Introduction of fintie element method

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$\Omega \subset \mathbb{R}^2, \partial\Omega = \Gamma_D \cup \Gamma_N$

$$-\triangle u = f \text{ in } \Omega,$$
$$u = g \text{ on } \Gamma_D,$$
$$\frac{\partial u}{\partial n} = \nabla u \cdot n = h \text{ on } \Gamma_N.$$

weak formulation
V : function space, $V(g) = \{u \in V \,;\, u = g \text{ on } \Gamma_D\}$. $V = C^1(\Omega) \cap C^0(\bar{\Omega})$?
Find $u \in V(g)$ s.t.

$$\int_\Omega -\triangle u \, v dx = \int_\Omega f \, v dx \quad \forall v \in V(0)$$

Gauss-Green's formula
$u, v \in V$, $n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$: outer normal to $\partial\Omega$

$$\int_\Omega (\partial_i u) v \, dx = -\int_\Omega u \partial_i v \, dx + \int_{\partial\Omega} u \, n_i v \, ds \,.$$

Poisson equation with mixed B.C. and a weak formulation: 2/2

$$\int_\Omega (-\partial_1^2 - \partial_2^2) u\, v\, dx = \int_\Omega (\partial_1 u \partial_1 v + \partial_2 u \partial_2 v)\, dx - \int_{\partial\Omega} (\partial_1 u\, n_1 + \partial_2 u\, n_2) v\, ds$$

$$= \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\Gamma_D \cup \Gamma_N} \nabla u \cdot n\, v\, ds$$

$$v = 0 \text{ on } \Gamma_D \Rightarrow\ = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\Gamma_N} h v\, ds$$

Find $u \in V(g)$ s.t.
$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f\, v dx + \int_{\Gamma_N} h\, v ds \quad \forall v \in V(0)$$

- $a(\cdot, \cdot) : V \times V \to \mathbb{R}$: bilinear form
- $F(\cdot) : V \to \mathbb{R}$: functional

Find $u \in V(g)$ s.t.

$$a(u, v) = F(v) \quad \forall v \in V(0)$$

discretization and matrix formulation : 1/3

finite element basis, span$[\varphi_1, \ldots, \varphi_N] = V_h \subset V$

$u_h \in V_h \Rightarrow u_h = \sum_{1 \le i \le N} u_i \varphi_i$

finite element nodes $\{P_j\}_{j=1}^N$, $\varphi_i(P_j) = \delta_{ij}$ Lagrange element

$\Lambda_D \subset \Lambda = \{1, \ldots, N\}$: index set of node on the Dirichlet boundary

Dirichlet data : $u(P_k) = g(P_k) \quad P_k \in \Gamma_D$

$$V_h(g) = \{u_h \in V_h \, ; \, u_h = \sum u_i \varphi_i, u_k = g_k \ (k \in \Lambda_D)\}$$

Find $u_h \in V_h(g)$ s.t.

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h(0).$$

Find $\{u_j\}, u_k = g_k (k \in \Lambda_D)$ s.t.

$$a(\sum_j u_j \varphi_j, \sum_i v_i \varphi_i) = F(\sum_i v_i \varphi_i) \quad \forall \{v_i\}, v_k = 0 (k \in \Lambda_D)$$

Find $\{u_j\}_{j \in \Lambda}$ s.t.

$$\sum_j a(\varphi_j, \varphi_i) u_j = F(\varphi_i) \qquad \forall i \in \Lambda \setminus \Lambda_D$$

$$u_k = g_k \qquad \forall k \in \Lambda_D$$

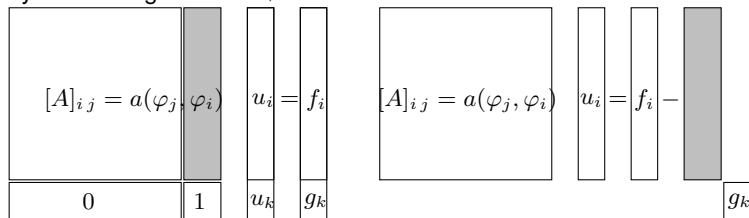## discretization and matrix formulation : 2/3

Find $\{u_j\}_{j \in \Lambda \setminus \Lambda_D}$ s.t.

$$\sum_{j \in \Lambda \setminus \Lambda_D} a(\varphi_j, \varphi_i) u_j = F(\varphi_i) - \sum_{k \in \Lambda_D} a(\varphi_k, \varphi_i) g_k \quad \forall i \in \Lambda \setminus \Lambda_D$$

$A = \{a(\varphi_j, \varphi_i)\}_{i,j \in \Lambda \setminus \Lambda_D}$: symmetric.

$A \in \mathbb{R}^{N \times N}$, $f \in \mathbb{R}^N$, $N = \#(\Lambda \setminus \Lambda_D)$

by re-indexing Λ into $N + m$ vectors

$[A]_{ij} = a(\varphi_j, \varphi_i)$ 0 1 $u_i = f_i$ u_k g_k $[A]_{ij} = a(\varphi_j, \varphi_i)$ $u_i = f_i -$ g_k

discretization and matrix formulation : 3/3

positivity of the matrix from coercivity of the bilinear form

A : (symmetric) positive definite i.e., $(A\vec{u}, \vec{u}) > 0 \,\forall \vec{u} \neq 0$

poof

$$(A\vec{u}, \vec{u}) = \sum_i \left(\sum_j a(\varphi_j, \varphi_i) u_j \right) u_i$$
$$= a(\sum_j \varphi_j u_j, \sum_i \varphi_i u_i) = a(u, u) \geq \alpha ||u||_1^2$$

corecivity of blinear form $a(\cdot, \cdot)$ ensures positivity of stiffness matrix A

corecivity is obtained by Poincare's inequality $|u|_1^2 \geq c||u||_0^2$

$$||u||_1^2 = ||u||_0^2 + |u|_1^2 = \int_\Omega u\, u + \int_\Omega \nabla u \cdot \nabla u$$
$$a(u, u) = \int_\Omega \nabla u \cdot \nabla u = ||\nabla u||_0^2 = |u|_1^2$$
$$|u|_1^2 = ((1 - \beta) + \beta)|u|_1^2 \geq c\beta||u||_0^2 + (1 - \beta)||u||_1^2 = \frac{c}{1 + c}||u||_1^2$$
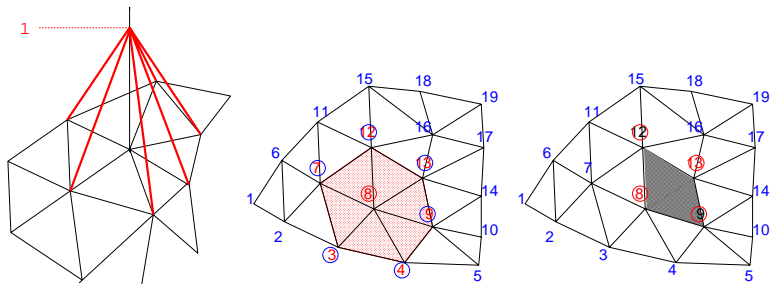
by taking $\beta = 1/(1 + c)$

P1 finite element and sparse matrix

$\mathcal{T}_h$: triangulation of a domain Ω, triangular element $K \in \mathcal{T}_h$

piecewise linear element : $\varphi_i|_K(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2$

$\varphi_i|_K(P_j) = \delta_{ij}$

$$[A]_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \nabla \varphi_j \cdot \nabla \varphi_i \, dx = \sum_{K \in \mathcal{T}_h} \int_K \nabla \varphi_j \cdot \nabla \varphi_i \, dx.$$
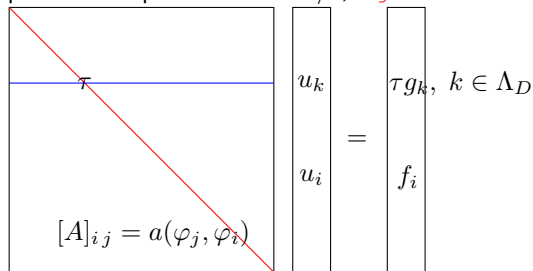

A : sparse matrix, CRS (Compressed Row Storage) format to store

penalty method to solve inhomogeneous Dirichlet problem

modification of diagonal entries of A where index $k \in \Lambda_D$
penalization parameter $\tau = 1/\varepsilon$; `tgv`

$$\tau u_k + \sum_{j \neq k} a_{k\,j} u_j = \tau g_k \iff u_k - g_k = \varepsilon(-\sum_{j \neq k} a_{k\,j} u_j),$$

$$\sum_j a_{i\,j} u_j = f_i \qquad \forall i \in \{1, \ldots, N\} \setminus \Lambda_D.$$

keeping symmetry of the matrix without changing index numbering.

abstract framework

V: Hilbert space with inner product $(\cdot, \cdot)$ and norm $|| \cdot ||$.

bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$

▶ coercive : $\exists \alpha > 0 \quad a(u, u) \geq \alpha ||u||^2 \ \forall u \in V.$

▶ continuous : $\exists \gamma > 0 \quad |a(u, v)| \leq \gamma ||u|| \, ||v|| \ \forall u, v \in V.$

functional $F(\cdot) : V \to \mathbb{R}.$

$$\text{find } u \in V \ \text{s.t.} \ a(u, v) = F(v) \quad \forall v \in V$$

has a unique solution : Lax-Milgram's theorem

most general case on $a(\cdot, \cdot)$

▶ inf-sup conditions

▶ $\exists \alpha_1 > 0 \quad \sup\limits_{v \in V, v \neq 0} \dfrac{a(u, v)}{||v||} \geq \alpha_1 ||u|| \ \forall u \in V.$

▶ $\exists \alpha_2 > 0 \quad \sup\limits_{u \in V, u \neq 0} \dfrac{a(u, v)}{||u||} \geq \alpha_2 ||v|| \ \forall v \in V.$

find $u \in V$ s.t. $a(u, v) = F(v) \quad \forall v \in V$ has a unique solution.

error estimate : theory 1 /2

V : Hilbert space, $V_h \subset V$: finite element space.

- $u \in V$, $a(u, v) = F(v) \ \forall v \in V$.
- $u_h \in V_h$, $a(u_h, v_h) = F(v_h) \ \forall v_h \in V_h \subset V$.

$a(u, v_h) = F(v_h) \ \forall v_h \in V_h \subset V$.

Galerkin orthogonality $\qquad a(u - u_h, v_h) = 0 \ \forall v_h \in V_h$.

assuming coercivity and continuity of $a(\cdot, \cdot)$.

Céa's lemma $\qquad ||u - u_h|| \leq \left(1 + \dfrac{\gamma}{\alpha}\right) \inf_{v_h \in V_h} ||u - v_h||$.

proof: $||u - u_h|| \leq ||u - v_h|| + ||v_h - u_h||$

$$\alpha ||u_h - v_h||^2 \leq a(u_h - v_h, u_h - v_h)$$
$$= a(u_h, u_h - v_h) - a(v_h, u_h - v_h)$$
$$= a(u, u_h - v_h) - a(v_h, u_h - v_h)$$
$$= a(u - v_h, u_h - v_h) \leq \gamma ||u - v_h|| ||u_h - v_h||.$$
$$||u_h - v_h|| \leq \frac{\gamma}{\alpha} ||u - v_h||$$

error estimate : theory 2 /2

$\Pi_h : C(\bar{\Omega}) \to V_h, \quad \Pi_h u = \sum_{1 \le i \le N} u(P_i)\phi_i,$
span$[\{\phi_i\}_{1 \le i \le N}] = S_h$, P_k finite element basis.

interpolation error by polynomial $K \in \mathcal{T}_h$, $P_k(K) \subset H^l(K)$, $v \in H^{k+1}(\Omega)$
$\Rightarrow$
$\exists c > 0 \quad |v - \Pi_h v|_{s,K} \le c\, h_K^{k+1-s}|v|_{k+1,K} \ (0 \le s \le \min\{k+1, l\}).$
<div align="right">cf. Ern-Guermond, 2021</div>

finite element error $u \in H^{k+1}$, u_h : finite element solution by P_k element.
$\Rightarrow$
$\exists c > 0 \quad ||u - u_h||_{1,\Omega} \le c\, h^k|u|_{k+1,\Omega}$

proof:

$$\begin{aligned}
||u - u_h||_{1,\Omega} &\le c \inf_{v_h \in V_h} ||u - v_h||_{1,\Omega} \\
&\le c||u - \Pi_h u||_{1,\Omega} \\
&\le \tilde{c}\sum_{K \in \mathcal{T}_h} (|u - \Pi_h u|_{0,K} + |u - \Pi_h u|_{1,K}) \\
&\le \check{c}\sum_{K \in \mathcal{T}_h} (h_K^{(k+1)} + h_K^k)|u|_{k+1,K} \\
&\le \hat{c}h^k|u|_{k+1,\Omega}
\end{aligned}$$

Sobolev space

P1 element element space does not belong to $C^1(\Omega)$.

$$H^1(\Omega) = \{u \in L^2(\Omega) \,;\, ||u||_1^2 = (u, u) < +\infty\}$$

$$(u, v) = \int_\Omega u\,v + \nabla u \cdot \nabla v,$$

$$||u||_0^2 = \int_\Omega u\,u, \quad |u|_1^2 = \int_\Omega \nabla u \cdot \nabla u$$

$H_0^1 = \{u \in H^1(\Omega) \,;\, u = 0 \text{ on } \partial\Omega\}$.

Poincaré's inequality $\exists C(\Omega)\ u \in H_0^1 \Rightarrow ||u||_0 \leq C(\Omega)|u|_1$.

proof for the case $\Omega \subset B = (0, s) \times (0, s)$

$v \in C_0^\infty(\Omega)$, extension with $\widetilde{v}(x) = 0 \ (x \in B \setminus \bar\Omega)$.

$$v(x_1, x_2) = v(0, x_2) + \int_0^{x_1} \partial_1 v(t, x_2)dt$$

$$|v(x_1, x_2)|^2 \leq \int_0^{x_1} 1^2 dt \int_0^{x_1} |\partial_1 v(t, x_2)|^2 dt \leq s \int_0^s |\partial_1 \widetilde{v}(t, x_2)|^2 dt$$

$$\int_0^s |v(x_1, x_2)|^2 dx_1 \leq \int_0^s s \int_0^s |\partial_1 \widetilde{v}(t, x_2)|^2 dt\,dx_1 \leq s^2 \int_0^s |\partial_1 \widetilde{v}(x_1, x_2)|^2 dx_1$$

$$\int_\Omega |v|^2 dx_1 dx_2 = \int_B |\widetilde{v}|^2 dx_1 dx_2 \leq s^2 \int_B |\partial_1 \widetilde{v}|^2 dx_1 dx_2 = s^2 \int_\Omega |\partial_1 v|^2 dx_1 dx_2$$

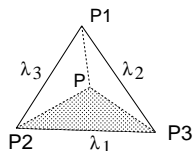## numerical integration

Numerical quadrature:
$\{P_i\}_{1 \leq i \leq m}$: integration points in K, $\{\omega_i\}_{1 \leq i \leq m}$: weights

$$|u - u_h|^2_{0,\Omega} = \sum_{K \in \mathcal{T}_h} \int_K |u - u_h|^2 dx \sim \sum_{K \in \mathcal{T}_h} \sum_{i=1}^m |(u - u_h)(P_i)|^2 \omega_i$$

formula : degree 5, 7 points,

P.C. Hammer, O.J. Marlowe, A.H. Stroud [1956]

area coordinates $\{\lambda_i\}_{i=1}^3$	weight	
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{9}{40}|K|$	$\times 1$
$(\frac{6-\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21}, \frac{9+2\sqrt{15}}{21})$	$\frac{155-\sqrt{15}}{1200}|K|$	$\times 3$
$(\frac{6+\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21}, \frac{9-2\sqrt{15}}{21})$	$\frac{155+\sqrt{15}}{1200}|K|$	$\times 3$

Remark
it is not good idea to use interpolation of continuous function to finite element space, for verification of convergence order.
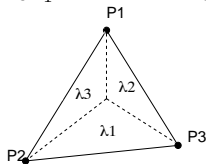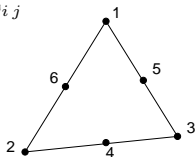$|\Pi_h u - u_h|_{1,\Omega}$ may be smaller (in extreme cases, super convergence)

P2 finite element

$\mathcal{T}_h$: triangulation of a domain Ω, triangular element $K \in \mathcal{T}_h$
piecewise quadratic element : 6 DOF on element K.

$\varphi_i|_K(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2$

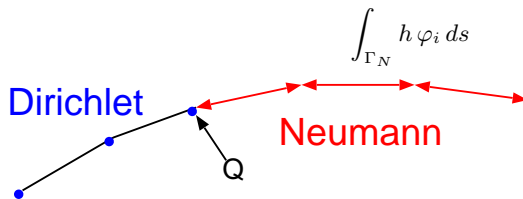$\varphi_i|_K(P_j) = \delta_{ij}$

by using area coordinates $\{\lambda_1, \lambda_2, \lambda_3\}$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{pmatrix} = \begin{pmatrix} 1 & & & & -1 & -1 \\ & 1 & & -1 & & -1 \\ & & 1 & -1 & -1 & \\ & & & 4 & & \\ & & & & 4 & \\ & & & & & 4 \end{pmatrix} \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \\ \lambda_2 \lambda_3 \\ \lambda_3 \lambda_1 \\ \lambda_1 \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(2\lambda_1 - 1) \\ \lambda_2(2\lambda_2 - 1) \\ \lambda_3(2\lambda_3 - 1) \\ 4\lambda_2 \lambda_3 \\ 4\lambda_3 \lambda_1 \\ 4\lambda_1 \lambda_2 \end{pmatrix}$$

treatment of Neumann data around mixed boundary

Neumann data is evaluated by line integral with FEM basis φ_i.

$$\int_{\Gamma_N} h\,\varphi_i\,ds$$

Dirichlet

Neumann

Q

For given discrete Neumann data, h is interpolated in FEM space,
$h = \sum_j h_j \varphi_j|_{\Gamma_N}$,

$$\sum_j h_j \int_{\Gamma_N} \varphi_j \varphi_i\,ds.$$

On the node $Q \in \bar\Gamma_D \cap \bar\Gamma_N$, both Dirichlet and Neumann are necessary.

advantages of finite element formulation

- ▶ weak formulation is obtained by integration by part with clear description on the boundary
- ▶ Dirichlet boundary condition is embedded in a functional space, called as essential boundary condition
- ▶ Neumann boundary condition is treated with surface/line integral by Gauss-Green's formula, called as natural boundary condition
- ▶ solvability of linear system is inherited from solvability of continuous weak formulation
- ▶ error of finite element solution is evaluated by approximation property of finite element space

linear and nonlinear solid mechanics : 1/3

a problem with the second Piola-Kirchhoff stress tensor Σ
in Ω with $\partial\Omega = \Gamma_D \cup \Gamma_N$

$$-\text{div}\left((I + \nabla u)\Sigma(u(x))\right) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma_D,$$
$$(I + \nabla u)\Sigma(u(x))n = g(x) \quad \text{on } \Gamma_N.$$

Green-St. Venant tensor $E(u)$

$$E(u) = \frac{1}{2}\left((\nabla u)^T + \nabla u + (\nabla u)^T(\nabla u)\right) = E_L(u) + E_{NL}(u)$$

Piola-Kirchhoff stress tensor with Lamé constants λ and μ

$$\Sigma(u) = \check{\Sigma}(E(u)) = \lambda\left(\text{tr}(E(u))\right)I + 2\mu\,E(u)$$

variational problem with test function $v,\ v = 0$ on Γ_D

$$-\int_\Omega \text{div}(I + \nabla u)\check{\Sigma}(E(u)) \cdot v = \int_\Omega f \cdot v$$
$$\Leftrightarrow \int_\Omega (I + \nabla u)\check{\Sigma}(E(u)) : \nabla v = \int_\Omega f \cdot v + \int_{\Gamma_N \cup \Gamma_D} (I + \nabla u)\check{\Sigma}(E(u))n \cdot v$$
$$= \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v$$

Dirchlet data is treated as the essential boundary condition

linear and nonlinear solid mechanics : 2/3

derivatives of Green-St. Venant tensor

- ▶ first order derivative $dE(u)[v]$: symmetric

$$
\begin{aligned}
dE(u)[v] &= \frac{1}{2}\left((\nabla v)^T + \nabla v\right) + \frac{1}{2}\left((\nabla u)^T \nabla v + (\nabla v)^T \nabla u\right) \\
&= E_L(v) + dE_{NL}(u)[v]
\end{aligned}
$$

calculated from variation as

$$
\begin{aligned}
E(u+v) - E(u) &= (E_L(u+v) + E_{NL}(u+v)) - ((E_L(u) + E_{NL}(u)) \\
&= E_L(v) + \frac{1}{2}(\nabla(u+v)^T \nabla(u+v)) - E_{NL}(u) \\
&= E_L(v) + \frac{1}{2}((\nabla u)^T \nabla v + (\nabla v)^T \nabla u + (\nabla v)^T \nabla v)
\end{aligned}
$$

- ▶ second order derivative $d^2 E(u)[v, w]$

$$
d^2 E(u)[v, w] = dE_{NL}(w)[v]
$$

calculated from variation as

$$
\begin{aligned}
dE(u+w)[v] - dE(u)[v] &= (E_L(v) + dE_{NL}(u+w)[v]) - (E_L(v) + dE_{NL}(u)[v]) \\
&= \frac{1}{2}\left((\nabla(u+w))^T \nabla v + (\nabla v)^T \nabla(u+w)\right. \\
&\quad \left. - (\nabla u)^T \nabla v + (\nabla v)^T \nabla u\right) \\
&= \frac{1}{2}\left((\nabla w)^T \nabla v + (\nabla v)^T \nabla w\right)
\end{aligned}
$$

linear and nonlinear solid mechanics : 3/3

The variational equation is re-written as

$$\int_\Omega \check{\Sigma}(E(u)) : dE(u)[v] = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v\,.$$

symmetry of $\check{\Sigma}(E)$ leads to

$$\begin{aligned}
(I + \nabla u)\check{\Sigma}(E(u)) : \nabla v &= \check{\Sigma}(E(u)) : (I + \nabla u)^T \nabla v \\
&= \frac{1}{2}\check{\Sigma}(E(u)) : \left((I + \nabla u)^T \nabla v + \nabla v^T (I + \nabla u)\right) \\
&= \frac{1}{2}\check{\Sigma}(E(u)) : \left(\nabla v + (\nabla u)^T \nabla v + \nabla v^T + \nabla v^T \nabla u\right) \\
&= \check{\Sigma}(E(u)) : dE(u)[v]
\end{aligned}$$

tensors $A, B, C \Rightarrow A\,B : C = B : A^T C$

$$\sum_{i\,j}\left(\sum_k [A]_{i\,k}[B]_{k\,j}\right)[C]_{i\,j} = \sum_{k\,j}[B]_{k\,j}\left(\sum_i [A^T]_{k\,i}[C]_{i\,j}\right)$$

▶ nonlinear solver by Newton iteration (linearization for iterative solver)
▶ linearization of the Green-St. Venant tensor $E(u)$ by $E_L(u)$

linear elasticity problem : 1/2

Piola-Kirchhoff stress tensor with Lamé constants λ and μ

$$\Sigma(u) = \check{\Sigma}(E(u)) = \lambda\left(\text{tr}(E(u))\right)I + 2\mu\,E(u)$$

linearization of Green-St. Venant tensor $E(u)$

$$E(u) = \frac{1}{2}\left((\nabla u)^T + \nabla u + (\nabla u)^T(\nabla u)\right) \simeq \frac{1}{2}\left((\nabla u)^T + \nabla u\right) = E_L(u) = e(u)$$

$$E_L(u+v) - E_L(u) = \frac{1}{2}\left((\nabla(u+v)^T + \nabla(u+v)\right) - \frac{1}{2}\left(((\nabla u)^T + \nabla u\right)$$

$$= \frac{1}{2}\left((\nabla v)^T + \nabla v\right) = dE_L(u)[v] = E_L(v)$$

$$\check{\Sigma}(E(u)) : dE_L(u)[v] = \lambda\left(\text{tr}(E(u))\right)I : E_L(v) + 2\mu\,E_L(u) : dE_L(v)$$

$$= \lambda\nabla\cdot u\nabla\cdot v + 2\mu E_L(u) : E_L(v)$$

variational problem of the linear elasticity with test function $v,\ v = 0$ on Γ_D

$$\int_\Omega 2\mu\,e(u) : e(v) + \lambda\nabla\cdot u\,\nabla\cdot v = \int_\Omega f\cdot v + \int_{\Gamma_N} g\cdot v$$
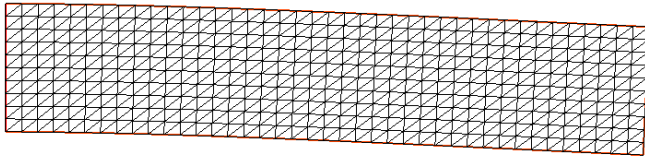
Korn's inequality $u \in H_0^1(\Omega)^3$

$$\Rightarrow \quad \exists\alpha(\Omega)\int_\Omega e(u) : e(u) \geq \alpha(\Omega)\int_\Omega u\cdot u + \nabla u : \nabla u$$
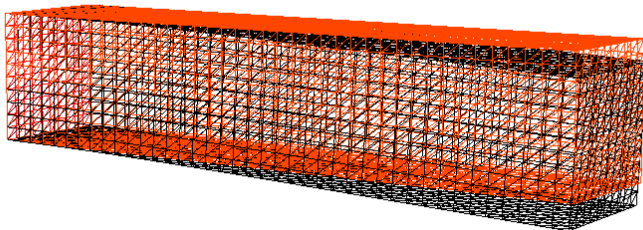
A : ternsor $\Rightarrow I : A = \text{tr}(A)$, $\text{tr}(e(u)) = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = \nabla\cdot u$

linear elasticity problem : 2/2

2D beam : displacement on the left wall is fixed

3D beam : displacement on the left wall is fixed

nonlinear iteration by Newton method : 1/3

nonlinear problem: $F : \mathbb{R}^N \to \mathbb{R}^N$, $\vec{x} \in \mathbb{R}^N$

$$\text{find } \vec{x} \quad F(\vec{x}) = \vec{0}$$

Fréchet derivative of F : $F(\vec{x} + \vec{\delta x}) - F(\vec{x}) \simeq \nabla F(\vec{x}) \vec{\delta x}$
Jacobian matrix $\nabla F(\vec{x}) \in \mathbb{R}^{N \times N}$
Algorithm : Newton iteration

$\vec{x}_0 \in \mathbb{R}^N$: initial guess
loop $n = 0, 1, 2, \cdots$

$\quad \nabla F(\vec{x}_n) \vec{\delta x} = F(\vec{x}_n)$

$\quad \vec{x}_{n+1} = \vec{x}_n - \vec{\delta x}$

nonlinear variational problem $f(\cdot, \cdot) : V \times V \to \mathbb{R}$

$$\text{find } x \in V(g) \ F(x, y) = 0 \ \forall y \in V(0)$$

Fréchet derivative of $F(\cdot, \cdot)$:
$F(x + \delta x, y) - F(x, y) \simeq (\nabla F(x)\delta x, y) \ \forall y \in V(0)$
Algorithm : Newton iteration

$x_0 \in V(g)$: initial guess
loop $n = 0, 1, 2, \cdots$

$\quad$ find $\delta x \in V(0) \ (\nabla F(x_n)\delta x, y) = F(x_n, y) \ \forall y \in V(0)$

$\quad x_{n+1} = x_n - \delta x$

nonlinear elasticity : 1/2

The variational equation

$$\int_\Omega \check{\Sigma}(E(u)) : dE(u)[v] = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v \,.$$

Newton iteration of the nonlinear system u^0 : given
loop $n = 1, 2, \cdots$
 solve linear system to find update w,

$$\int_\Omega \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2 E(u^n)[v, w]$$

$$= \int_\Omega \check{\Sigma}(E(u^n)) : dE(u^n)[v] - \int_\Omega f \cdot v - \int_{\Gamma_N} g \cdot v \quad \forall v$$

 update $u^{n+1} = u^n - w$

Jacobian of the variational problem is calculated as

$$\int_\Omega \check{\Sigma}(E(u^n + w)) : dE(u^n + w)[v] - \int_\Omega \check{\Sigma}(E(u^n)) : dE(u^n)[v]$$

$$\simeq \int_\Omega \check{\Sigma}\left(E(u^n) + dE(u^n)[w]\right) : \left(dE(u^n)[v] + d^2 E(u^n)[v, w]\right)$$

$$- \int_\Omega \check{\Sigma}(E(u^n)) : dE(u^n)[v] \quad \leftarrow \text{1st order term of } w$$

$$\simeq \int_\Omega \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2 E(u^n)[v, w]$$

nonlinear elasticity : 2/2

Jacobian for Newton iteration, w : unknown, v : test function, u^n : given

$$\int_\Omega \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2 E(u^n)[v, w].$$

is symmetric, which is shown from

▶ $d^2 E(u^n)[v, w] = dE_{NL}(w)[v] = \frac{1}{2}\left((\nabla w)^T \nabla v + (\nabla v)^T \nabla w\right)$ is sym.

▶ by recalling $I : A = \text{tr}(A)$ and for symmetric tensors η and ζ

$$\check{\Sigma}(\eta) : \zeta = (\lambda \text{tr}(\eta)I + 2\mu\eta) : \zeta = \lambda \text{tr}(\eta)\text{tr}(\zeta) + 2\mu\eta : \zeta$$
$$= \lambda \text{tr}(\zeta)\text{tr}(\eta) + 2\mu\zeta : \eta = \check{\Sigma}(\zeta) : \eta$$

coercivity depends on previous iteration step u^n

▶ the first half part :   inequality is indept. of u^n

$$\int_\Omega \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[w] \geq 2 \int_\Omega \mu E_L(w) : E_L(w)$$

▶ the second half part :   coercivity depends on $\check{\Sigma}(E(u^n))$

$$\int_\Omega \check{\Sigma}(E(u^n)) : d^2 E(u^n)[w, w]$$
$$= \int_\Omega \check{\Sigma}(E(u^n)) : dE_{NL}(w)[w] = \int_\Omega \check{\Sigma}(E(u^n)) : (\nabla w)^T (\nabla w)$$
$$= \int_\Omega \check{\Sigma}(E(u^n))(\nabla w)^T : (\nabla w)^T$$

nonlinear iteration by Newton method : 2/3

Jacobian and RHS are calculated from the previous u^n in Newton step

$$(\nabla F(u^n)w, v) = \int_\Omega \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2E(u^n)[v, w] \quad \forall v$$

$$(F(u^n), v) = \int_\Omega \check{\Sigma}(E(u^n)) : dE(u^n)[v] \quad \forall v$$

initial guess of the Newton iteration by linear elasticity with small load given by inhomogeneous Dirichlet data g
solution space $V(g) := \{u \in H^1(\Omega)^3 \,; \, u = g \text{ on } \Gamma_D\}$

linear elasticity problem is solved as

$$\text{to find } u^0 \in V(g) \quad \int_\Omega \check{\Sigma}(e(u^0)) : e(v) = 0 \quad \forall v \in V(0)$$

and the perform Newton iteration
loop $n = 0, 1, 2, \cdots$

  find $w \in V(0)$ $\quad (\nabla F(u^n)w, v) = (F(u^n), v) \quad \forall v \in V_0$

  $u^{n+1} = u^n - w$

The update w in the Newton iteration supposed to have homogeneous Dirichlet boundary condition, i.e., $w \in V(0)$.

nonlinear iteration by Newton method : 3/3

a method as incremental loading:
sequence of in homogeneous Dirichlet data whose magnitude is increasing

$$||g^{(0)}|| < ||g^{(1)}|| < \cdots < ||g^{(N)}||, \ g = g^{(N)}$$

The procedure consists of nested loops for increment and Newton iteration.

perform incremental loading::
loop $m = 0, 1, 2, \cdots, N-1$

if $m = 0$ then solve linear elasticity

$$\text{to find } u^0 \in V(g^0) \quad \int_\Omega \check{\Sigma}(e(u^0)) : e(v) = 0 \quad \forall v \in V(0)$$

otherwise $(m > 0)$ set $u^{m,0} \in V(g^{(m)})$ from previous $u^{m-1} \in V(g^{(m-1)})$.

perform Newton iteration::
loop $n = 0, 1, 2, \cdots$

find $w \in V(0) \quad (\nabla F(u^{m,n})w, v) = (F(u^{m,n}), v) \quad \forall v \in V_0$

$$u^{m,n+1} = u^{m,n} - w$$

description of complex geometry

- ▶ 2D : `bamg` embedded in FreeFEM generates isotropic triangulation of the domain consisting of parameterized segments. By changing of the metric based on finite element solution, an-isotropic triangulation is also performed

- ▶ 3D : `Gmsh` can handle parameterized surface segments and/or treat `STEP` file by `OepnCASCADE` geometry engine. Surface mesh generator and volume mesh generator constructs isotropic tetrahedral mesh decomposition
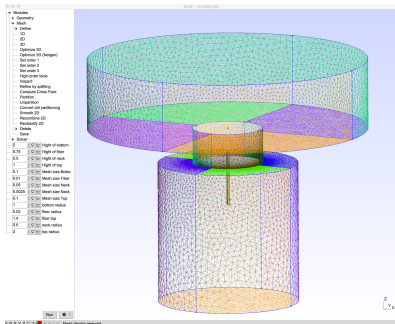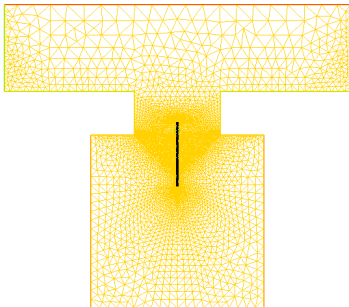
2D geometry definition by FreeFEM

```
real rb = 1.0; real lb = 2.0; // ....
border c1(t=0,1){x=-rb + 2*rb*t; y = 0.0; label = 2;};
border c2(t=0,1){x=rb; y = lb * t; label = 3;};
border c3(t=0,1){x=(rb + (r0 - rb) * t); y = lb; label = 4;};
border c4(t=0,1){x=r0; y = lb + l0 * t*t; label = 5;};
border c5(t=0,1){x=r0 + (rc - r0) * t; y = lb+l0; label = 6;};
border c6(t=0,1){x=rc; y = lb + l0 + lc * t; label = 7;};
border c7(t=0,1){x=-rc + 2*rc * t; y = lb+l0+lc; label = 1;};
//
border c8(t=0,1){x=-rb; y = lb * t; label = 3;};
border c9(t=0,1){x=-(rb + (r0 - rb) * t); y = lb; label = 4;};
// ...
// fiber
border d1(t=0,1){x=-rd+2*rd * t + re; y = le; label = 8;};
border d2(t=0,1){x = rd+re; y = le + ld * t; label = 8; };
border d3(t=0,1){x=rd-2*rd * t + re; y = le+ld; label=8;};
border d4(t=0,1){x = -rd+re; y = le+ld * t; label = 8; };

int n = 20; int nn = 20; int nnn = 50;
mesh Th = buildmesh(c1(n) + c2(n) + c3(nn) + c4(nn)
             + c5(n) + c6(nn) + c7(-n)
             + c8(-n) + c9(-nn) + c10(-nn) + c11(-n) + c12(-nn)
             + d1(nn) + d2(nnn) + d3(nn) +d4(-nnn));
```
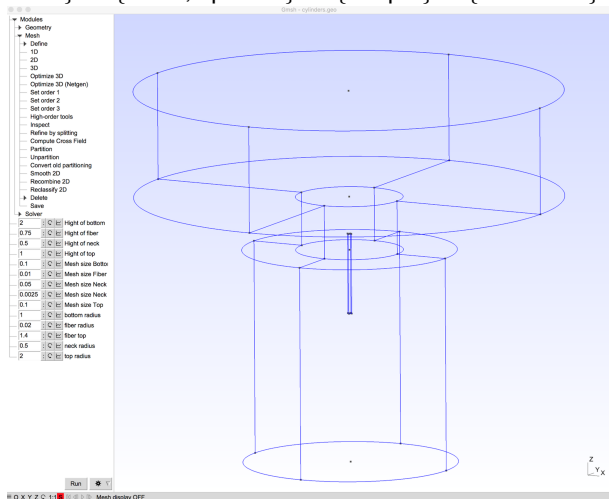
3D geometry definition by Gmsh : 1/4

```
pext = newp;
Point(pext+0) = {0, 0, 0, hBottom};
Point(pext+1) = {RRb, 0, 0, hBottom};
Point(pext+2) = {0, RRb, 0, hBottom};
Point(pext+3) = {-RRb, 0, 0, hBottom};
Point(pext+4) = {0, -RRb, 0, hBottom};
lext = newl;
Circle(lext+0) = {pext+1, pext+0, pext+2};
Circle(lext+4) = {pext+6, pext+5, pext+7};
Line(lext+8) = {pext+1, pext+6};
Line(lext+9) = {pext+2, pext+7};
llext = newll;
Line Loop(llext+0) = {lext+0, lext+9, -(lext+4), -(lext+8)};
// ..
sext = news;
Ruled Surface(sext+0) = {llext+0};         Side[0] = sext+0;
// ..
Physical Surface(SIDE) = {Side[],Neck[]};
// ..
sl = newsl;
Surface Loop(sl+1)={Neck[],NeckFine[],Side[],Top[],Bottom[]};
v = newv;
Volume(v+0) = {sl+1, sl+2};
```

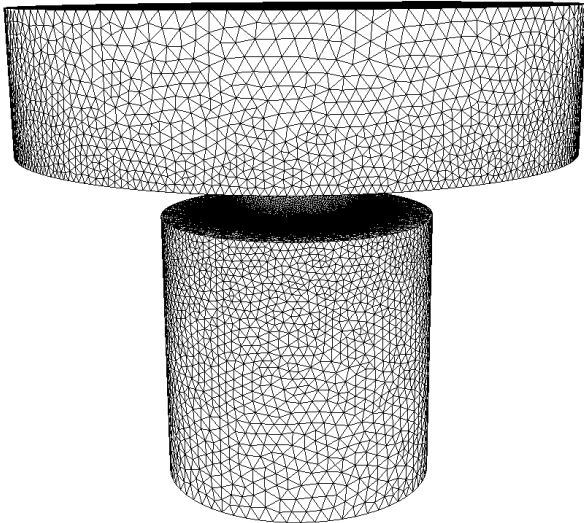# 3D geometry definition by Gmsh : 2/4

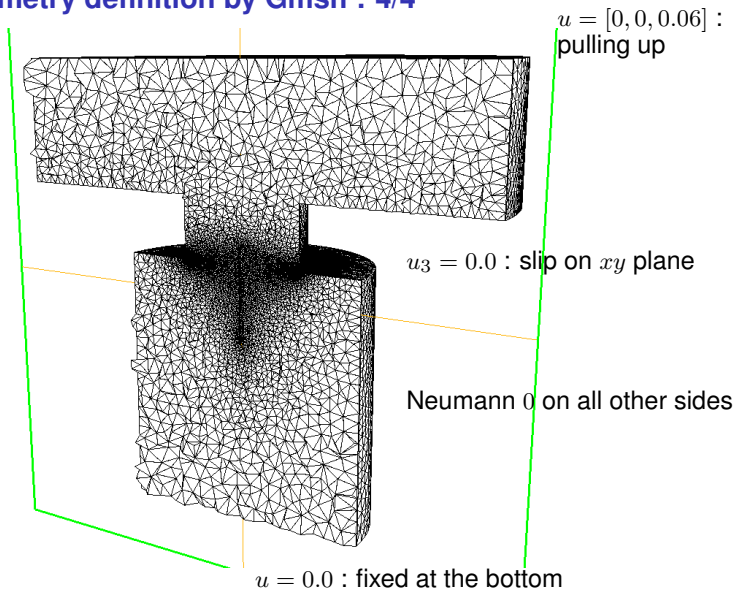{ points } → { lines, splines } → { loops } ≃ { surfaces } → { volumes }

surface segments and generated mesh can be verified by GUI
command line option can generate mesh readable by FreeFEM

```
gmsh -3 -format msh2 -o test.msh test.geo
```
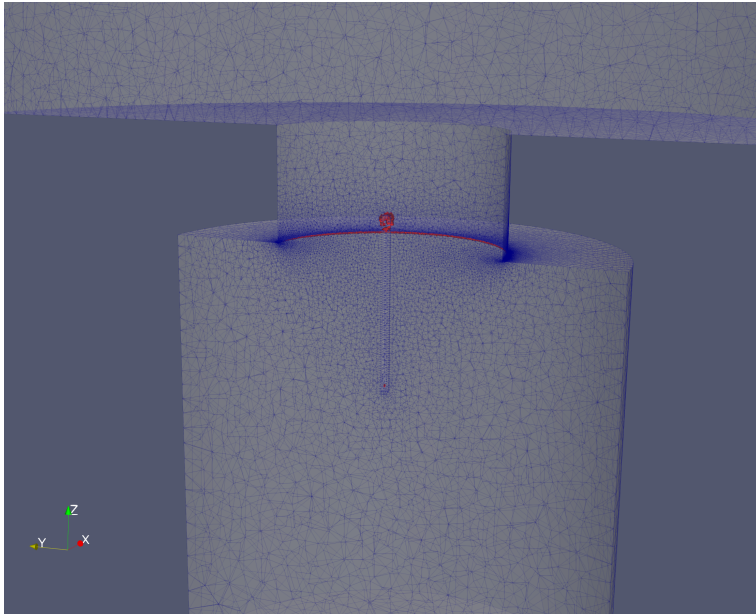
3D geometry definition by Gmsh : 4/4

$u = [0, 0, 0.06]$:
pulling up

$u_3 = 0.0$: slip on xy plane

Neumann 0 on all other sides

$u = 0.0$: fixed at the bottom

- ▶ height of the domain is 2.5, radius of the top is 2, bottom 1
- ▶ internal fiber has 100 times large Lamé constants than the surround

stress concentration

$708,947$ vertexes, $2,076,809$ DOF with P1 element

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