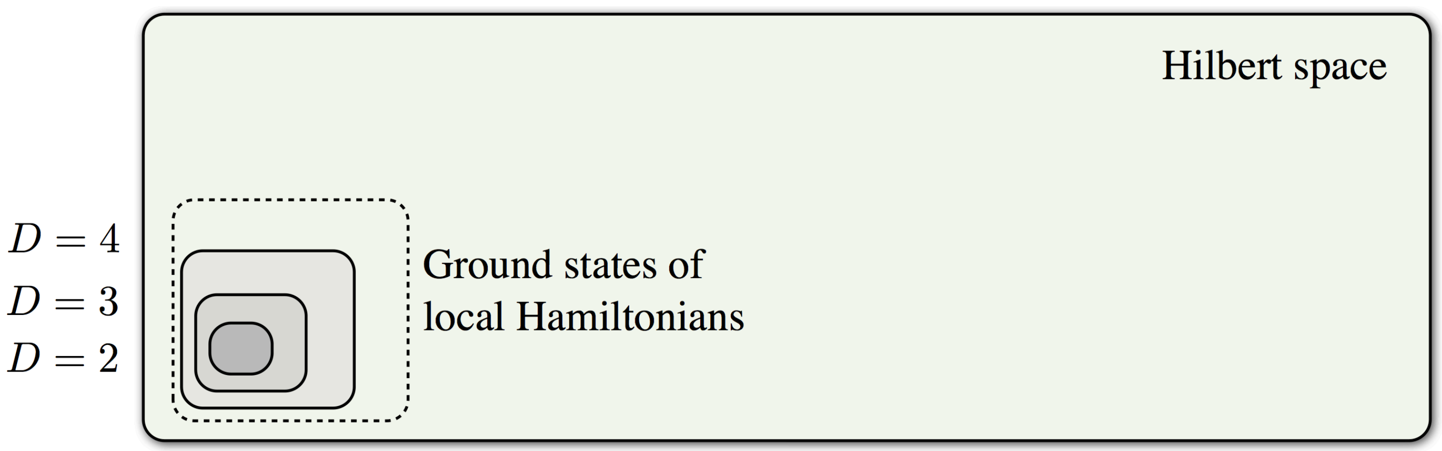
Issue of large Hilbert space & DMRG/MPS

S. White, 1992; M.B. Hasting, 2004; F. Verstraeten and I. Cirac, 2006; …

For a spin system of size n and local dimension d, $\dim(\mathcal{H}) = O(d^n)$.

$$|\psi\rangle = \sum_{j_1,\ldots,j_n=1}^{d} c_{j_1,\ldots,j_n} |j_1,\ldots,j_n\rangle = \sum_{j_1,\ldots,j_n=1}^{d} c_{j_1,\ldots,j_n} |j_1\rangle \otimes \cdots \otimes |j_n\rangle$$



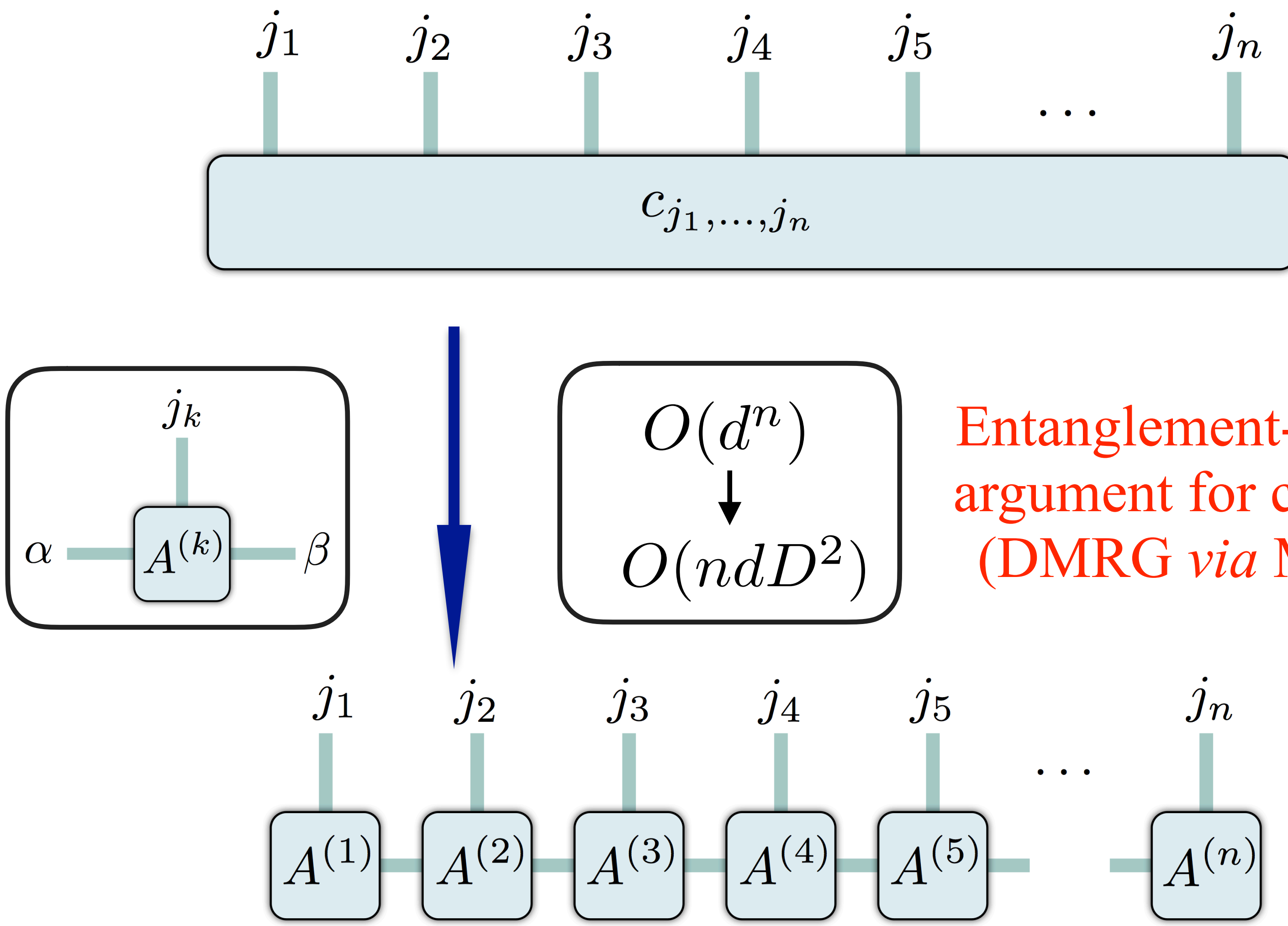


Entanglement-based truncation of the Hilbert space

(Area law of the entanglement entropy)

Matrix product states in a nutshell

$$|\psi\rangle = \sum_{j_1,\ldots,j_n=1}^{d} c_{j_1,\ldots,j_n} |j_1,\ldots,j_n\rangle = \sum_{j_1,\ldots,j_n=1}^{d} c_{j_1,\ldots,j_n} |j_1\rangle \otimes \cdots \otimes |j_n\rangle$$



Entanglement-based argument for choosing D (DMRG *via* MPS)

Bond dim

$$c_{j_1,\ldots,j_n} = \sum_{\alpha,\ldots,\omega=1}^{D} A^{(1)}_{\alpha;j_1} A^{(2)}_{\beta,\gamma;i_2} \ldots A^{(n)}_{\omega;j_n} = A^{(1)}_{j_1} A^{(2)}_{j_2} \ldots A^{(n)}_{j_n}$$

Matrix Product Operator

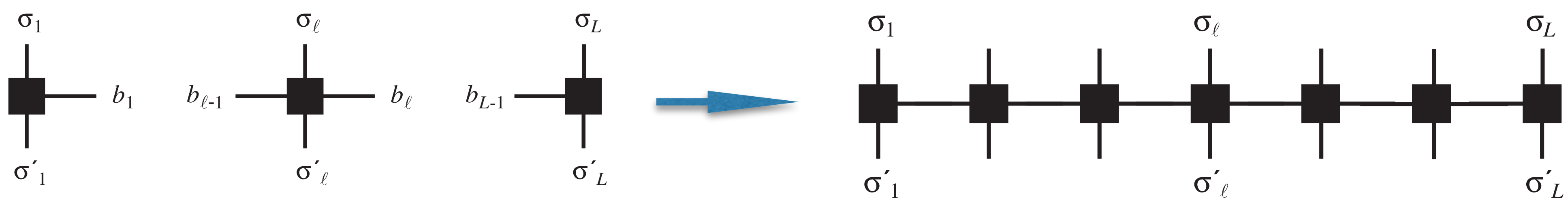
$$\hat{O} = \sum_i \left(\hat{A}_i \hat{B}_{i+1} + \hat{B}_i \hat{A}_{i+1} \right)$$
$$= \hat{A} \otimes \hat{B} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$+ \mathbb{1} \otimes \hat{A} \otimes \hat{B} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots$$
$$+ \hat{B} \otimes \hat{A} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$+ \mathbb{1} \otimes \hat{B} \otimes \hat{A} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots$$

b_{l-1} (→)

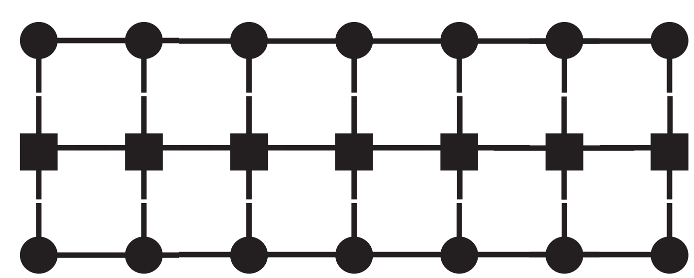
$$M = \begin{pmatrix} \mathbb{1} & \hat{A} & \hat{B} & 0 \\ 0 & 0 & 0 & \hat{B} \\ 0 & 0 & 0 & \hat{A} \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

b_{l-1} (↓)

$$\hat{O} = \sum_{b_1,\ldots,b_{L-1}} M^{\sigma_1,\sigma'_1}_{1,b_1} M^{\sigma_2,\sigma'_2}_{b_1,b_2} M^{\sigma_3,\sigma'_3}_{b_2,b_3} \ldots M^{\sigma_{L-1},\sigma'_{L-1}}_{b_{L-3},b_{L-1}} M^{\sigma_L,\sigma'_L}_{b_{L-1},1}$$



matrix elements



It is simple to compute local operator matrix elements with canonical states.

Simulation details for the phase structure

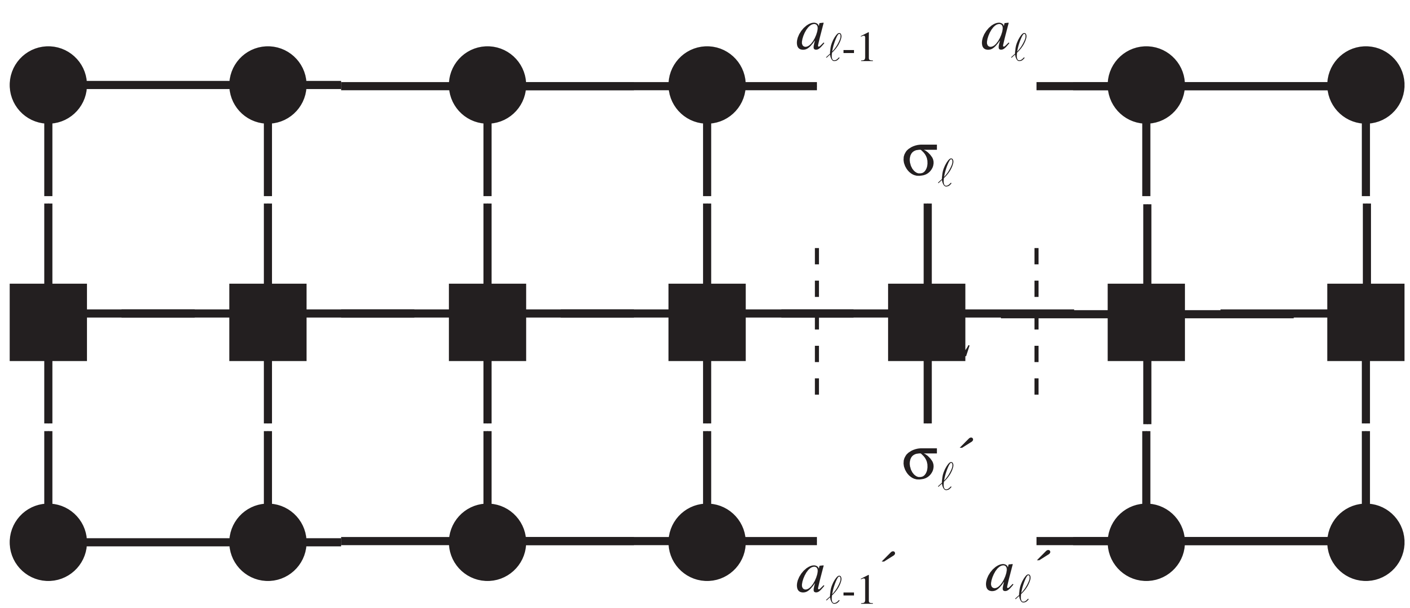
- Matrix product operator for the Hamiltonian (bulk)

$$W^{[n]} = \begin{pmatrix} 1_{2\times 2} & -\frac{1}{2}S^{+} & -\frac{1}{2}S^{-} & 2\lambda S^{z} & \Delta S^{z} & \beta_n S^{z} + \alpha 1_{2\times 2} \\ 0 & 0 & 0 & 0 & 0 & S^{-} \\ 0 & 0 & 0 & 0 & 0 & S^{+} \\ 0 & 0 & 0 & 1 & 0 & S^{z} \\ 0 & 0 & 0 & 0 & 0 & S^{z} \\ 0 & 0 & 0 & 0 & 0 & 1_{2\times 2} \end{pmatrix}$$

$$\beta_n = \Delta + (-1)^n \tilde{m}_0 a - 2\lambda S_{\text{target}} \,, \; \alpha = \lambda \left(\frac{1}{4} + \frac{S_{\text{target}}^2}{N} \right) + \frac{\Delta}{4}$$

- Simulation parameters
  - ★ Twenty values of $\Delta(g)$, ranging from -0.9 to 1.0
  - ★ Fourteen values of $\tilde{m}_0 a$, ranging from 0 to 0.4
  - ★ Bond dimension $D = 50, 100, 200, 300, 400, 500, 600$
  - ★ System size $N = 400, 600, 800, 1000$

Practice of MPS for DMRG

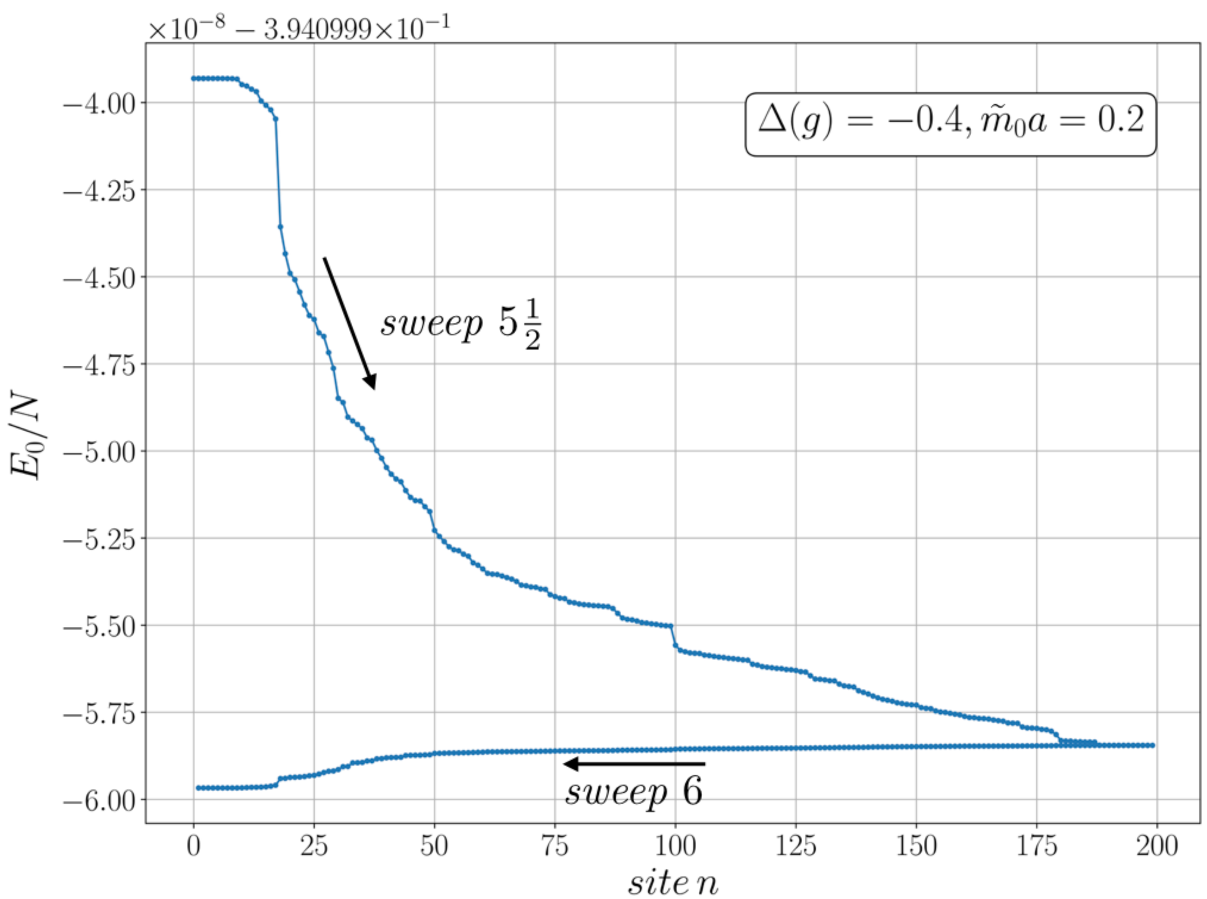


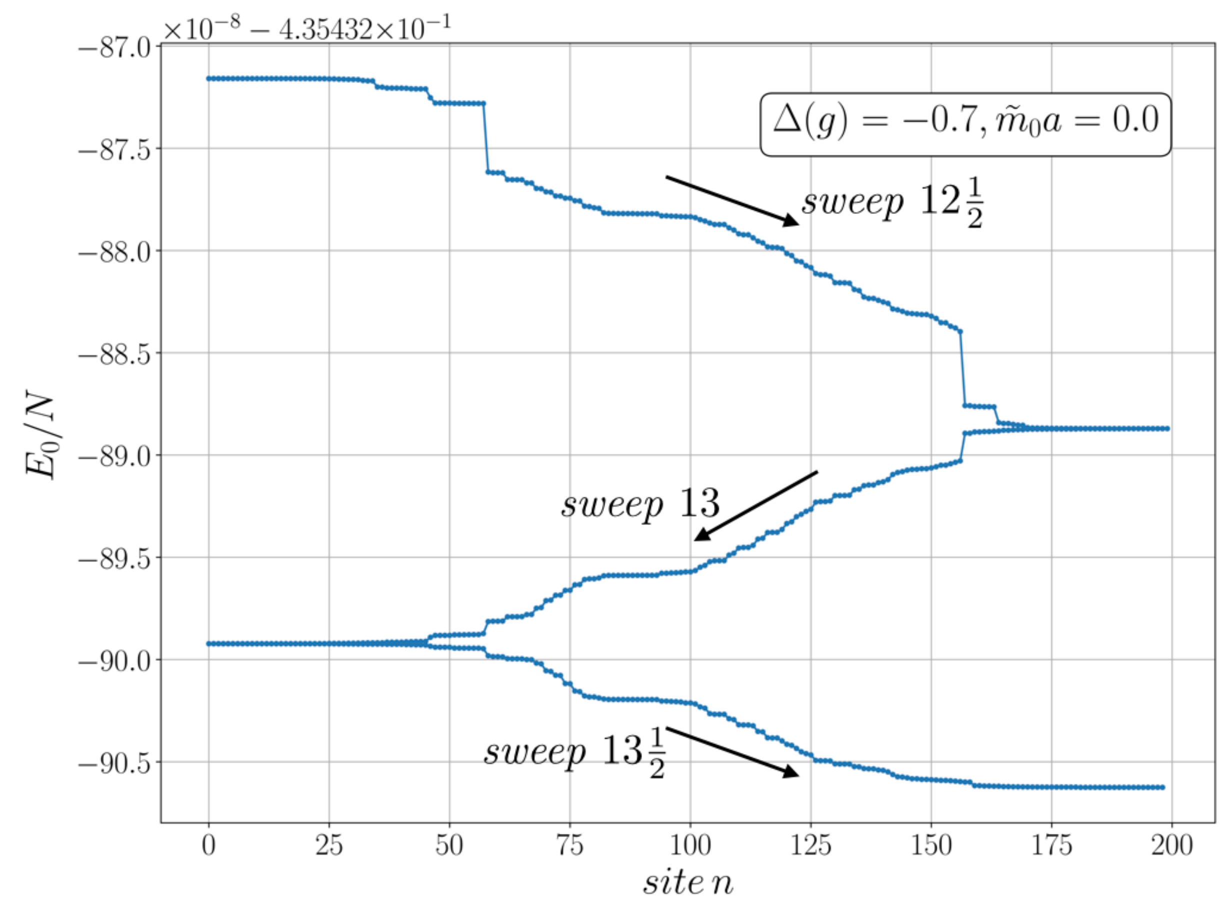
One step in a sweep of finite-size DMRG

Simulations and numerical results

Convergence of DMRG

- Start from random tensors at D=50, then go up in D
- DMRG converges fast at $\tilde{m}_0 a \neq 0$ and $\Delta(g) \gtrsim -0.7$

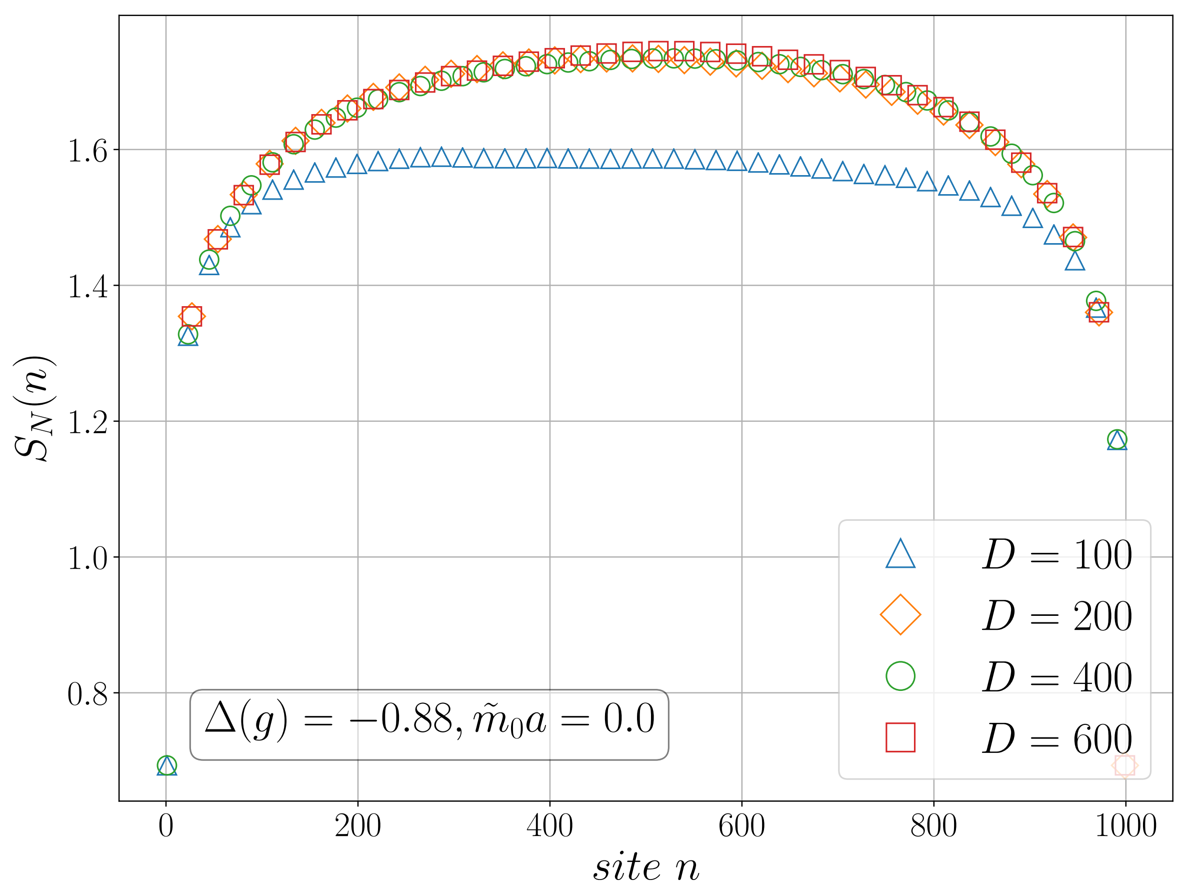


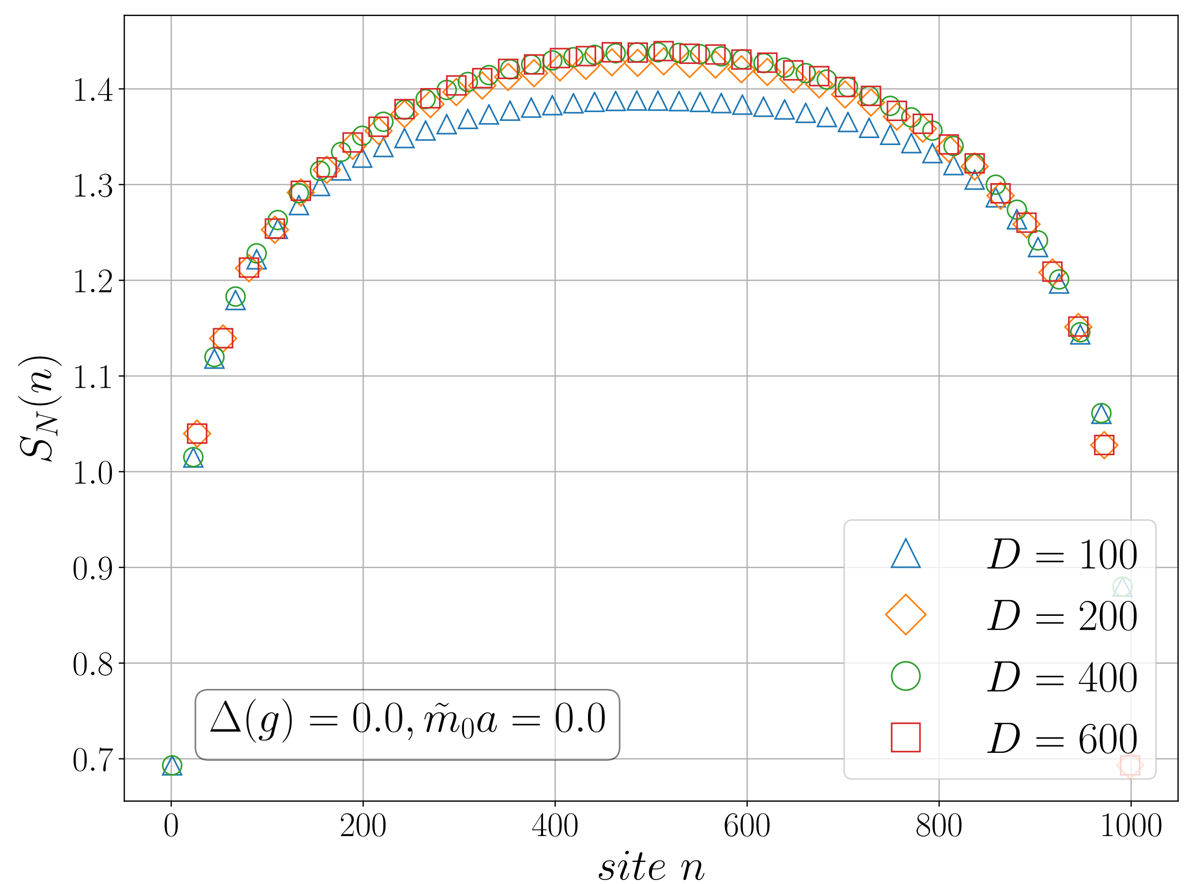


Entanglement entropy

Calabrese-Cardy scaling and the central charge

$$S_N(n) = \frac{c}{6}\ln\left[\frac{N}{\pi}\sin\left(\frac{\pi n}{N}\right)\right] + k$$





★ Calabrese-Cardy scaling observed at all values of $\Delta(g)$ for $\tilde{m}_0 a = 0$

Entanglement entropy

Calabrese-Cardy scaling and the central charge

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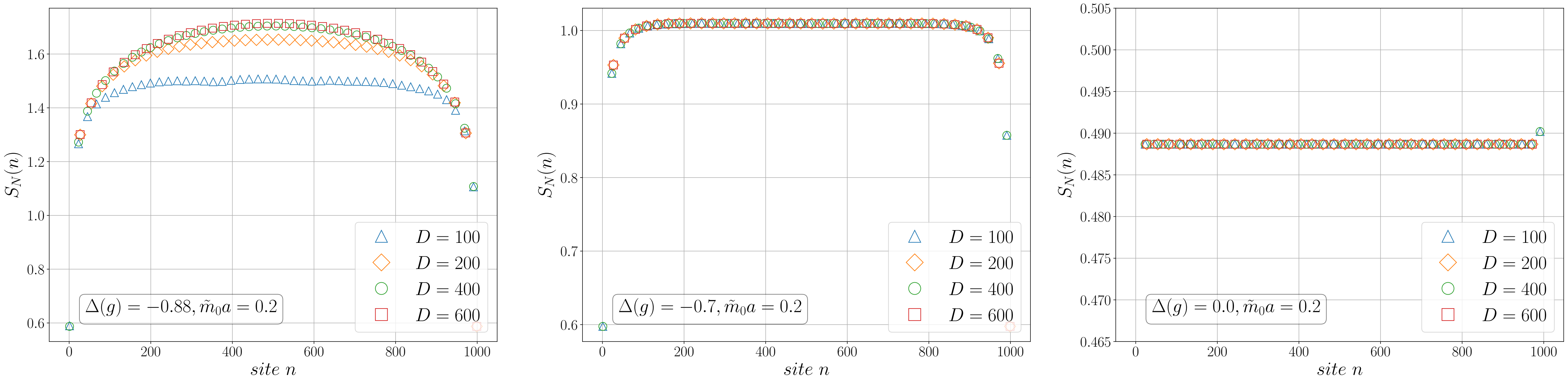
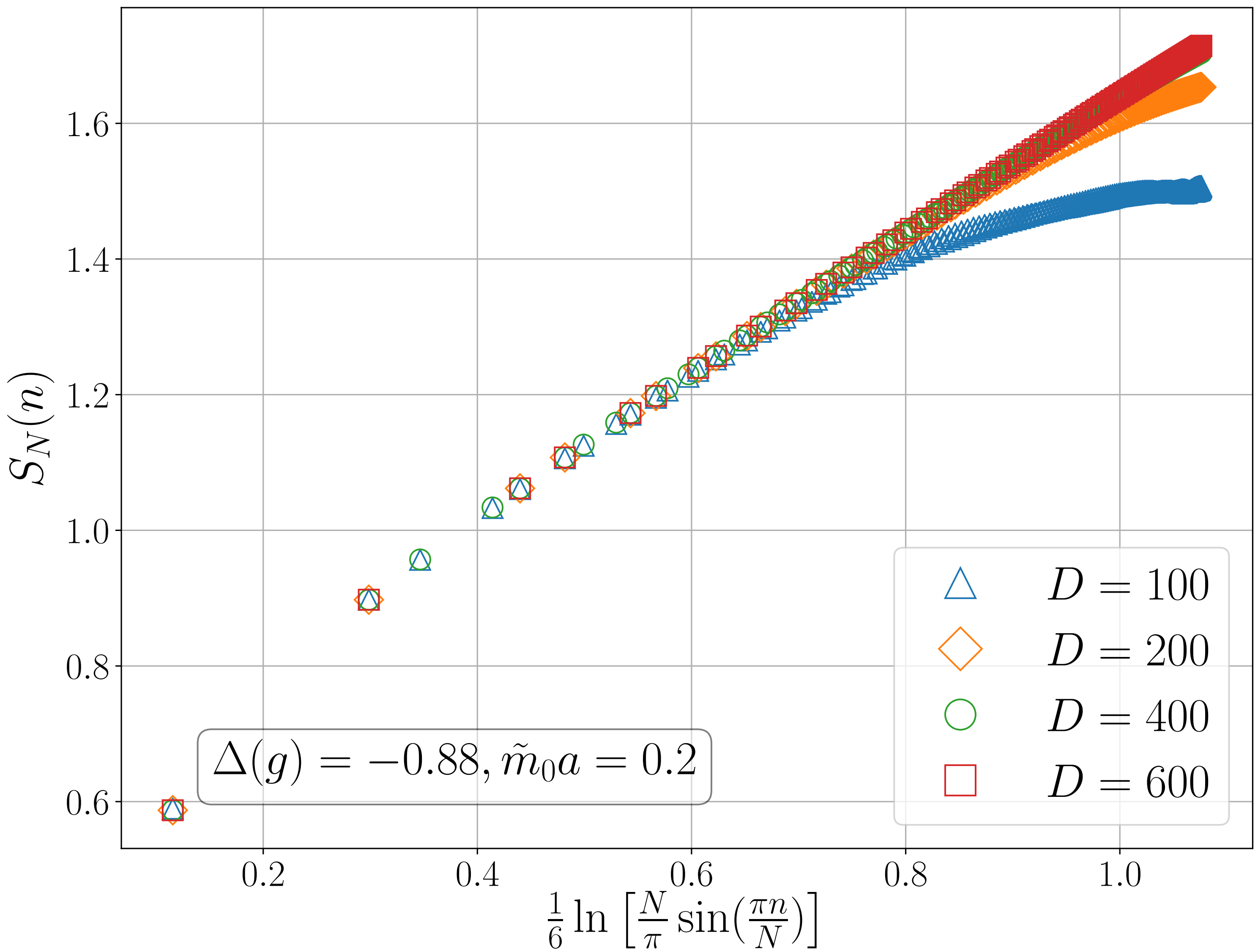


★ Calabrese-Cardy scaling observed at $\Delta(g) \lesssim -0.7$ for $\tilde{m}_0 a \neq 0$

Entanglement entropy

Calabrese-Cardy scaling and the central charge

$$S_N(n) = \frac{c}{6}\,\ln\left[\frac{N}{\pi}\sin\left(\frac{\pi n}{N}\right)\right] + k$$



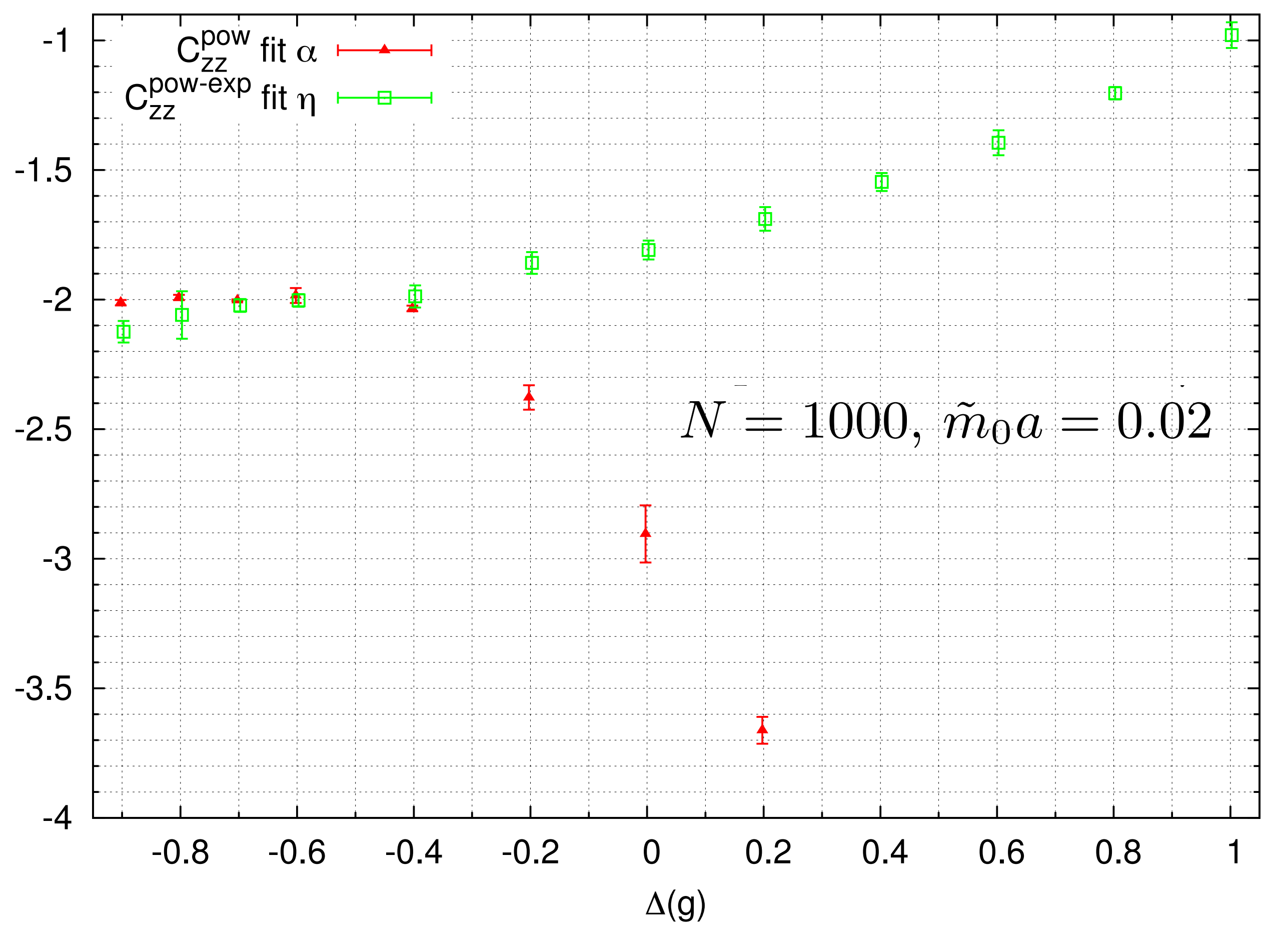
★ Central charge is unity in the critical phase

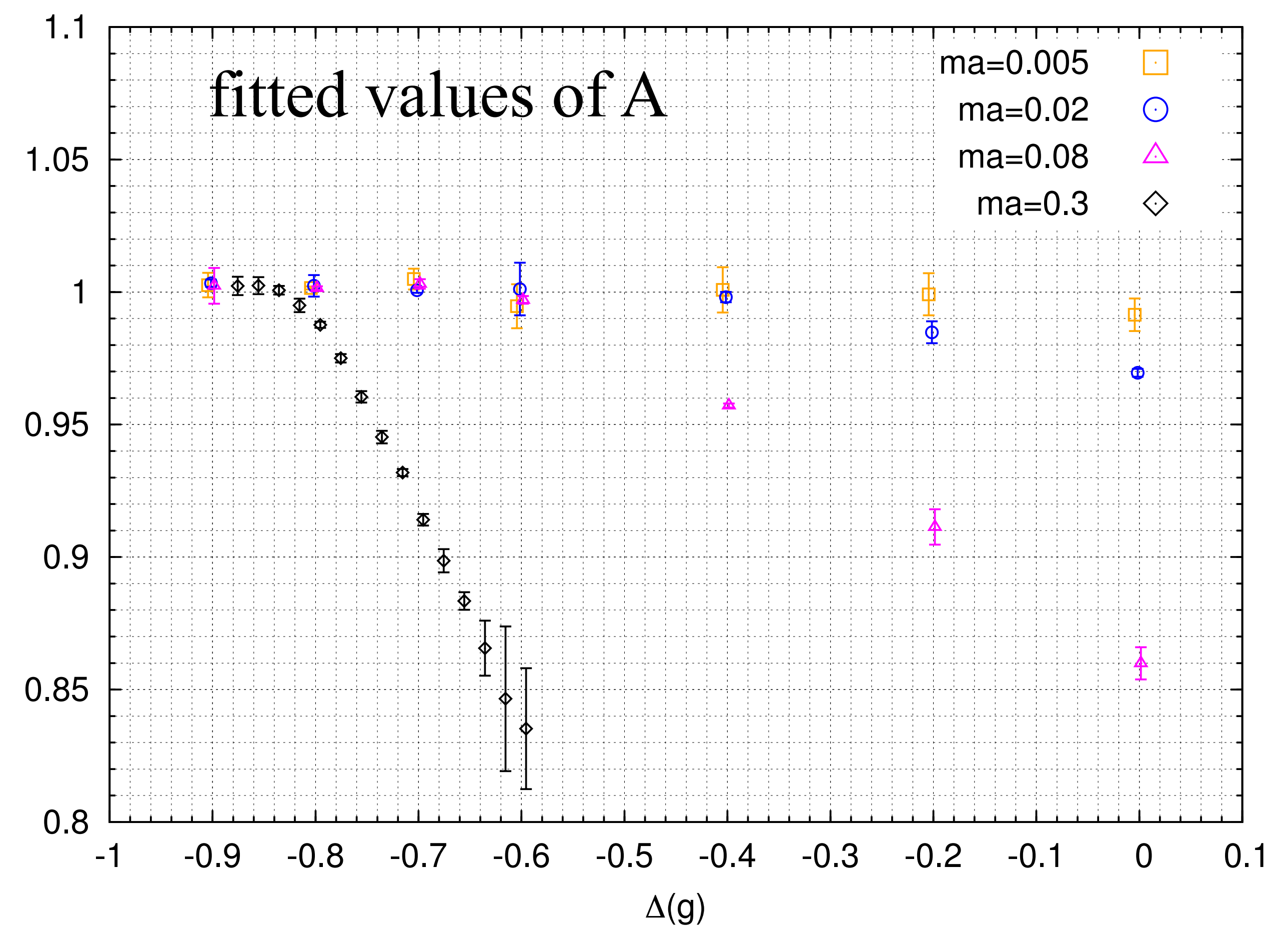
Density-density correlators

$$C_{zz}(x) = \langle\bar{\psi}\psi(x_0+x)\bar{\psi}\psi(x_0)\rangle_{\rm conn} \xrightarrow{\rm JW\ trans} \frac{1}{N_x}\sum_n S^z(n)S^z(n+x) - \frac{1}{N_0}\sum_n S^z(n)\sum_n S^z(n+1)$$

try fitting to

$$C_{zz}^{\rm pow}(x) = \beta x^\alpha \text{ and } C_{zz}^{\rm pow-exp}(x) = Bx^\eta A^x$$





★ Evidence for a critical phase

Soliton correlators

S. Mandelstam, 1975; E. Witten, 1978

$$\psi_\alpha^\dagger(x)\psi_\alpha(y) = \mp i|2\pi(x-y)|^{-1}|c\mu(x-y)|^{-\beta^2 g^2/(2\pi)^3}$$

$$\times : \exp\left\{-2\pi i\beta^{-1}\int_x^y d\xi\,\dot{\phi}(\xi) \mp \frac{1}{2}i\beta\left[\phi(y)-\phi(x)\right] + O(x-y)^2\right\} :$$

$\alpha = \pm$

Soliton operators

connecting vortex and anti-vortex

★ Power-law in the critical phase

★ Exponential-law in the gapped phase

Vertex operators

Power-law

Jordan-Wigner transformation

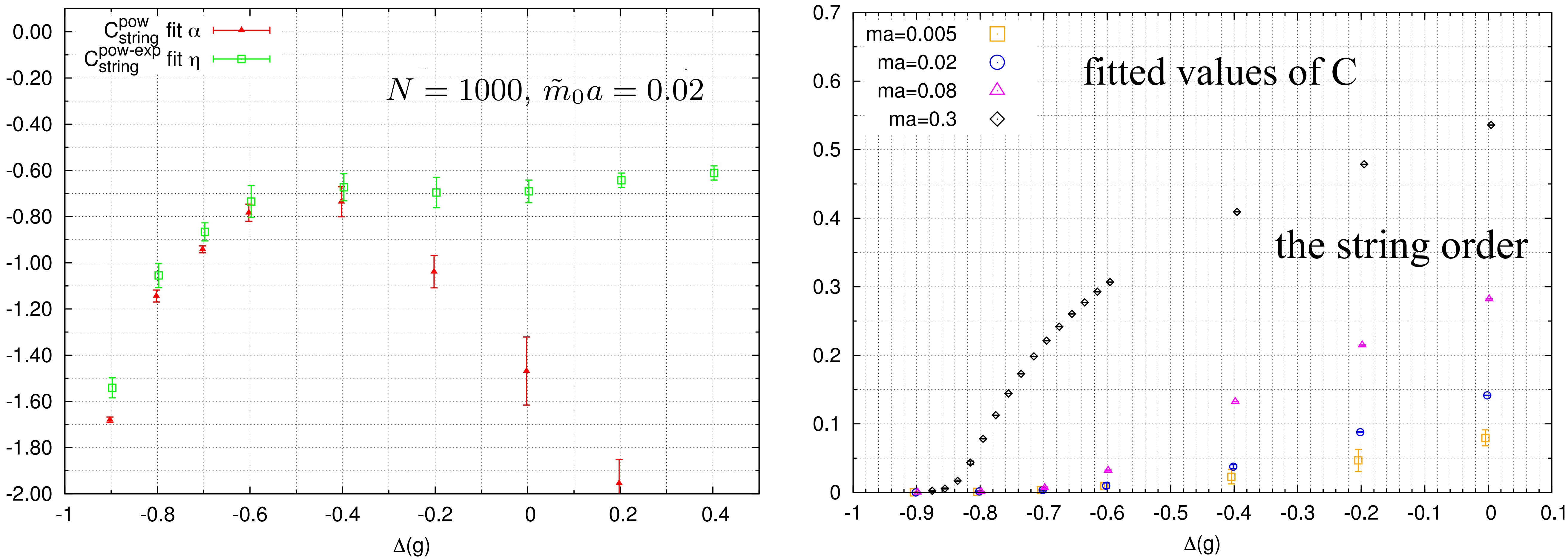
$$S_m^+ e^{i\pi\sum_{j=m+1}^{n-1} S_j^z} S_n^-$$

Soliton (string) correlators

$$C_{\text{string}}(x) = \langle \psi^\dagger(x_0+x)\psi(x_0)\rangle \xrightarrow{\text{JW trans}} \frac{1}{N_x}\sum_n S^+(n)S^z(n+1)\cdots S^z(n+x-1)S^-(n+x)$$

try fitting to

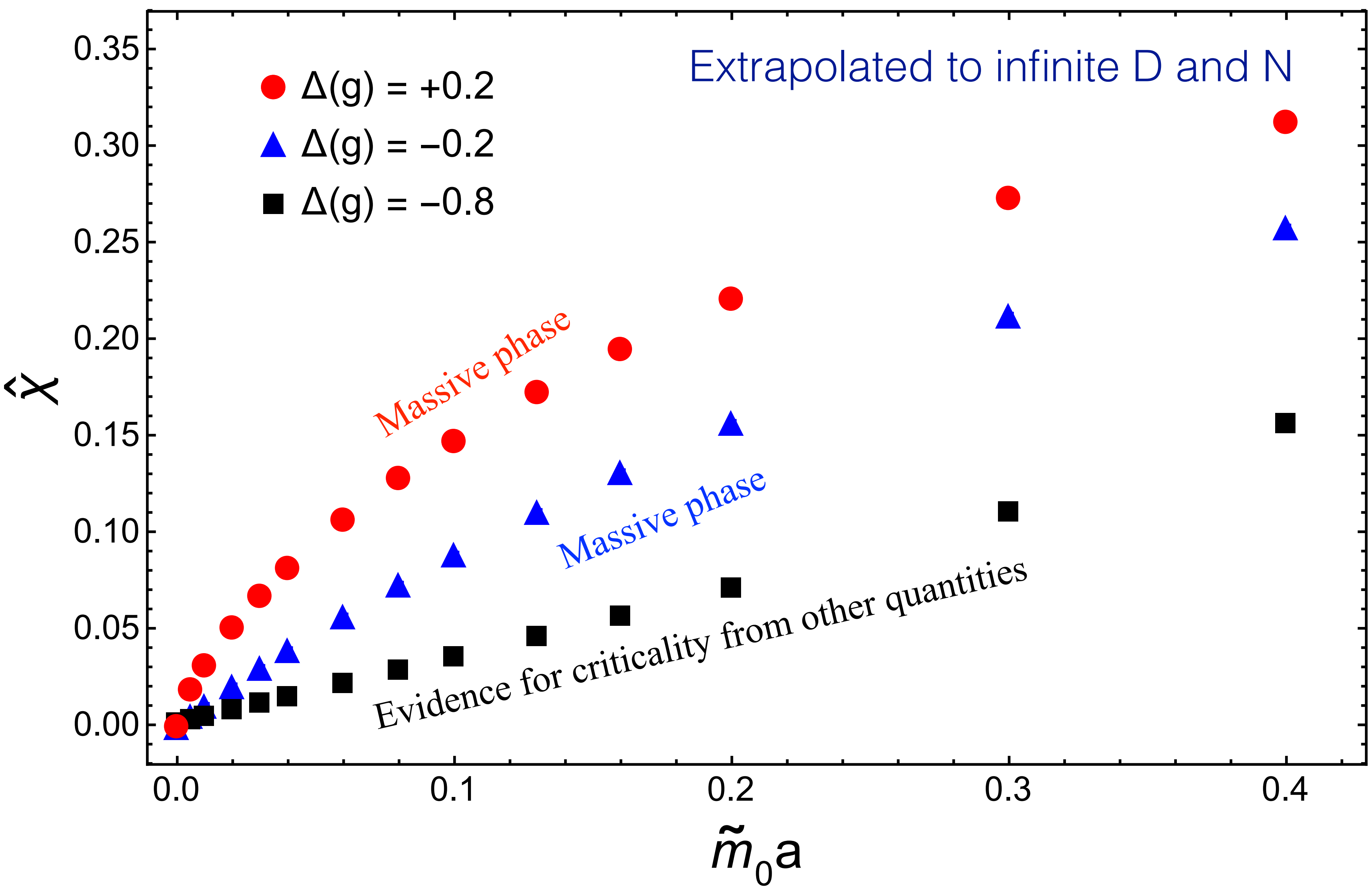
$$C^{\text{pow}}_{\text{string}}(x) = \beta x^\alpha + C \quad \text{and} \quad C^{\text{pow-exp}}_{\text{string}}(x) = B x^\eta A^x + C$$



★ Similar behaviour in A. Evidence for a critical phase

Chiral condensate

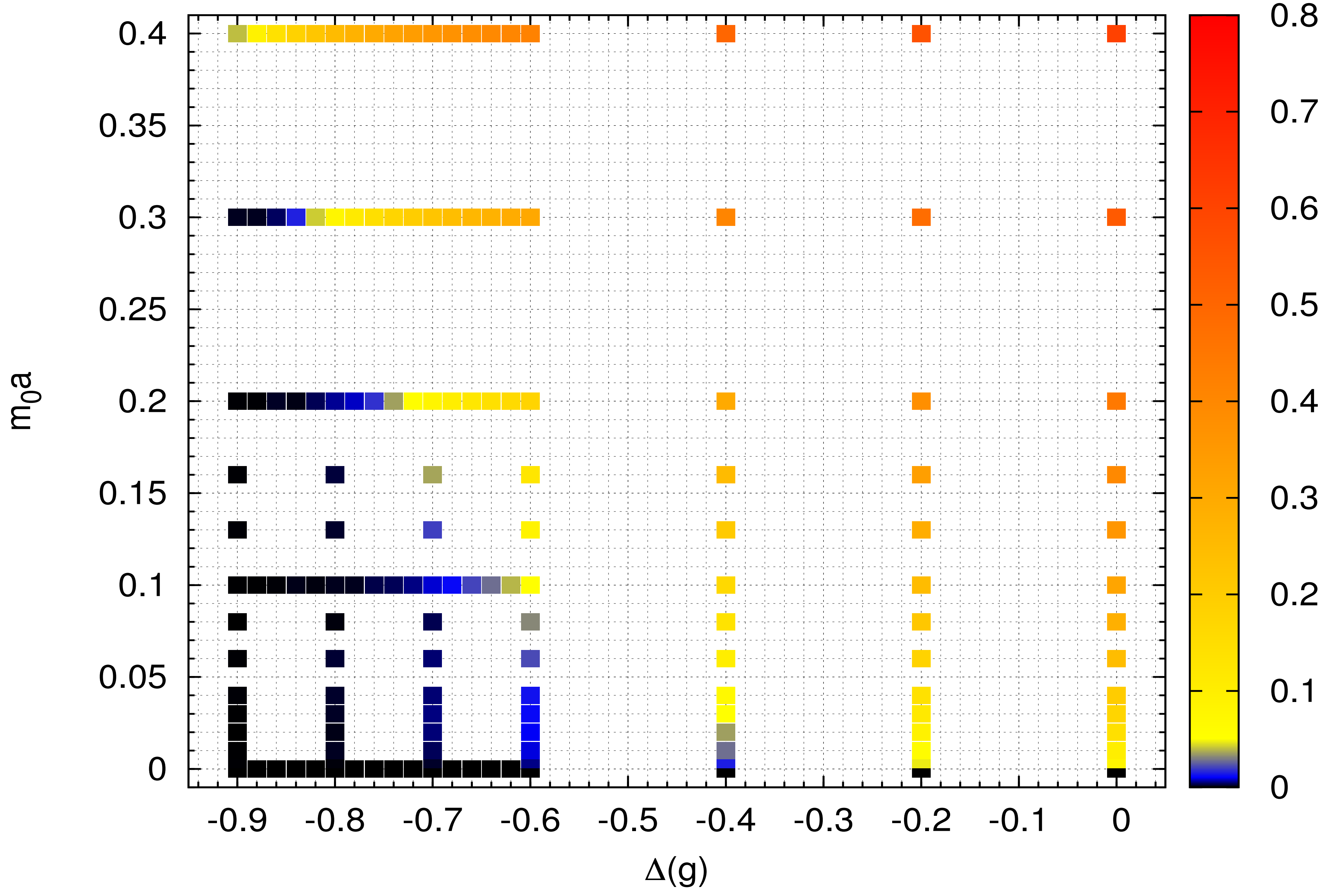
$$\hat{\chi} = a\left|\langle\bar{\psi}\psi\rangle\right| = \frac{1}{N}\left|\sum_n (-1)^n S_n^z\right|$$



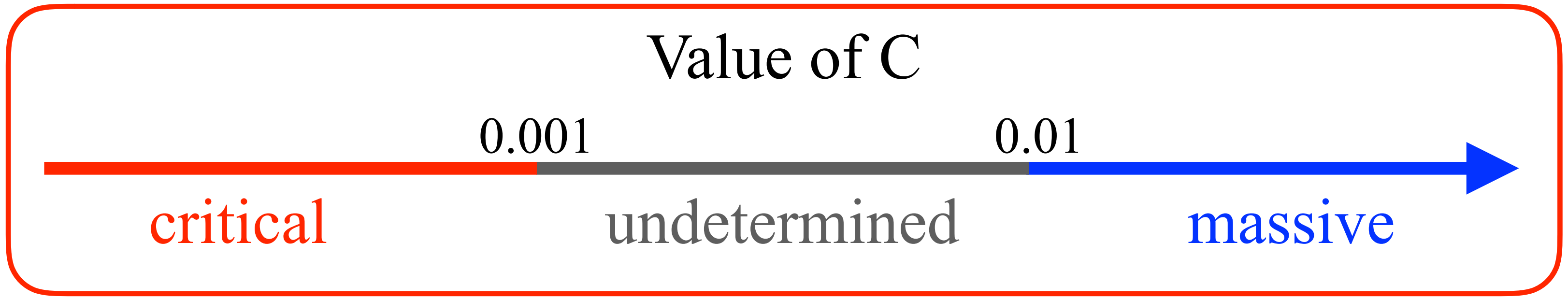
★ Chiral condensate is not an order parameter

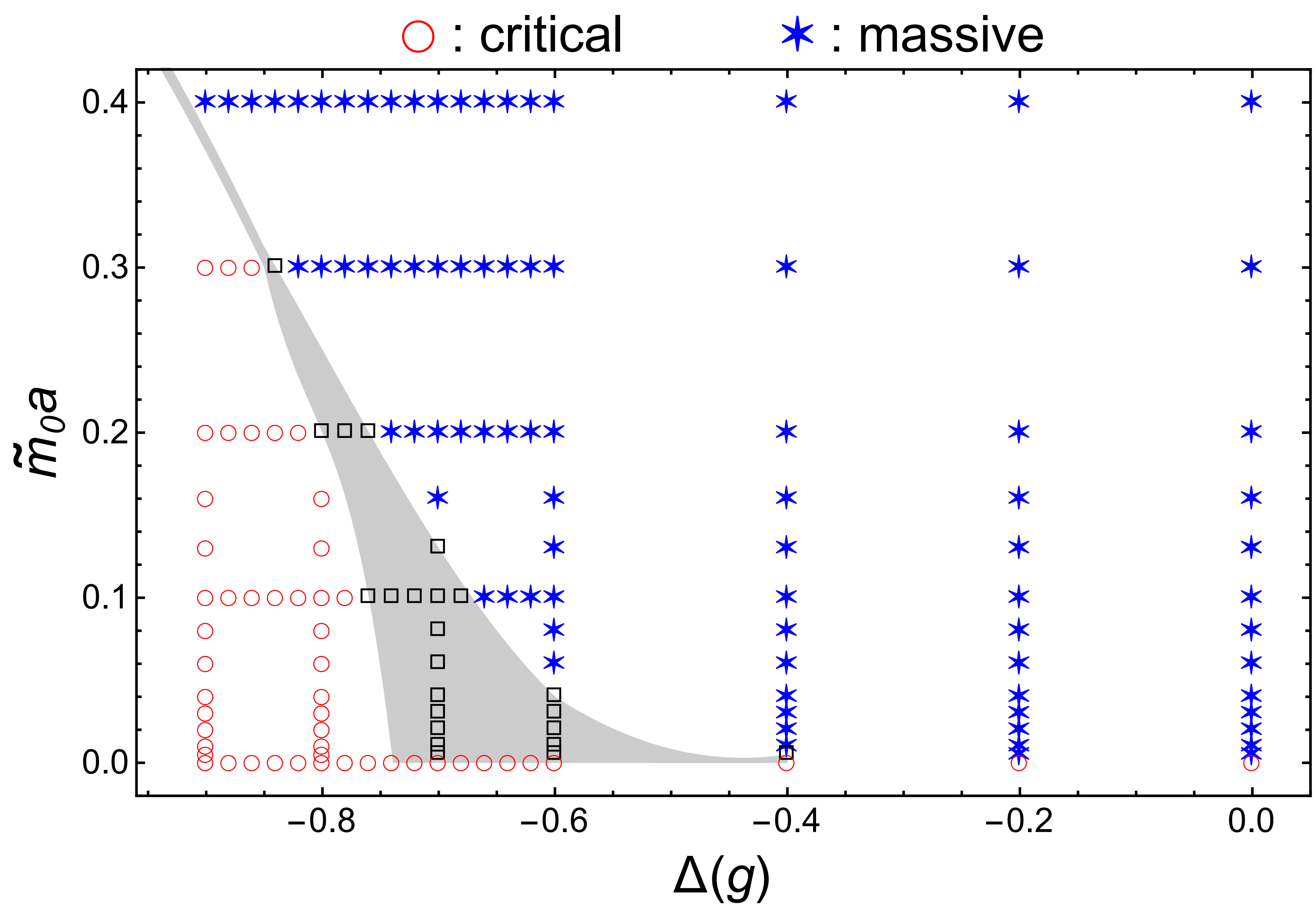
Probing the phase structure

$$C_{\text{string}}^{\text{pow}-\text{exp}}(x) = Bx^{\eta}A^{x} + C$$



Results for the phase structure





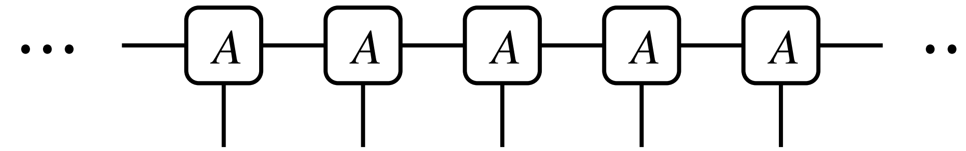
Conclusion and outlook

- Concluding results for phase structure
  - ★ KT-type transition observed using the MPS
- Current and future work
  - ★ Excited-state spectrum and the continuum limit
    - → Exploratory spectrum results presented at Lattice 2017
  - ★ Real-time dynamics and dynamical phase transition
    - → Exploratory results presented at Lattice 2019

Backup slides

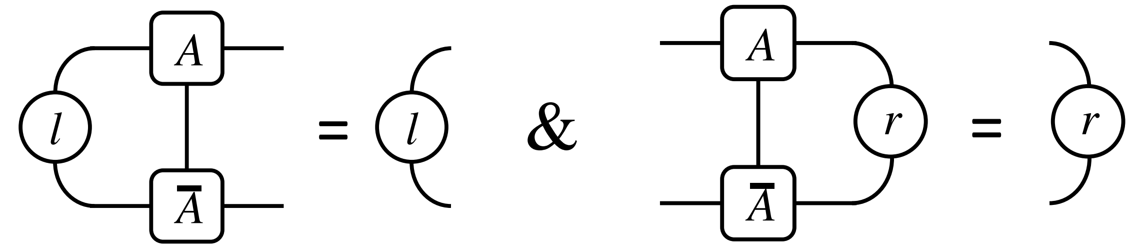
Uniform MPS and real-time evolution

★ Translational invariance in MPS



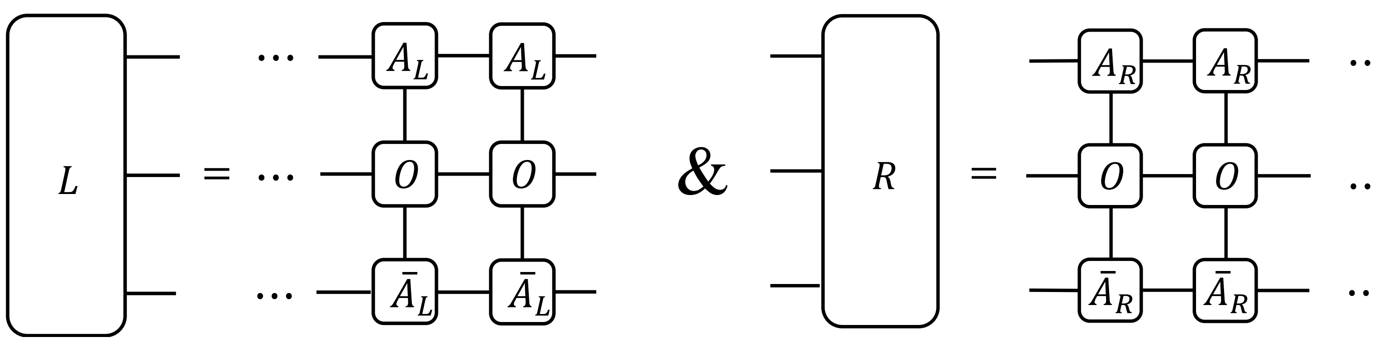
★ Finding the infinite BC for amplitudes

(largest eigenvalue normalised to be 1)



H.N. Phien, G. Vidal and I.P. McCulloch, Phys. Rev. B86, 2012

★ Similar (more complicated) procedure in the variation search for the ground state



V. Zauner-Stauber *et al*, Phys. Rev. B97, 2018

★ Real-time evolution *via* time-dependent variational principle

➤ Key: projection to MPS in $i\frac{d}{dt}|\Psi(A(t))\rangle = P_{|\Psi(A)\rangle}\hat{H}|\Psi(A(t))\rangle$

J. Haegeman *et al*, Phys. Rev. Lett.107, 2011

Dynamical quantum phase transition

★ "Quenching" : Sudden change of coupling strength in time evolution

$$H(g_1)|0_1\rangle = E_0^{(1)}|0_1\rangle \quad \text{and} \quad |\psi(t)\rangle = \mathrm{e}^{-iH(g_2)t}|0_1\rangle$$

★ Questions: Any singular behaviour?   Related to equilibrium PT?

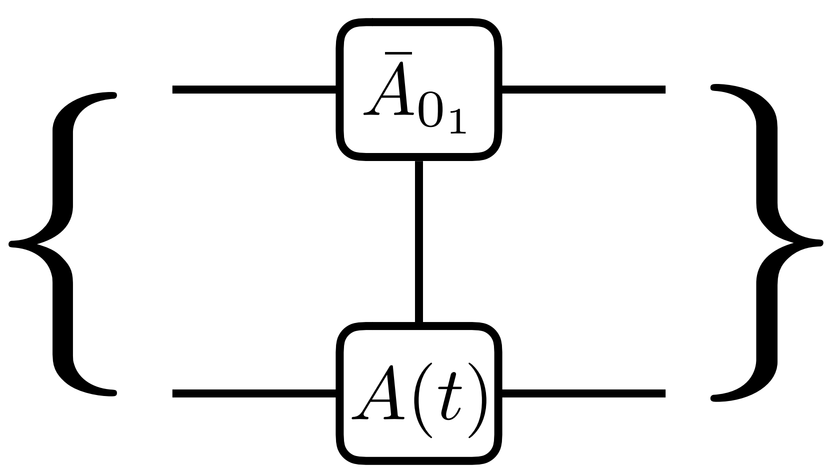
★ The Loschmidt echo and the return rate

$$L(t) = \langle 0_1|\mathrm{e}^{-iH(g_2)t}|0_1\rangle \quad \& \quad g(t) = -\lim_{N\to\infty}\frac{1}{N}\ln L(t)$$

➤ *c.f.*, the partition function and the free energy

➤ In uMPS computed from the largest eigenvalue of the "transfer matrix"

$T_{i,j}(t) = i$



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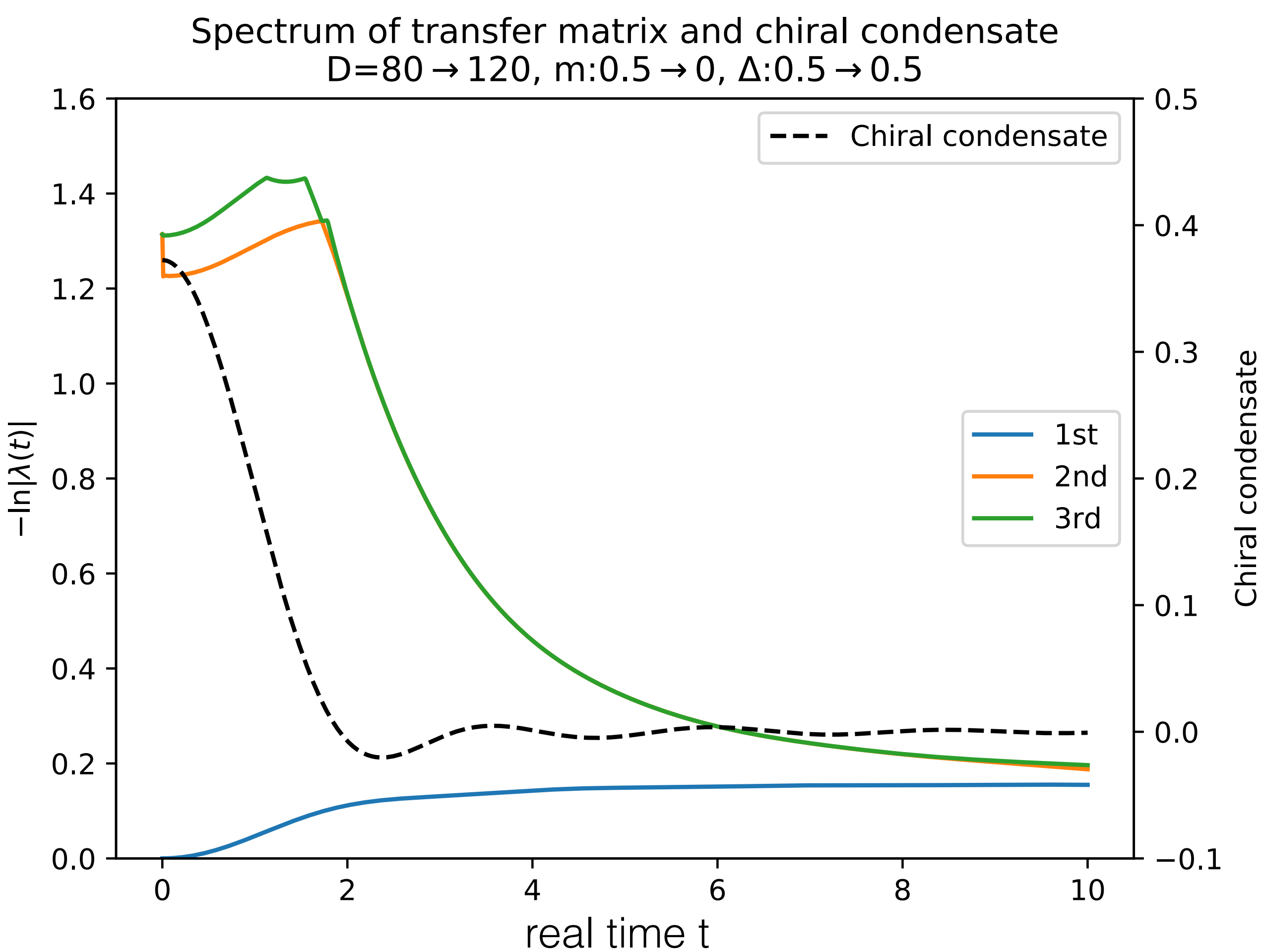
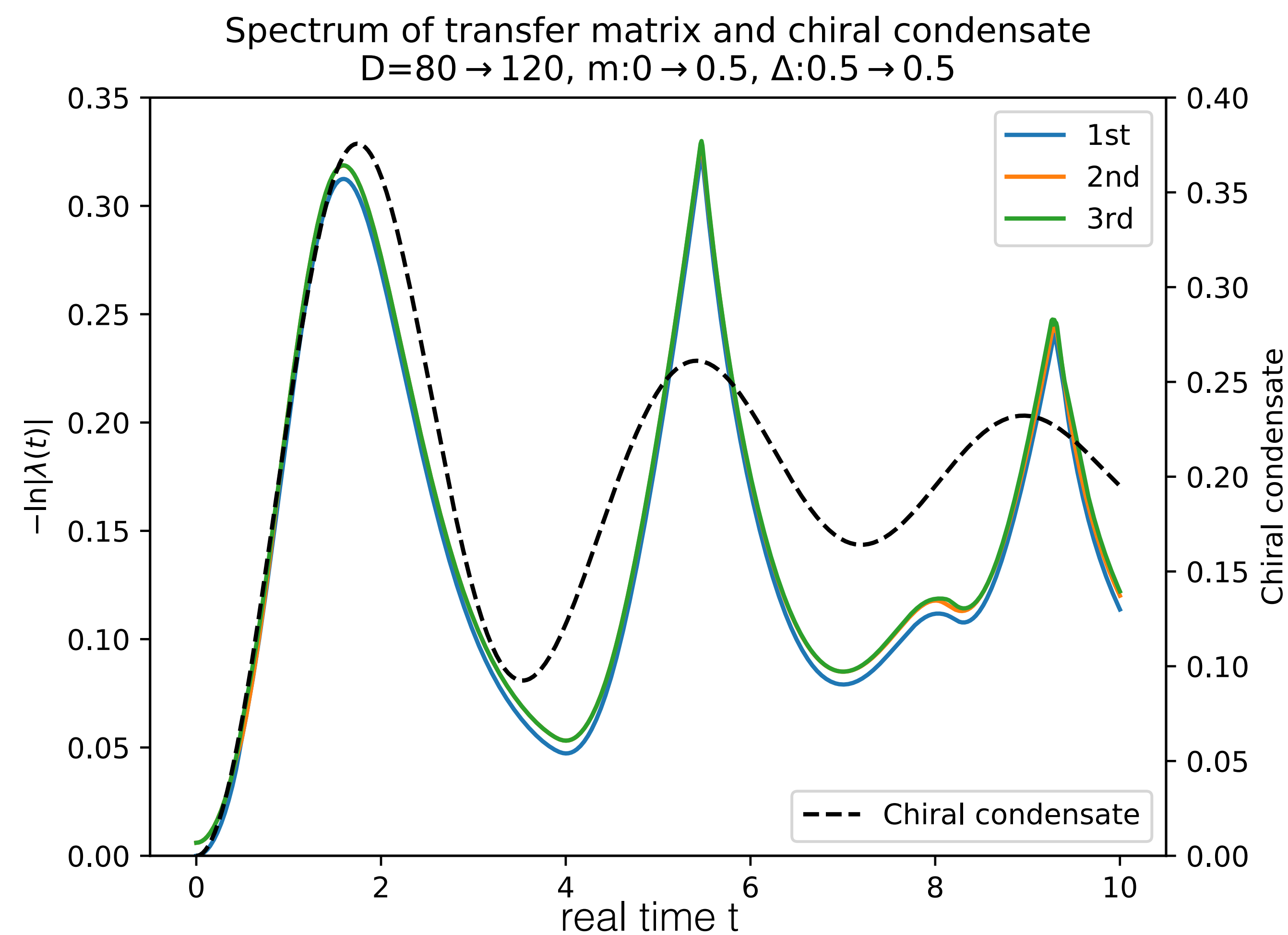
Observing DQPT



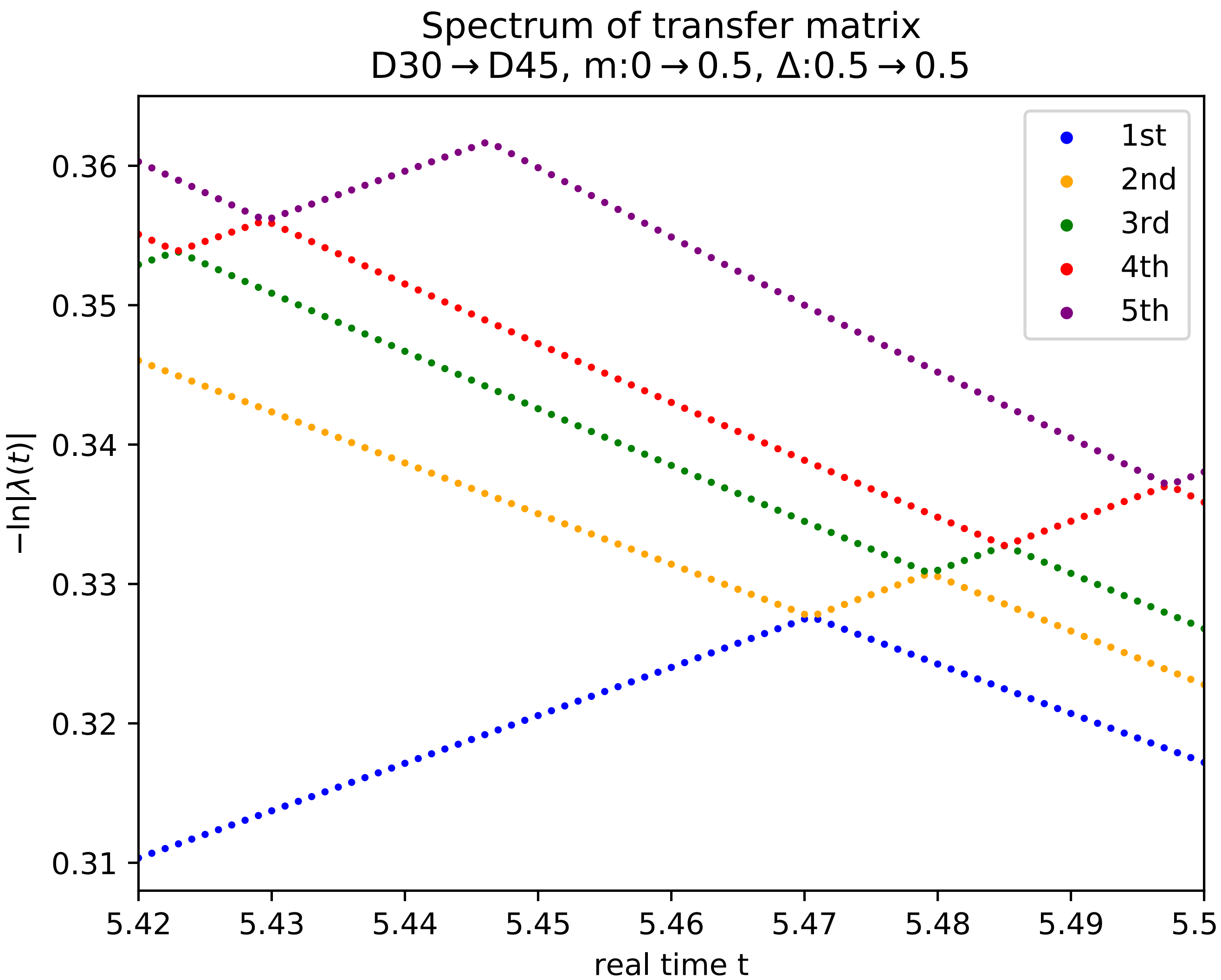
massive → critical

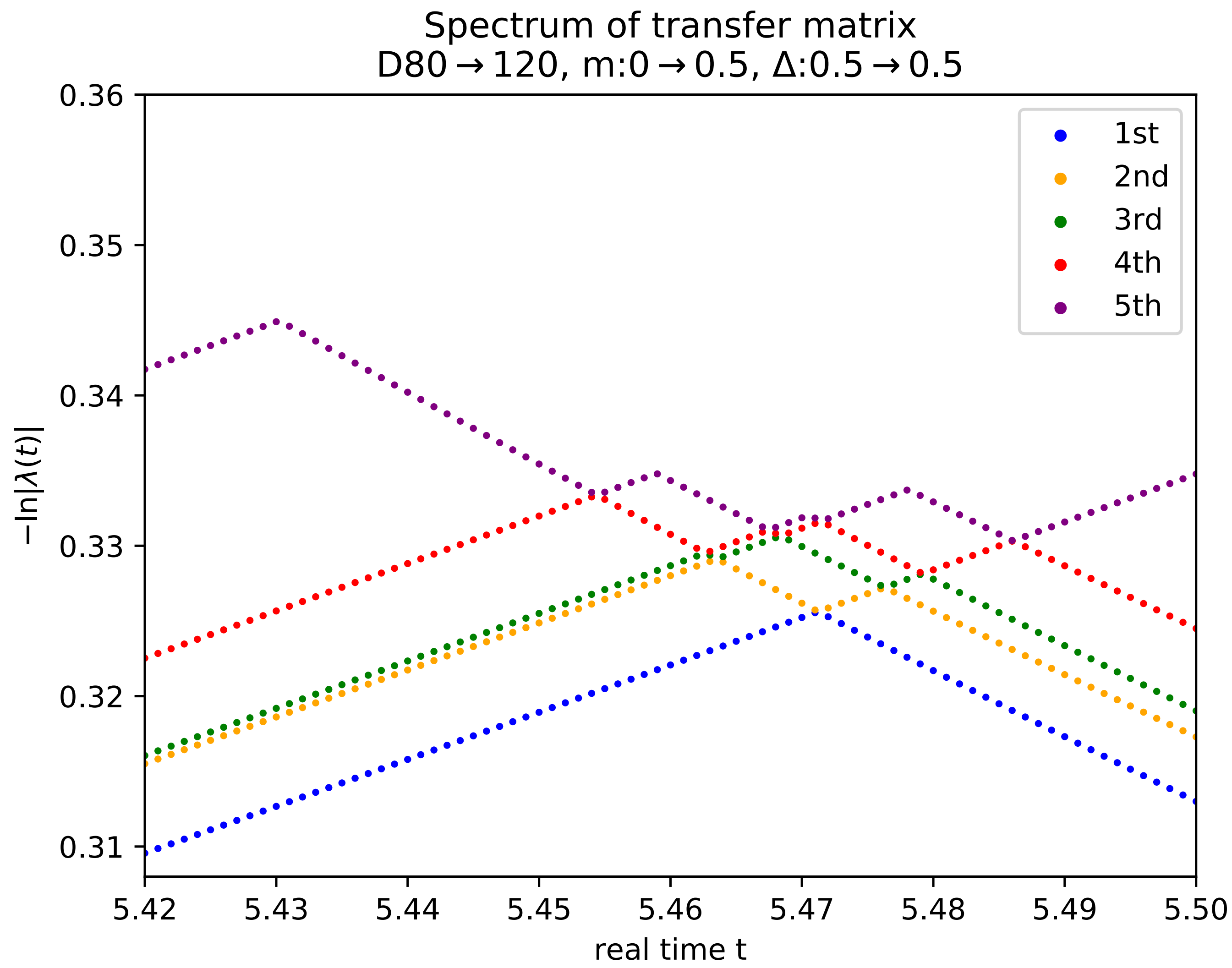


critical → massive

★ DQPT is a "one-way" transition…

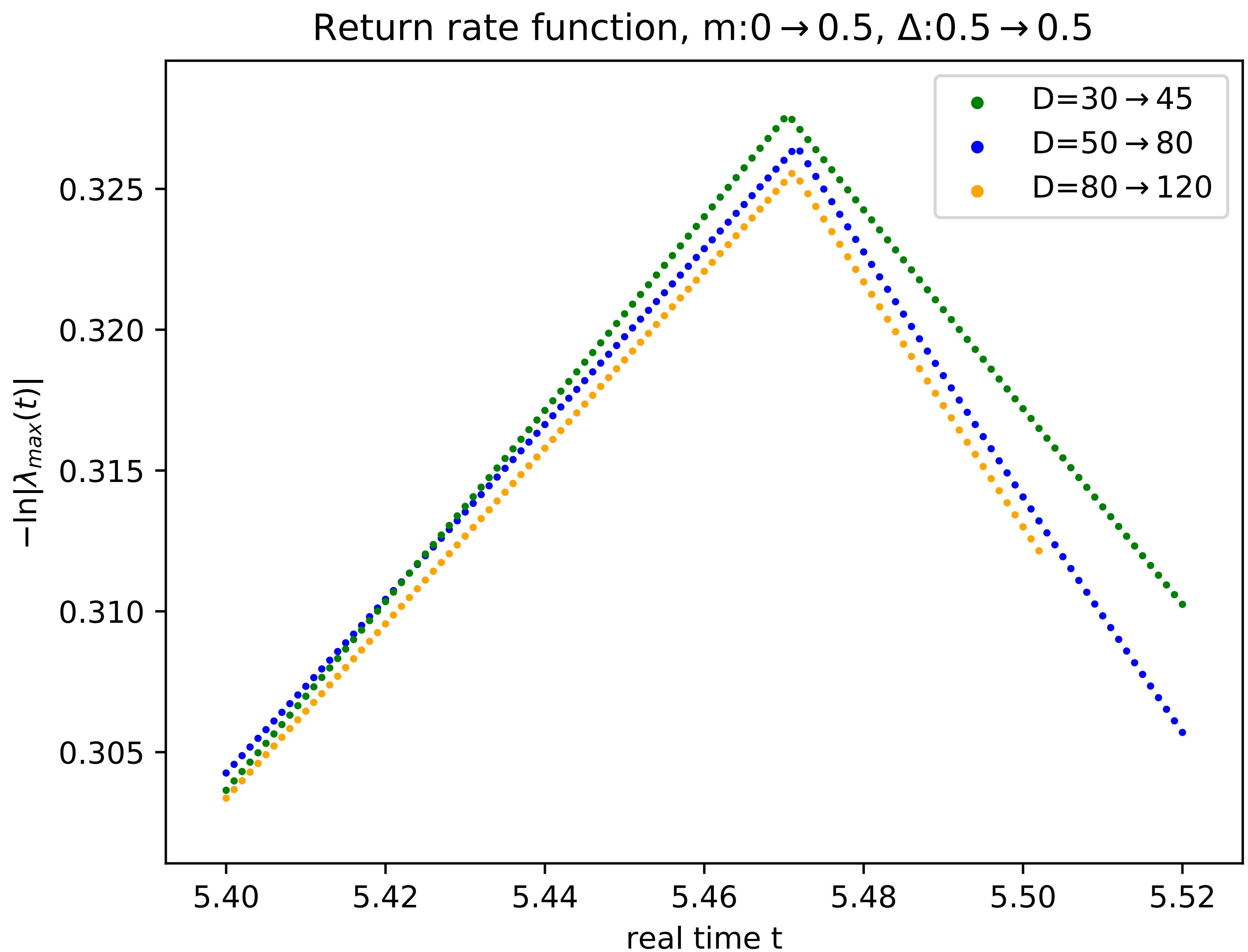
DQPT and eigenvalue crossing





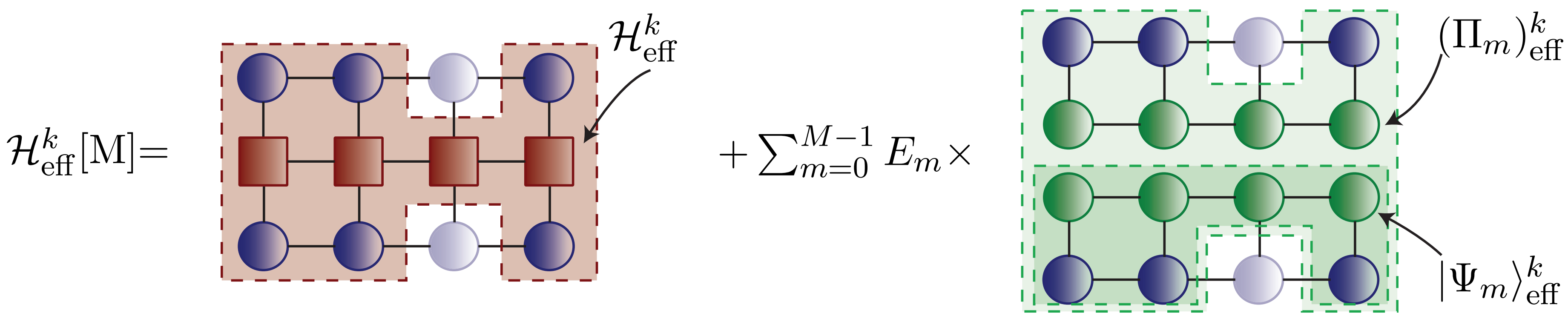
★ D-dependence in the crossing points

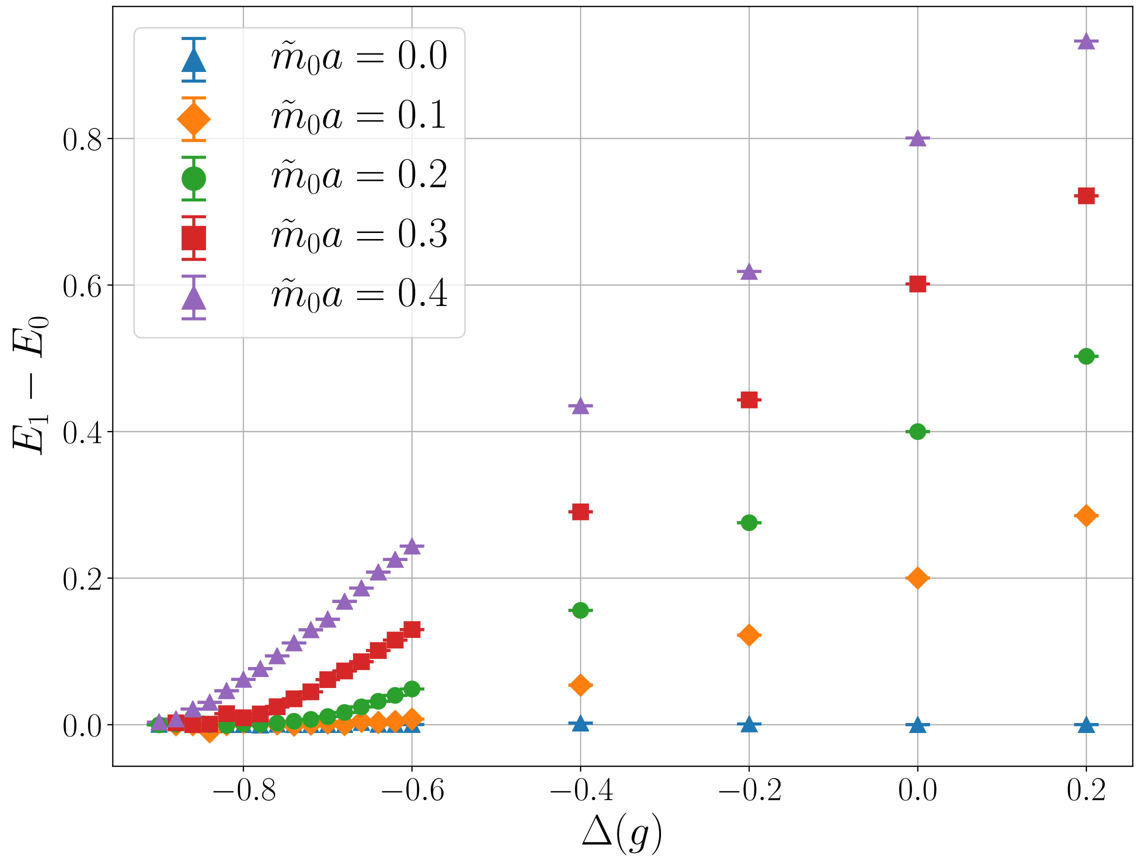
“Universality” in DQPT?



Mass gap

$$H_{\text{eff}}[M] = \Pi_{M-1}\ldots\Pi_0 H \Pi_0 \ldots \Pi_{M-1} = H - \sum_{k=0}^{M-1} E_k |\Psi_k\rangle\langle\Psi_k|$$





The Jordan-Wigner transformation

- The fermion fields satisfy

$$\{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0, \;\; \{c_n, c_m^\dagger\} = \delta_{n,m} \;\; .$$

- The Jordan-Wigner transformation

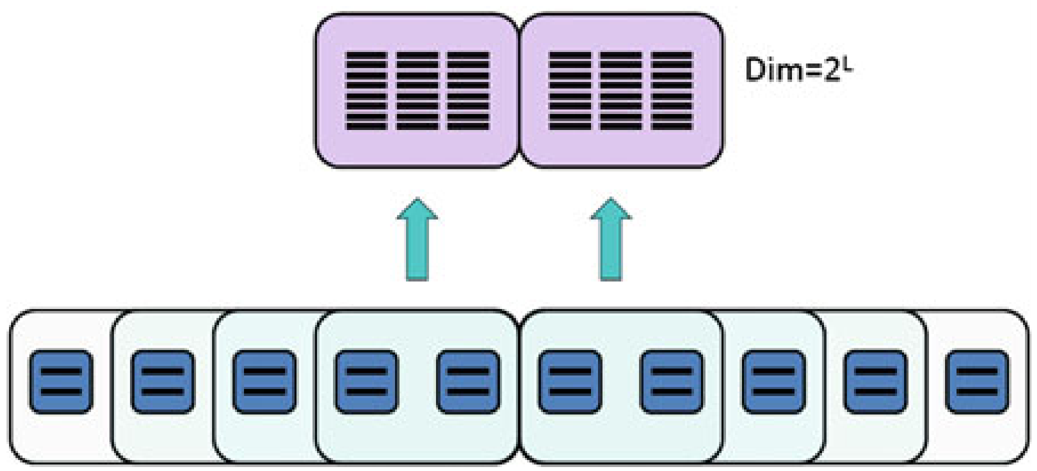
$$c_n = \exp\left(i\pi \sum_{j=1}^{n-1} S_j^z\right) S_n^-, \;\; c_n^\dagger = S_n^+ \exp\left(-i\pi \sum_{j=1}^{n-1} S_j^z\right)$$

expresses the the fermions fields in spins,

$$S_j^\pm = S_j^x \pm i S_j^y, \;\; \left[S_i^a, S_j^b\right] = i\delta_{i,j}\epsilon^{abc} S_i^c \;\; .$$

The singular value decomposition

$$|\Psi\rangle = \sum_{i=1}^{D_A} \sum_{j=1}^{D_B} \Psi_{i,j} |i\rangle \otimes |j\rangle$$



$\Psi_{i,j}$ can be regarded as elements of a $D_A \times D_B$ (assuming $(D_A \geq D_B)$ matrix.

SVD

$$\Psi_{i,j} = \sum_{\alpha}^{D_B} U_{i,\alpha} \lambda_\alpha \left(V^\dagger\right)_{\alpha,j}$$

$$U^\dagger U = 1, \; V V^\dagger = 1$$

Discard small singular values

$$\Psi_{i,j} = \sum_{\alpha}^{D'_B < D_B} U_{i,\alpha} \lambda_\alpha \left(V^\dagger\right)_{\alpha,j}$$

Schmidt decomposition and entanglement

$$|\Psi\rangle = \sum_{i=1}^{D_A}\sum_{j=1}^{D_B}\sum_{\alpha}^{D_B} U_{i,\alpha}\lambda_\alpha V^*_{\alpha,j}|i\rangle \otimes |j\rangle = \sum_{\alpha}^{D_B}\lambda_\alpha \left(\sum_{i=1}^{D_A} U_{i,\alpha}|i\rangle\right) \otimes \left(\sum_{j}^{D_B} V^*_{\alpha,j}|j\rangle\right) = \sum_{\alpha}^{D_B}\lambda_\alpha|\alpha\rangle_A \otimes |\alpha\rangle_B$$

Reduced density matrices

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = \sum_\alpha \lambda_\alpha^2|\alpha\rangle_A\ {}_A\langle\alpha|\ ,\quad \rho_B = \mathrm{Tr}_A|\Psi\rangle\langle\Psi| = \sum_\alpha \lambda_\alpha^2|\alpha\rangle_B\ {}_B\langle\alpha|$$

von Neumann entanglement entropy

$$S = -\mathrm{Tr}\left[\rho_A\ \log\left(\rho_A\right)\right] = -\mathrm{Tr}\left[\rho_B\ \log\left(\rho_B\right)\right] = -\sum_\alpha \lambda_\alpha^2 \log \lambda_\alpha^2$$

★ Truncating the Hilbert space by omitting small singular values

→ Throwing away small-entanglement states