

Introduction of finite element method

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Poisson equation with mixed B.C. and a weak formulation: 1/2

$$\Omega \subset \mathbb{R}^2, \partial\Omega = \Gamma_D \cup \Gamma_N$$

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma_D, \end{aligned}$$

$$\frac{\partial u}{\partial n} = \nabla u \cdot n = h \text{ on } \Gamma_N.$$

weak formulation

V : function space, $V(g) = \{u \in V ; u = g \text{ on } \Gamma_D\}$. $V = C^1(\Omega) \cap C^0(\bar{\Omega})$?

Find $u \in V(g)$ s.t.

$$\int_{\Omega} -\Delta u v dx = \int_{\Omega} f v dx \quad \forall v \in V(0)$$

Gauss-Green's formula

$u, v \in V, n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$: outer normal to $\partial\Omega$

$$\int_{\Omega} (\partial_i u) v dx = - \int_{\Omega} u \partial_i v dx + \int_{\partial\Omega} u n_i v ds.$$

Poisson equation with mixed B.C. and a weak formulation: 2/2

$$\begin{aligned}\int_{\Omega} (-\partial_1^2 - \partial_2^2)u v \, dx &= \int_{\Omega} (\partial_1 u \partial_1 v + \partial_2 u \partial_2 v) \, dx - \int_{\partial\Omega} (\partial_1 u n_1 + \partial_2 u n_2) v \, ds \\ &= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Gamma_D \cup \Gamma_N} \nabla u \cdot n v \, ds \\ v = 0 \text{ on } \Gamma_D \Rightarrow &= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Gamma_N} h v \, ds\end{aligned}$$

Find $u \in V(g)$ s.t.

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_N} h v \, ds \quad \forall v \in V(0)$$

- ▶ $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$: bilinear form
- ▶ $F(\cdot) : V \rightarrow \mathbb{R}$: functional

Find $u \in V(g)$ s.t.

$$a(u, v) = F(v) \quad \forall v \in V(0)$$

discretization and matrix formulation : 1/2

finite element basis, $\text{span}[\varphi_1, \dots, \varphi_N] = V_h \subset V$

$$u_h \in V_h \Rightarrow u_h = \sum_{1 \leq i \leq N} u_i \varphi_i$$

finite element nodes $\{P_j\}_{j=1}^N$, $\varphi_i(P_j) = \delta_{ij}$ Lagrange element

$\Lambda_D \subset \Lambda = \{1, \dots, N\}$: index set of node on the Dirichlet boundary

Dirichlet data : $u(P_k) = g(P_k)$ $P_k \in \Gamma_D$

$$V_h(g) = \{u_h \in V_h ; u_h = \sum u_i \varphi_i, u_k = g_k (k \in \Lambda_D)\}$$

Find $u_h \in V_h(g)$ s.t.

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h(0).$$

Find $\{u_j\}$, $u_k = g_k (k \in \Lambda_D)$ s.t.

$$a\left(\sum_j u_j \varphi_j, \sum_i v_i \varphi_i\right) = F\left(\sum_i v_i \varphi_i\right) \quad \forall \{v_i\}, v_k = 0 (k \in \Lambda_D)$$

Find $\{u_j\}_{j \in \Lambda}$ s.t.

$$\begin{aligned} \sum_j a(\varphi_j, \varphi_i) u_j &= F(\varphi_i) & \forall i \in \Lambda \setminus \Lambda_D \\ u_k &= g_k & \forall k \in \Lambda_D \end{aligned}$$

discretization and matrix formulation : 2/2

Find $\{u_j\}_{j \in \Lambda \setminus \Lambda_D}$ s.t.

$$\sum_{j \in \Lambda \setminus \Lambda_D} a(\varphi_j, \varphi_i) u_j = F(\varphi_i) - \sum_{k \in \Lambda_D} a(\varphi_k, \varphi_i) g_k \quad \forall i \in \Lambda \setminus \Lambda_D$$

$A = \{a(\varphi_j, \varphi_i)\}_{i,j \in \Lambda \setminus \Lambda_D}$: symmetric.

$A \in \mathbb{R}^{N \times N}$, $f \in \mathbb{R}^N$, $N = \#(\Lambda \setminus \Lambda_D)$

positivity of the matrix from coercivity of the bilinear form

A : (symmetric) positive definite i.e., $(A\vec{u}, \vec{u}) > 0 \quad \forall \vec{u} \neq 0$

poof

$$\begin{aligned}(A\vec{u}, \vec{u}) &= \sum_i \left(\sum_j a(\varphi_j, \varphi_i) u_j \right) u_i \\ &= a\left(\sum_j \varphi_j u_j, \sum_i \varphi_i u_i\right) = a(u, u) \geq \alpha \|u\|_1^2\end{aligned}$$

corecivity of bilinear form $a(\cdot, \cdot)$ ensures positivity of stiffness matrix A

corecivity is obtained by the Poincare's inequality $|u|_1^2 \geq c \|u\|_0^2$

$$a(u, u) = \int_{\Omega} \nabla u \cdot \nabla u = \|\nabla u\|_0^2 = |u|_1^2$$

$$|u|_1^2 = ((1 - \beta) + \beta) |u|_1^2 \geq c\beta \|u\|_0^2 + (1 - \beta) |u|_1^2 = \frac{c}{1 + c} \|u\|_1^2$$

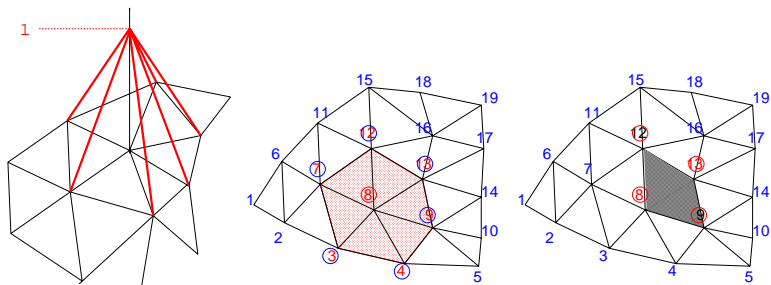
P1 finite element and sparse matrix

\mathcal{T}_h : triangulation of a domain Ω , triangular element $K \in \mathcal{T}_h$

piecewise linear element : $\varphi_i|_K(x_1, x_2) = a_0 + a_1x_1 + a_2x_2$

$\varphi_i|_K(P_j) = \delta_{ij}$

$$[A]_{ij} = a(\varphi_j, \varphi_i) = \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \, dx = \sum_{K \in \mathcal{T}_h} \int_K \nabla \varphi_j \cdot \nabla \varphi_i \, dx.$$



A : sparse matrix, CRS (Compressed Row Storage) format to store

penalty method to solve inhomogeneous Dirichlet problem

modification of diagonal entries of A where index $k \in \Lambda_D$

penalization parameter $\tau = 1/\varepsilon$; τg_k

$$[A]_{ij} = a(\varphi_j, \varphi_i)$$
$$\begin{matrix} u_k \\ u_i \end{matrix} = \begin{matrix} \tau g_k, k \in \Lambda_D \\ f_i \end{matrix}$$

$$\tau u_k + \sum_{j \neq k} a_{kj} u_j = \tau g_k \Leftrightarrow u_k - g_k = \varepsilon \left(- \sum_{j \neq k} a_{kj} u_j \right),$$

$$\sum_j a_{ij} u_j = f_i \quad \forall i \in \{1, \dots, N\} \setminus \Lambda_D.$$

keeping symmetry of the matrix without changing index numbering.

abstract framework

V : Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$.

bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$

▶ coercive : $\exists \alpha > 0 \quad a(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V$.

▶ continuous : $\exists \gamma > 0 \quad |a(u, v)| \leq \gamma \|u\| \|v\| \quad \forall u, v \in V$.

functional $F(\cdot) : V \rightarrow \mathbb{R}$.

find $u \in V$ s.t. $a(u, v) = F(v) \quad \forall v \in V$

has a unique solution : Lax-Milgram's theorem

most general case on $a(\cdot, \cdot)$

▶ inf-sup conditions

▶ $\exists \alpha_1 > 0 \quad \sup_{v \in V, v \neq 0} \frac{a(u, v)}{\|v\|} \geq \alpha_1 \|u\| \quad \forall u \in V$.

▶ $\exists \alpha_2 > 0 \quad \sup_{u \in V, u \neq 0} \frac{a(u, v)}{\|u\|} \geq \alpha_2 \|v\| \quad \forall v \in V$.

find $u \in V$ s.t. $a(u, v) = F(v) \quad \forall v \in V$ has a unique solution.

error estimate : theory 1 /2

V : Hilbert space, $V_h \subset V$: finite element space.

► $u \in V, a(u, v) = F(v) \quad \forall v \in V.$

► $u_h \in V_h, a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \subset V.$

$a(u, v_h) = F(v_h) \quad \forall v_h \in V_h \subset V.$ Galerkin orthogonality

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

assuming coercivity and continuity of $a(\cdot, \cdot).$

Céa's lemma $\|u - u_h\| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|.$

proof: $\|u - u_h\| \leq \|u - v_h\| + \|v_h - u_h\|$

$$\begin{aligned} \alpha \|u_h - v_h\|^2 &\leq a(u_h - v_h, u_h - v_h) \\ &= a(u_h, u_h - v_h) - a(v_h, u_h - v_h) \\ &= a(u, u_h - v_h) - a(v_h, u_h - v_h) \\ &= a(u - v_h, u_h - v_h) \leq \gamma \|u - v_h\| \|u_h - v_h\|. \end{aligned}$$

error estimate : theory 2 /2

$\Pi_h : C(\bar{\Omega}) \rightarrow V_h$, $\Pi_h u = \sum_{1 \leq i \leq N} u(P_i) \phi_i$,
 $\text{span}[\{\phi_i\}_{1 \leq i \leq N}] = S_h$, P_k finite element basis.

interpolation error by polynomial $K \in \mathcal{T}_h$, $P_k(K) \subset H^l(K)$, $v \in H^{k+1}(\Omega)$

\Rightarrow

$\exists c > 0 \quad |v - \Pi_h v|_{s,K} \leq c h_K^{k+1-s} |v|_{k+1,K} \quad (0 \leq s \leq \min\{k+1, l\})$.

cf. Ern-Guermond, 2021

finite element error $u \in H^{k+1}$, u_h : finite element solution by P_k element.

\Rightarrow

$\exists c > 0 \quad \|u - u_h\|_{1,\Omega} \leq c h^k |u|_{k+1,\Omega}$

proof:

$$\begin{aligned} \|u - u_h\|_{1,\Omega} &\leq c \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \\ &\leq c \|u - \Pi_h u\|_{1,\Omega} \\ &\leq \tilde{c} \sum_{K \in \mathcal{T}_h} (|u - \Pi_h u|_{0,K} + |u - \Pi_h u|_{1,K}) \\ &\leq \hat{c} \sum_{K \in \mathcal{T}_h} (h_K^{(k+1)} + h_K^k) |u|_{k+1,K} \\ &\leq \hat{c} h^k |u|_{k+1,\Omega} \end{aligned}$$

Sobolev space

P1 element element space does not belong to $C^1(\Omega)$.

$$H^1(\Omega) = \{u \in L^2(\Omega); \|u\|_1^2 = (u, u) < +\infty\}$$

$$(u, v) = \int_{\Omega} u v + \nabla u \cdot \nabla v,$$

$$\|u\|_0^2 = \int_{\Omega} u u, \quad |u|_1^2 = \int_{\Omega} \nabla u \cdot \nabla u$$

$$H_0^1 = \{u \in H^1(\Omega); u = 0 \text{ on } \partial\Omega\}.$$

Poincaré's inequality $\exists C(\Omega) u \in H_0^1 \Rightarrow \|u\|_0 \leq C(\Omega)|u|_1$.

proof for the case $\Omega \subset B = (0, s) \times (0, s)$

$v \in C_0^\infty(\Omega)$,

$$v(x_1, x_2) = v(0, x_2) + \int_0^{x_1} \partial_1 v(t, x_2) dt$$

$$|v(x_1, x_2)|^2 \leq \int_0^{x_1} 1^2 dt \int_0^{x_1} |\partial_1 v(t, x_2)|^2 dt \leq s \int_0^s |\partial_1 v(t, x_2)|^2 dt$$

$$\int_0^s |v(x_1, x_2)|^2 dx_1 \leq s^2 \int_0^s |\partial_1 v(x)|^2 dx_1$$

$$\int_{\Omega} |v|^2 = \int_B |v|^2 dx_1 dx_2 \leq s^2 \int_B |\partial_1 v|^2 dx_1 dx_2 = s^2 \int_{\Omega} |\partial_1 v|^2.$$

numerical integration

Numerical quadrature:

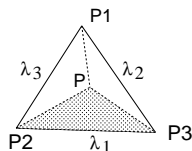
$\{P_i\}_{i \leq m}$: integration points in K , $\{\omega_i\}_{i \leq m}$: weights

$$|u - u_h|_{0,\Omega}^2 = \sum_{K \in \mathcal{T}_h} \int_K |u - u_h|^2 dx \sim \sum_{K \in \mathcal{T}_h} \sum_{i=1}^m |(u - u_h)(P_i)|^2 \omega_i$$

formula : degree 5, 7 points,

P.C. Hammer, O.J. Marlowe, A.H. Stroud [1956]

area coordinates $\{\lambda_i\}_{i=1}^3$	weight	
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{9}{40} K $	$\times 1$
$(\frac{6-\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21}, \frac{9+2\sqrt{15}}{21})$	$\frac{155-\sqrt{15}}{1200} K $	$\times 3$
$(\frac{6+\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21}, \frac{9-2\sqrt{15}}{21})$	$\frac{155+\sqrt{15}}{1200} K $	$\times 3$



Remark

it is not good idea to use interpolation of continuous function to finite element space, for verification of convergence order.

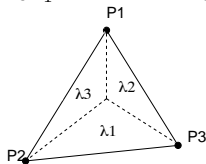
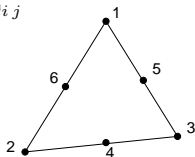
$|\Pi_h u - u_h|_{1,\Omega}$ may be smaller (in extreme cases, super convergence)

P2 finite element

\mathcal{T}_h : triangulation of a domain Ω , triangular element $K \in \mathcal{T}_h$
piecewise quadratic element : 6 DOF on element K .

$$\varphi_i|_K(x_1, x_2) = a_0 + a_1x_1 + a_2x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2$$

$$\varphi_i|_K(P_j) = \delta_{ij}$$

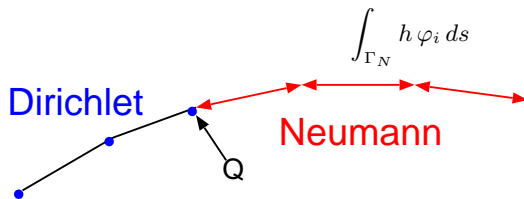


by using area coordinates $\{\lambda_1, \lambda_2, \lambda_3\}$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{pmatrix} = \begin{pmatrix} 1 & & & -1 & -1 \\ & 1 & & -1 & -1 \\ & & 1 & -1 & -1 \\ & & & 4 & \\ & & & & 4 \\ & & & & & 4 \end{pmatrix} \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \\ \lambda_2\lambda_3 \\ \lambda_3\lambda_1 \\ \lambda_1\lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(2\lambda_1 - 1) \\ \lambda_2(2\lambda_2 - 1) \\ \lambda_3(2\lambda_3 - 1) \\ 4\lambda_2\lambda_3 \\ 4\lambda_3\lambda_1 \\ 4\lambda_1\lambda_2 \end{pmatrix}$$

treatment of Neumann data around mixed boundary

Neumann data is evaluated by line integral with FEM basis φ_i .



For given discrete Neumann data, h is interpolated in FEM space,
 $h = \sum_j h_j \varphi_j|_{\Gamma_N}$,

$$\sum_j h_j \int_{\Gamma_N} \varphi_j \varphi_i ds.$$

On the node $Q \in \bar{\Gamma}_D \cap \bar{\Gamma}_N$, both Dirichlet and Neumann are necessary.

advantages of finite element formulation

- ▶ weak formulation is obtained by integration by part with clear description on the boundary
- ▶ Dirichlet boundary condition is embedded in a functional space, called as essential boundary condition
- ▶ Neumann boundary condition is treated with surface/line integral by Gauss-Green's formula, called as natural boundary condition
- ▶ solvability of linear system is inherited from solvability of continuous weak formulation
- ▶ error of finite element solution is evaluated by approximation property of finite element space

linear and nonlinear solid mechanics : 1/2

a problem with the second Piola-Kirchhoff stress tensor Σ
in Ω with $\partial\Omega = \Gamma_D \cup \Gamma_N$

$$\begin{aligned} -\operatorname{div}((I + \nabla u)\Sigma(u(x))) &= f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ (I + \nabla u)\Sigma(u(x))n &= g(x) && \text{on } \Gamma_N. \end{aligned}$$

Green-St. Venant tensor $E(u)$

$$E(u) = \frac{1}{2} \left((\nabla u)^T + \nabla u + (\nabla u)^T (\nabla u) \right) = E_L(u) + E_{NL}(u)$$

Piola-Kirchhoff stress tensor with Lamé constants λ and μ

$$\Sigma(u) = \check{\Sigma}(E(u)) = \lambda (\operatorname{tr}(E(u))) I + 2\mu E(u)$$

variational problem with test function v , $v = 0$ on Γ_D

$$\begin{aligned} - \int_{\Omega} \operatorname{div}(I + \nabla u) \check{\Sigma}(E(u)) \cdot v &= \int_{\Omega} f \cdot v \\ \Leftrightarrow \int_{\Omega} (I + \nabla u) \check{\Sigma}(E(u)) : \nabla v &= \int_{\Omega} f \cdot v + \int_{\Gamma_N \cup \Gamma_D} (I + \nabla u) \check{\Sigma}(E(u))n \cdot v \\ &= \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v \end{aligned}$$

Dirichlet data is treated as the essential boundary condition

linear and nonlinear solid mechanics : 2/2

The variational equation is re-written as

$$\int_{\Omega} \check{\Sigma}(E(u)) : dE(u)[v] = \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v.$$

symmetry of $\check{\Sigma}(E)$ leads to

$$\begin{aligned}(I + \nabla u) \check{\Sigma}(E(u)) : \nabla v &= \check{\Sigma}(E(u)) : (I + \nabla u)^T \nabla v \\ &= \frac{1}{2} \check{\Sigma}(E(u)) : \left((I + \nabla u)^T \nabla v + \nabla v^T (I + \nabla u) \right) \\ &= \frac{1}{2} \check{\Sigma}(E(u)) : \left(\nabla v + (\nabla u)^T \nabla v + \nabla v^T + \nabla v^T \nabla u \right) \\ &= \check{\Sigma}(E(u)) : dE(u)[v]\end{aligned}$$

tensors $A, B, C \Rightarrow AB : C = B : A^T C$

$$\sum_{i,j} \left(\sum_k [A]_{ik} [B]_{kj} \right) [C]_{ij} = \sum_{k,j} [B]_{kj} \left(\sum_i [A^T]_{ki} [C]_{ij} \right)$$

- ▶ nonlinear solver by Newton iteration (linearization for iterative solver)
- ▶ linearization of the Green-St. Venant tensor $E(u)$ by $E_L(u)$

linearized elasticity

Piola-Kirchhoff stress tensor with Lamé constants λ and μ

$$\Sigma(u) = \check{\Sigma}(E(u)) = \lambda (\operatorname{tr}(E(u))) I + 2\mu E(u)$$

linearization of Green-St. Venant tensor $E(u)$

$$E(u) = \frac{1}{2} \left((\nabla u)^T + \nabla u + (\nabla u)^T (\nabla u) \right) \simeq \frac{1}{2} \left((\nabla u)^T + \nabla u \right) = E_L(u) = e(u)$$

$$\begin{aligned} E_L(u+v) - E_L(u) &= \frac{1}{2} \left((\nabla(u+v))^T + \nabla(u+v) \right) - \frac{1}{2} \left((\nabla u)^T + \nabla u \right) \\ &= \frac{1}{2} \left((\nabla v)^T + \nabla v \right) = dE_L(u)[v] = E_L(v) \end{aligned}$$

$$\begin{aligned} \check{\Sigma}(E(u)) : dE_L(u)[v] &= \lambda (\operatorname{tr}(E(u))) I : E_L(v) + 2\mu E_L(u) : dE_L(v) \\ &= \lambda \nabla \cdot u \nabla \cdot v + 2\mu E_L(u) : E_L(v) \end{aligned}$$

variational problem of the linear elasticity with test function v , $v = 0$ on Γ_D

$$\int_{\Omega} 2\mu e(u) : e(v) + \lambda \nabla \cdot u \nabla \cdot v = \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v$$

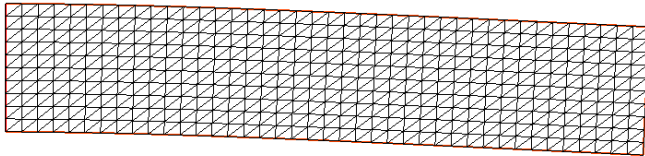
Korn's inequality $u \in H_1^1(\Omega)^3$

$$\Rightarrow \exists \alpha(\Omega) \int_{\Omega} e(u) : e(u) \geq \alpha(\Omega) \int_{\Omega} u \cdot u + \nabla u : \nabla u$$

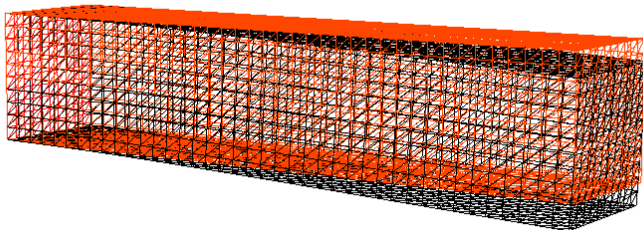
A : tensor $\Rightarrow I : A = \operatorname{tr}(A)$, $\operatorname{tr}(e(u)) = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = \nabla \cdot u$

linear elasticity problem in 3D : 4/4

2D beam : displacement on the left wall is fixed



3D beam : displacement on the left wall is fixed



solution of nonlinear problem by Newton iteration

nonlinear problem: $F : \mathbb{R}^N \rightarrow \mathbb{R}^N, \vec{x} \in \mathbb{R}^N$

$$\text{find } \vec{x} \quad F(\vec{x}) = \vec{0}$$

Fréchet derivative of $F : F(\vec{x} + \delta\vec{x}) - F(\vec{x}) \simeq \nabla F(\vec{x})\delta\vec{x}$

Jacobian matrix $\nabla F(\vec{x}) \in \mathbb{R}^{N \times N}$

Algorithm : Newton iteration

$\vec{x}_0 \in \mathbb{R}^N$: initial guess

loop $n = 0, 1, 2, \dots$

$$\nabla F(\vec{x}_{n-1})\delta\vec{x} = F(\vec{x}_{n-1})$$

$$\vec{x}_n = \vec{x}_{n-1} - \delta\vec{x}$$

nonlinear variational problem $f(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$

$$\text{find } x \in V(g) \quad F(x, y) = 0 \quad \forall y \in V(0)$$

Fréchet derivative of $F(\cdot, \cdot) :$

$$F(x + \delta x, y) - F(x, y) \simeq (\nabla F(x)\delta x, y) \quad \forall y \in V(0)$$

Algorithm : Newton iteration

$x_0 \in V(g)$: initial guess

loop $n = 0, 1, 2, \dots$

$$\text{find } \delta x \in V(0) \quad (\nabla F(x_{n-1})\delta x, y) = F(x_{n-1}, y) \quad \forall y \in V(0)$$

$$x_n = x_{n-1} - \delta x$$

nonlinear elasticity : 1/3

derivatives of Green-St. Venant tensor

- ▶ first order derivative $dE(u)[v]$

$$\begin{aligned}dE(u)[v] &= \frac{1}{2} \left((\nabla v)^T + \nabla v \right) + \frac{1}{2} \left((\nabla u)^T \nabla v + (\nabla v)^T \nabla u \right) \\ &= E_L(v) + dE_{NL}(u)[v]\end{aligned}$$

calculated from variation as

$$\begin{aligned}E(u+v) - E(u) &= (E_L(u+v) + E_{NL}(u+v)) - (E_L(u) + E_{NL}(u)) \\ &= E_L(v) + \frac{1}{2} (\nabla(u+v))^T \nabla(u+v) - E_{NL}(u) \\ &= E_L(v) + \frac{1}{2} ((\nabla u)^T \nabla v + (\nabla v)^T \nabla u + (\nabla v)^T \nabla v)\end{aligned}$$

- ▶ second order derivative $d^2 E(u)[v, w]$

$$d^2 E(u)[v, w] = dE_{NL}(w)[v]$$

calculated from variation as

$$\begin{aligned}dE(u+w)[v] - dE(u)[v] &= (E_L(v) + dE_{NL}(u+w)[v]) - (E_L(v) + dE_{NL}(u)[v]) \\ &= \frac{1}{2} \left((\nabla(u+w))^T \nabla v + (\nabla v)^T \nabla(u+w) \right. \\ &\quad \left. - (\nabla u)^T \nabla v + (\nabla v)^T \nabla u \right) \\ &= \frac{1}{2} \left((\nabla w)^T \nabla v + (\nabla v)^T \nabla w \right)\end{aligned}$$

nonlinear elasticity : 2/3

The variational equation

$$\int_{\Omega} \check{\Sigma}(E(u)) : dE(u)[v] = \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v.$$

Newton iteration of the nonlinear system u^0 : given

loop $n = 0, 1, 2, \dots$

 solve linear system to find update w ,

$$\begin{aligned} & \int_{\Omega} \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2E(u^n)[v, w] \\ & = \int_{\Omega} \check{\Sigma}(E(u^n)) : dE(u^n)[v] - \int_{\Omega} f \cdot v - \int_{\Gamma_N} g \cdot v \quad \forall v \end{aligned}$$

 update $u^{n+1} = u^n - w$

Jacobian of the variational problem is calculated as

$$\begin{aligned} & \int_{\Omega} \check{\Sigma}(E(u^n + w)) : dE(u^n + w)[v] - \int_{\Omega} \check{\Sigma}(E(u^n)) : dE(u^n)[v] \\ & \simeq \int_{\Omega} \check{\Sigma}(E(u^n) + dE(u^n)[w]) : (dE(u^n)[v] + d^2E(u^n)[v, w]) \\ & \quad - \int_{\Omega} \check{\Sigma}(E(u^n)) : dE(u^n)[v] \\ & \simeq \int_{\Omega} \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2E(u^n)[v, w] \end{aligned}$$

nonlinear elasticity : 3/3

Jacobian for Newton iteration

$$\int_{\Omega} \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2E(u^n)[v, w].$$

is symmetric, which is shown from

- ▶ $d^2E(u^n)[v, w] = dE_{NL}(w)[v] = \frac{1}{2} ((\nabla w)^T \nabla v + (\nabla v)^T \nabla w)$ is sym.
- ▶ by recalling $I : A = \text{tr}(A)$ and for symmetric tensors η and ζ

$$\begin{aligned}\check{\Sigma}(\eta) : \zeta &= (\lambda \text{tr}(\eta) I + 2\mu \eta) : \zeta = \lambda \text{tr}(\eta) \text{tr}(\zeta) + 2\mu \eta : \zeta \\ &= \lambda \text{tr}(\zeta) \text{tr}(\eta) + 2\mu \zeta : \eta = \check{\Sigma}(\zeta) : \eta\end{aligned}$$

coercivity depends on previous iteration step u^n

- ▶ the first half part :

$$\int_{\Omega} \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[w] \geq 2 \int_{\Omega} \mu E_L(w) : E_L(w)$$

- ▶ the second half part : coercivity depends on $\check{\Sigma}(E(u^n))$

$$\begin{aligned}& \int_{\Omega} \check{\Sigma}(E(u^n)) : d^2E(u^n)[w, w] \\ &= \int_{\Omega} \check{\Sigma}(E(u^n)) : dE_{NL}(w)[w] = \int_{\Omega} \check{\Sigma}(E(u^n)) : (\nabla w)^T (\nabla w) \\ &= \int_{\Omega} \check{\Sigma}(E(u^n)) (\nabla w)^T : (\nabla w)^T\end{aligned}$$

nonlinear iteration by Newton method : 1/2

Jacobian and RHS are calculated from the previous u^n in Newton step

$$(\nabla F(u^n)w, v) = \int_{\Omega} \check{\Sigma}(dE(u^n)[w]) : dE(u^n)[v] + \check{\Sigma}(E(u^n)) : d^2E(u^n)[v, w] \quad \forall v$$

$$(F(u^n), v) = \int_{\Omega} \check{\Sigma}(E(u^n)) : dE(u^n)[v] \quad \forall v$$

initial guess of the Newton iteration by linear elasticity with small load given by inhomogeneous Dirichlet data g

solution space $V(g) := \{u \in H^1(\Omega)^3 ; u = g \text{ on } \Gamma_D\}$

linear elasticity problem is solved as

$$\text{to find } u^0 \in V(g) \quad \int_{\Omega} \check{\Sigma}(e(u^0)) : e(v) = 0 \quad \forall v \in V(0)$$

and then perform Newton iteration

loop $n = 0, 1, 2, \dots$

$$\text{find } w \in V(0) \quad (\nabla F(u^n)w, v) = (F(u^n), v) \quad \forall v \in V_0$$

$$u^{n+1} = u^n - w$$

The update w in the Newton iteration is supposed to have homogeneous Dirichlet boundary condition, i.e., $w \in V(0)$.

nonlinear iteration by Newton method : 2/2

a method as incremental loading:

sequence of in homogeneous Dirichlet data whose magnitude is increasing

$$\|g^{(0)}\| < \|g^{(1)}\| < \dots < \|g^{(N)}\|, \quad g = g^{(N)}$$

The procedure consists of nested loops for increment and Newton iteration.

perform incremental loading::

loop $m = 0, 1, 2, \dots, N - 1$

if $m = 0$ then solve linear elasticity

$$\text{to find } u^0 \in V(g^0) \quad \int_{\Omega} \check{\Sigma}(e(u^0)) : e(v) = 0 \quad \forall v \in V(0)$$

otherwise ($m > 0$) set $u^{m,0} \in V(g^{(m)})$ from previous $u^{m-1} \in V(g^{(m-1)})$.

perform Newton iteration::

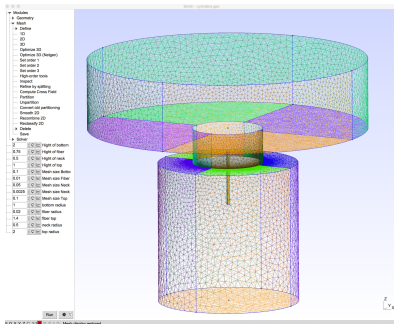
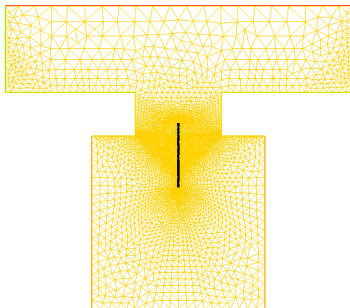
loop $n = 0, 1, 2, \dots$

$$\text{find } w \in V(0) \quad (\nabla F(u^{m,n})w, v) = (F(u^{m,n}), v) \quad \forall v \in V_0$$

$$u^{m,n+1} = u^{m,n} - w$$

description of complex geometry

- ▶ 2D : bamg embedded in FreeFEM generates isotropic triangulation of the domain consisting of parameterized segments. By changing of the metric based on finite element solution, an-isotropic triangulation is also performed
- ▶ 3D : Gmsh can handle parameterized surface segments and/or treat STEP file by OpenCASCADE geometry engine. Surface mesh generator and volume mesh generator constructs isotropic tetrahedral mesh decomposition



2D geometry definition by FreeFEM

```
real rb = 1.0; real lb = 2.0; // ....
border c1(t=0,1){x=-rb + 2*rb*t; y = 0.0; label = 2;};
border c2(t=0,1){x=rb; y = lb * t; label = 3;};
border c3(t=0,1){x=(rb + (r0 - rb) * t); y = lb; label = 4;};
border c4(t=0,1){x=r0; y = lb + l0 * t*t; label = 5;};
border c5(t=0,1){x=r0 + (rc - r0) * t; y = lb+l0; label = 6;};
border c6(t=0,1){x=rc; y = lb + l0 + lc * t; label = 7;};
border c7(t=0,1){x=-rc + 2*rc * t; y = lb+l0+lc; label = 1;};
//
border c8(t=0,1){x=-rb; y = lb * t; label = 3;};
border c9(t=0,1){x=-(rb + (r0 - rb) * t); y = lb; label = 4;};
// ...
// fiber
border d1(t=0,1){x=-rd+2*rd * t + re; y = le; label = 8;};
border d2(t=0,1){x = rd+re; y = le + ld * t; label = 8; };
border d3(t=0,1){x=rd-2*rd * t + re; y = le+ld; label=8;};
border d4(t=0,1){x = -rd+re; y = le+ld * t; label = 8; };

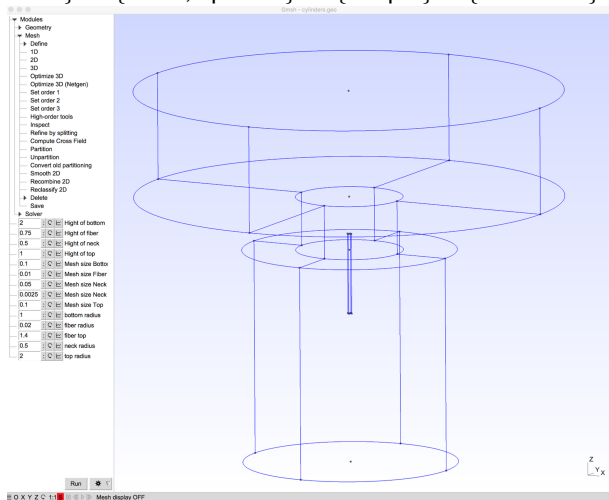
int n = 20; int nn = 20; int nnn = 50;
mesh Th = buildmesh(c1(n) + c2(n) + c3(nn) + c4(nn)
    + c5(n) + c6(nn) + c7(-n)
    + c8(-n) + c9(-nn) + c10(-nnn) + c11(-n) + c12(-nn)
    + d1(nn) + d2(nnn) + d3(nn) +d4(-nnn));
```

3D geometry definition by Gmsh : 1/4

```
pext = newp;  
Point(pext+0) = {0, 0, 0, hBottom};  
Point(pext+1) = {RRb, 0, 0, hBottom};  
Point(pext+2) = {0, RRb, 0, hBottom};  
Point(pext+3) = {-RRb, 0, 0, hBottom};  
Point(pext+4) = {0, -RRb, 0, hBottom};  
lext = newl;  
Circle(lext+0) = {pext+1, pext+0, pext+2};  
Circle(lext+4) = {pext+6, pext+5, pext+7};  
Line(lext+8) = {pext+1, pext+6};  
Line(lext+9) = {pext+2, pext+7};  
llex = newll;  
Line Loop(llex+0) = {lext+0, lext+9, -(lext+4), -(lext+8)};  
// ..  
sext = news;  
Ruled Surface(sext+0) = {llex+0};          Side[0] = sext+0;  
// ..  
Physical Surface(SIDE) = {Side[], Neck[]};  
// ..  
sl = newsl;  
Surface Loop(sl+1) = {Neck[], NeckFine[], Side[], Top[], Bottom[]};  
v = newv;  
Volume(v+0) = {sl+1, sl+2};
```


3D geometry definition by Gmsh : 2/4

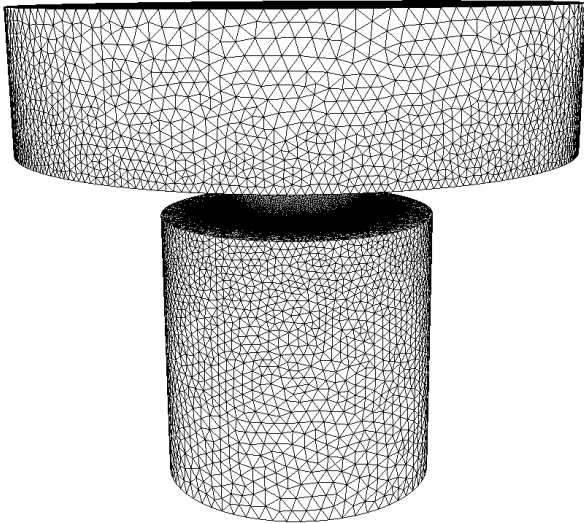
{ points } → { lines, splines } → { loops } ≈ { surfaces } → { volumes }



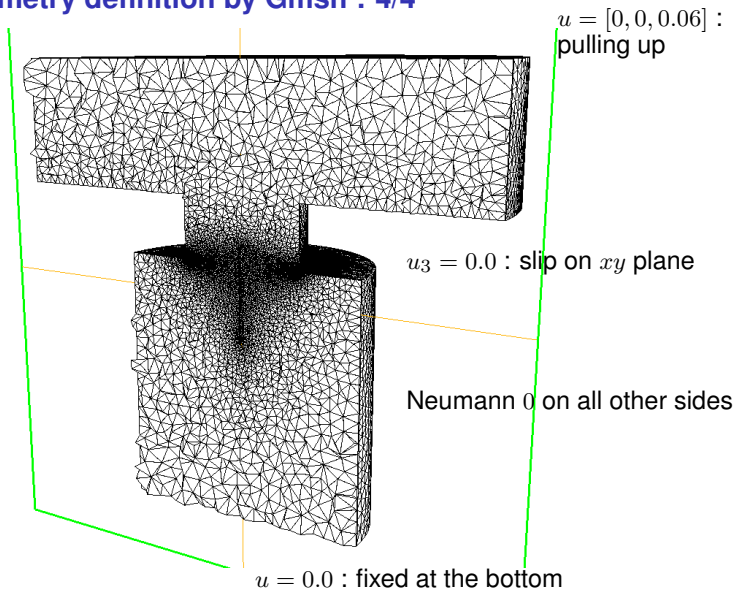
surface segments and generated mesh can be verified by GUI
command line option can generate mesh readable by FreeFEM

```
gmsht -3 -format msh2 -o test.msh test.geo
```

3D geometry definition by Gmsh : 3/4

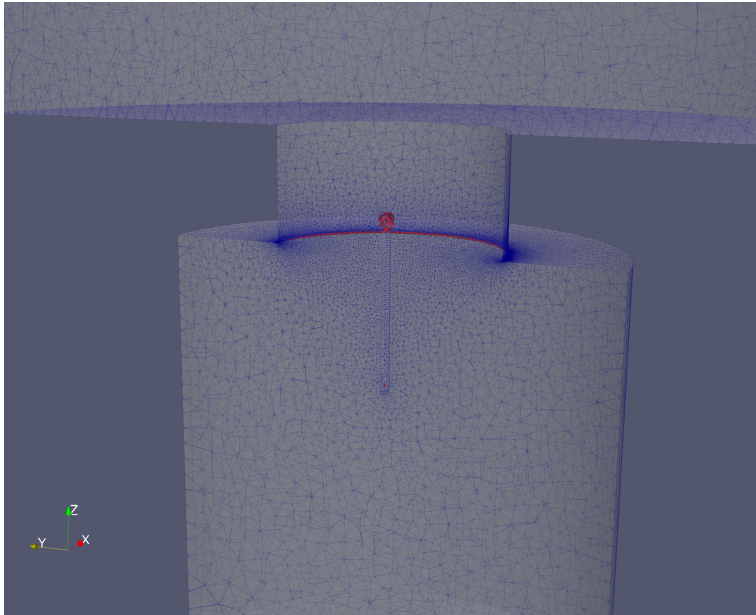


3D geometry definition by Gmsh : 4/4



- ▶ height of the domain is 2.5, radius of the top is 2, bottom 1
- ▶ internal fiber has 100 times large Lamé constants than the surround

stress concentration



708,947 vertexes, 2,076,809 DOF with P1 element

References

- ▶ Finite Elements: Theory, Fast Solvers, and Applications in Elasticity Theory, 3rd ed. D. Braess Cambridge University Press, 2010
- ▶ Numerical Models for Differential Problems, 2nd ed. A. Quarteroni Springer, 2014 ISBN 978-88-470-5521-6
- ▶ Finite Elements I, A. Ern, J.-L. Guermond Springer, 2021 ISBN 978-3-030-56340-0
- ▶ The Mathematical Theory of Finite Element Methods, 3rd ed. S. Brenner, R. Scott Springer, 2008 ISBN 978-0-387-75933-3
- ▶ Numerical Approximation of Partial Differential Equations A. Quarteroni, A. Valli Springer, 2008
- ▶ Mathematical Elasticity: Three-Dimensional Elasticity. P. Ciarlet, SIAM Classics in Applied Mathematics, 2021